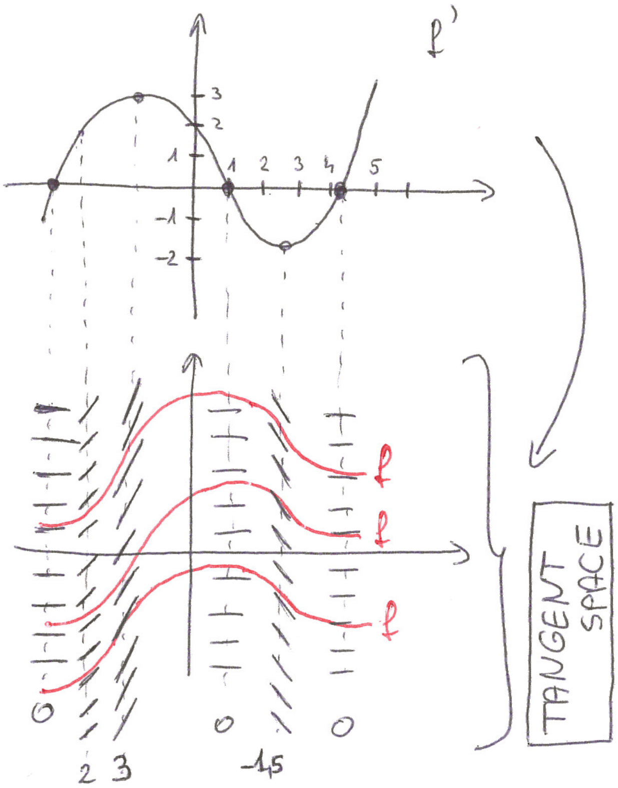
Practice 7

1. Graphical integral

General problem: $f'(x)$ is given but what can be $f(x)$?

What can we say about $f(x)$, when $f'(x)$ is given by its graph?



f' measures the values of the slope of the tangents.

- When f' has zero point, then f has horizontal tangent lines in that points.

!! We don't know the value of f, we know only the value of the slope!!

- In the non-zero points we sketch the segments of the tangents with the appropriate slopes.

After the tangent space, we can sketch "some original $f(x)$".

Notes:
- From the tangent space we cannot reconstruct a unique original $f(x)$.
- The original functions f has the same form, they differs only in a vertical shifting.

Definition $F(x)$ is the primitive function of the function $f(x)$, if $F' = f$.

Remarks:
- There are infinitely many primitive functions.
- If F_1 and F_2 are primitive functions of the same $f(x)$, the $F_1 = F_2 + C$, where c is a constant.

Definition: Indefinite integral is the set of primitive functions.

$$\int f(x)\,dx = \{F(x)+c,\ c\in\mathbb{R}\}$$

2. Formal integral

$f(x)$	$\int f(x)\,dx$
c	cx
$(\alpha\neq -1)$ x^α	$\frac{x^{\alpha+1}}{\alpha+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$
a^x	$\frac{a^x}{\ln a}$
e^x	e^x
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\frac{1}{\cos^2 x}$	$\tan x$

(all primitives $+C$)

Rules for the integral

- $\int f+g\,dx = \int f\,dx + \int g\,dx$
- $\int f-g\,dx = \int f\,dx - \int g\,dx$
- $\int c\cdot f\,dx = c\cdot\int f\,dx \quad (c\in\mathbb{R})$
- $\int f\cdot g\,dx \neq \int f\,dx \cdot \int g\,dx$

This is a difficult problem.

Problem 1 Give the indefinite integral of the following functions.

$$\int (x^2-3x+2)\,dx = ?\quad \int x^2 dx - \int 3x\,dx + \int 2\,dx = \frac{x^3}{3} - 3\frac{x^2}{2} + 2x + C$$

$$\int \frac{x\cdot 2^x - x^3 + \sqrt[3]{x}}{x}\,dx = ?\ \int\left(\frac{x\,2^x}{x} - \frac{x^3}{x} + \frac{\sqrt[3]{x}}{x}\right)dx = \int\left(2^x - x^2 + x^{-2/3}\right)dx = \frac{2^x}{\ln 2} - \frac{x^3}{3} + \frac{x^{1/3}}{1/3} + C$$

$$\int \frac{x^2\cos x + x - 1 + \sqrt{x}}{x^2}\,dx = ?\ \int\left(\cos x + \frac{1}{x} - \frac{1}{x^2} + x^{-3/2}\right)dx = \sin x + \ln|x| - \frac{x^{-1}}{-1} + \frac{x^{-1/2}}{-1/2} + C$$

$$\int (e^x + 2 - \cos x)\,dx = ?\ e^x + 2x - \sin x + C$$

3.) Initial value problem

General problem: Using the technic of indefinite integral, we get a lot of primitive functions.
If we fix a point (x_0, y_0), then there exists only one primitive function passing through that point.

Problem 2. Determine the primitive function of $f(x) = \dfrac{2x^3 - x^2}{x^3}$

such that $F(1) = 3$.

1.) $\displaystyle\int f(x)\,dx = \int \frac{2x^3 - x^2}{x^3}\,dx = \int \left(2 - \frac{1}{x}\right)dx = 2x - \ln|x| + C$

2.) $F(x) = 2x - \ln|x| + C$

3.) $F(1) = 2 \cdot 1 - \ln 1 + C \overset{!}{=} 3$

$$2 - 0 + C = 3$$

$$C = 1 \quad \Rightarrow \quad \underline{\underline{F(x) = 2x - \ln|x| + 1}}$$

4.) Integration by substitution

Linear substitution

We use this method if we know the $F(x)$ primitive function of $f(x)$. Then $\boxed{\displaystyle\int f(ax+b)\,dx = \frac{F(ax+b)}{a} + C.}$

Problem 3.

$$\int e^{3x-3}\,dx =? \quad \frac{e^{3x-3}}{3} + C$$

$$\int \sin(2x-1)\,dx =? \quad \frac{-\cos(2x-1)}{2} + C$$

$$\int \frac{1}{3x+9}\,dx = \frac{\ln(3x+9)}{3} + C$$

$$\int \sqrt[3]{2x-5}\,dx = \frac{\dfrac{(2x-5)^{4/3}}{4/3}}{2} + C$$

$$\int \cos(2-x)\,dx = \frac{\sin(2-x)}{-1} + C$$

$$\int 2^{1-3x}\,dx = \frac{\dfrac{2^{1-3x}}{\ln 2}}{-3} + C = \frac{2^{1-3x}}{-3 \cdot \ln 2} + C$$