Practice 6

Applications of the derivative

① Equation of the tangent line.

$$\boxed{y - f(x_0) = f'(x_0)\cdot(x-x_0)}$$ — The equation of the tangent to f at the point x_0.

Problem 1. Give the equation of the tangent line to the function $f(x) = x^3 - 3\sqrt{x} + \frac{1}{x^2}$ at the point $x_0 = 1$.

1.) $f(x_0) = 1^3 - 3\cdot\sqrt{1} + \frac{1}{1^2} = 1 - 3 + 1 = \underline{\underline{-1}}$

2.) $f'(x) = 3x^2 - 3\cdot\frac{1}{2}\cdot x^{-1/2} + (-2)\cdot x^{-3} = 3x^2 - \frac{3}{2}\cdot\frac{1}{\sqrt{x}} - 2\cdot\frac{1}{x^3}$

3.) $f'(x_0) = 3 - \frac{3}{2} - 2 = \underline{\underline{-\frac{1}{2}}}$

Tangent: $y - (-1) = -\frac{1}{2}\cdot(x-1)$

Problem 2. $f(x) = xe^{-x^2}$, $x_0 = 0$

1.) $f(x_0) = 0\cdot e^{0^2} = \underline{\underline{0}}$

2.) $f'(x) = \left[x\cdot e^{-x^2}\right]' = e^{-x^2} + x\cdot e^{-x^2}\cdot(-2x) = e^{-x^2} - 2x^2e^{-x^2}$
$= e^{-x^2}(1-2x^2)$

3.) $f'(x_0) = e^0\cdot(1-0) = \underline{\underline{1}}$

Tangent: $y - 0 = 1\cdot(x-0)$

$\underline{\underline{y = x}}$

② Investigation of the function

Problem 3. Investigate the shape of the function: $f(x) = 3x^5 - 10x^3 - 4$ according to monotonicity and convexity.

1.) Zero point: $3x^5 - 10x^3 - 4 = 0$.

$\uparrow$ We cannot solve it.

2.) First derivative: $f' = 15x^4 - 30x^2$

$f' = 0$

$15x^4 - 30x^2 = 0$

$15x^2(x^2 - 2) = 0$

Critical points: $-\sqrt{2}$, 0, $\sqrt{2}$

$x^2 = 0$ or $x^2 - 2 = 0$

$x = 0$ $\quad$ $x^2 = 2$

$x = \pm\sqrt{2}$

	-2	$\approx -1{,}4$	-1		1		2
	$x <$	$-\sqrt{2}$	$< x <$	0	$< x <$	$\sqrt{2}$	$< x$
f'	$+$	0	$-$	0	$-$	0	$+$
f	$\nearrow$	MAX	$\searrow$		$\searrow$	MIN	$\nearrow$

NO EXTR.

$f'(-2) = 15 \cdot (-2)^2 ((-2)^2 - 2)$

$= 15 \cdot 4 \cdot 2 = \oplus$

$f'(-1) = 15 \cdot (-1)^2 \cdot ((-1)^2 - 2)$

$= 15 \cdot 1 \cdot (-1) = \ominus$

$f'(1) = 15 \cdot 1^2 \cdot (-1) = \ominus$

$f'(2) = 15 \cdot 4 \cdot 2 = \oplus$

$f(-\sqrt{2}) = 3 \cdot (-\sqrt{2})^5 - 10(-\sqrt{2})^3 - 4$

$= -12\sqrt{2} + 20\sqrt{2} - 4$

$= 8\sqrt{2} - 4$

$f(\sqrt{2}) = 3 \cdot (\sqrt{2})^5 - 10(\sqrt{2})^3 - 4$

$= 12\sqrt{2} - 20\sqrt{2} - 4$

$= -8\sqrt{2} - 4$

MAX: $x = -\sqrt{2}$, $y = 8\sqrt{2} - 4$

MIN: $x = \sqrt{2}$, $y = -8\sqrt{2} - 4$

3.) Second derivative: $f'' = 60x^3 - 60x = 60(x)\cdot(x^2-1)$

$f''=0 \Leftrightarrow 60x(x^2-1)=0$

$x=0$ or $x^2-1=0$

$x^2=1$

$x=\pm 1$

Critical points: $-1, 0, 1$

	-2		$1/2$		$1/2$		2
	$x<$	-1	$<x<$	0	$<x<$	1	$<x$
f''	$-$	0	$+$	0	$-$	0	$+$
f	$\cap$	ip	$\cup$	ip	$\cap$	ip	$\cup$

$f''(-2) = 60\cdot(-2)(4-1) = \ominus$ $\qquad f''(2) = 60\cdot 2\cdot(4-1) = \oplus$

$f''(1/2) = 60\cdot\frac{1}{2}\cdot\left(\frac{1}{4}-1\right) = \ominus$ $\qquad f''\left(-\frac{1}{2}\right) = 60\cdot\left(-\frac{1}{2}\right)\left(\frac{1}{4}-1\right) = \oplus$

Inflection points:

$x=-1 \qquad f(-1) = -3+10-4 = 3$

$x=0 \qquad f(0) = -4$

$x=1 \qquad f(1) = 3-10-4 = -11$

Problem 4. $f(x) = e^{-\frac{x^2}{2}}$

1.) Zero points: $f=0 \Leftrightarrow e^{-\frac{x^2}{2}} = 0$

No solution $\Rightarrow$ No zero points.

2.) First derivative: $f' = e^{-\frac{x^2}{2}}\cdot\left(-\frac{1}{2}\right)\cdot 2x = -xe^{-\frac{x^2}{2}}$

$f'=0 \Leftrightarrow -xe^{-\frac{x^2}{2}} = 0 \Leftrightarrow x=0$

Critical point(~~s~~): 0

	$x<$ (-1)	0	$<x$ (1)
f'	$+$	0	$-$
f	$\nearrow$	$\uparrow$	$\searrow$

MAX

$f(0) = e^0 = 1$

$f'(-1) = -(-1)\cdot e^{\square} = 1\cdot \oplus = \oplus$

$f'(1) = -1\cdot e^{\square} = -1\cdot\oplus = \ominus$

3.) Second derivative: $f'' = (-1)e^{-\frac{x^2}{2}} + (-x)\, e^{-\frac{x^2}{2}}\cdot\left(-\frac{1}{2}\right)\cdot 2x$

$= -e^{-\frac{x^2}{2}} + x^2 e^{-\frac{x^2}{2}} = e^{-\frac{x^2}{2}}\cdot\left(x^2-1\right)$

$f''=0 \Leftrightarrow e^{-\frac{x^2}{2}}(x^2-1) = 0 \Leftrightarrow x^2-1=0$

$x^2 = 1$

$x = \pm 1$

Critical points: $-1,\ 1$

	$x<$ (-2)	-1	$<x<$ (0)	1	$<x$ (2)
f''	$+$	0	$-$	0	$+$
f	$\cup$	$\uparrow$	$\cap$	$\uparrow$	$\cup$

IP IP

$f(-1) = e^{-1/2}$ $f(1) = e^{1/2}$

$f''(0) = e^{\square}(-1) = \oplus\cdot(-1) = \ominus$

$f''(2) = e^{\square}(4-1) = \oplus\cdot 3 = \oplus$

$f''(-2) = e^{\square}(4-1) = \oplus$