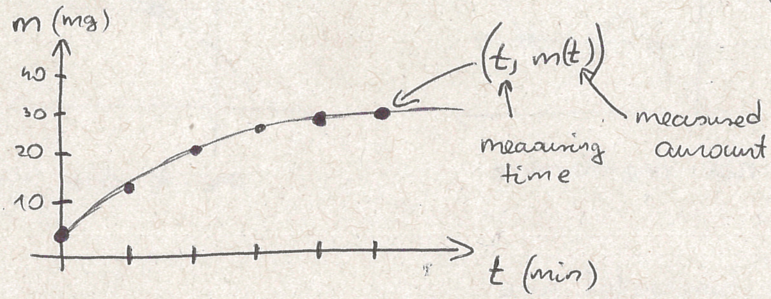
Practice 2.

Functions

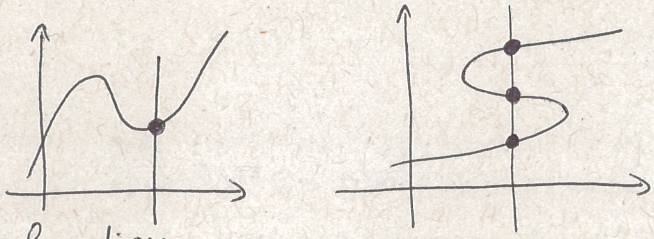
1.) Exemplification (introduction, motivation)

During the time we measure the amount of something: $t \mapsto m(t)$.



In high school: P point has (x, y) coordinates.

Function: each input is related to exactly one output.



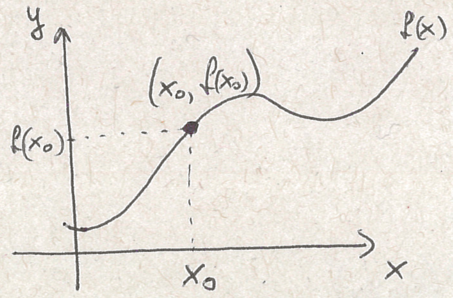
function

not function

← If a vertical line has more than one intersection, the it is not function.

2.) Graph of a function

(x, y) pairs on the x-y plane, where $y = f(x)$, and $f(x)$ is a function with variable x.



$x \rightarrow$ arguments

$y \rightarrow$ values

$f(2) = 3$ "f of 2 is 3"

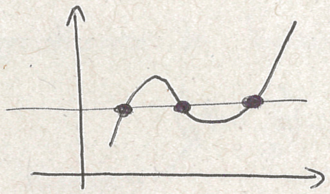
"the image of 2 is 3 under f"

DOMAIN : those x-values where f has y-value

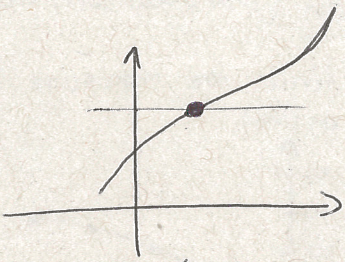
RANGE : those y-values that can be got by $y = f(x)$.

3.) Special functions

One-to-one function: every element of the range (y) of the function corresponds to exactly one element of the domain.



not one-to-one

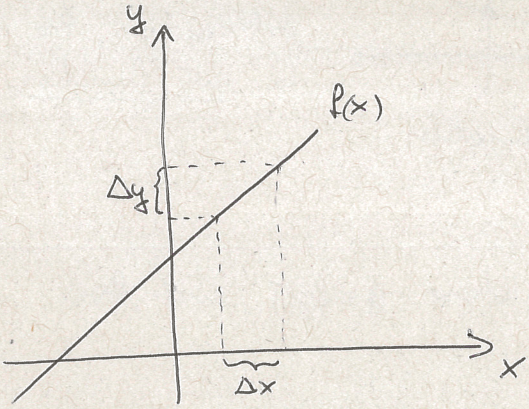


one-to-one → Eg: $x \mapsto ax+b$

If a horizontal line intersects the function more than once, it's not one-to-one.

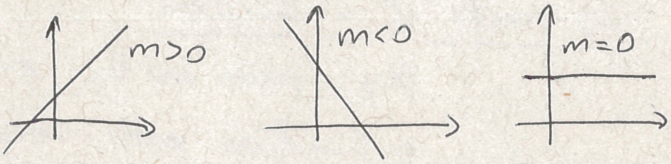
Eg. $x \mapsto x^2$

Linear Function (Line)



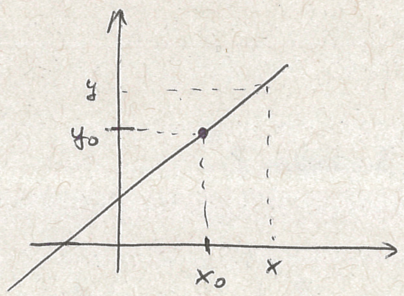
Slope: $m = \dfrac{\Delta y}{\Delta x}$

A line has constant slope.



Equations of a line

1. Given a point and the slope.



$P(x_0, y_0)$

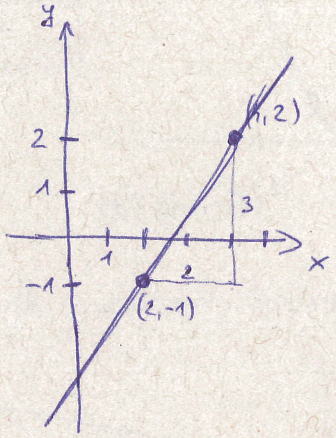
m

$$\frac{y-y_0}{x-x_0} = m \Rightarrow y - y_0 = m(x-x_0)$$

$$\boxed{y = m(x-x_0) + y_0}$$

Eg. $P(2,-1)$
$m = \frac{3}{2}$

Find the equation of the line. Plot it (without its equation).



$m = \frac{3 \to 3 \text{ to up}}{2 \to 2 \text{ to the right}}$

Equation:

$$y-(-1) = \frac{3}{2}(x-2)$$

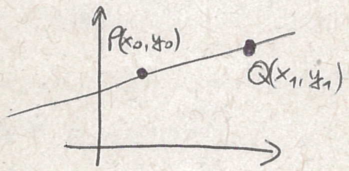
$$y+1 = \frac{3}{2}x - \frac{6}{2}$$

$$y = \frac{3}{2}x - 3 - 1$$

$$y = \frac{3}{2}x - 4$$

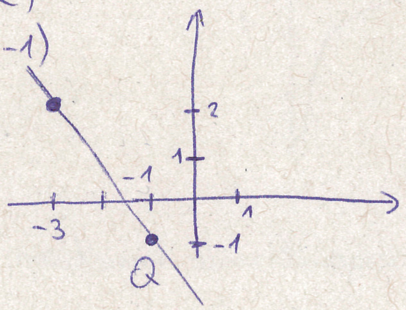
slope y-intersection

② Given to points.



$$m = \frac{y_1 - y_0}{x_1 - x_0} \Rightarrow ①$$

Eg. $P(-3,2)$
$Q(-1,-1)$



$$m = \frac{2-(-1)}{-3-(-1)} = \frac{3}{-2} = -\frac{3}{2}$$

$$(y-2) = -\frac{3}{2}(x-(-3))$$

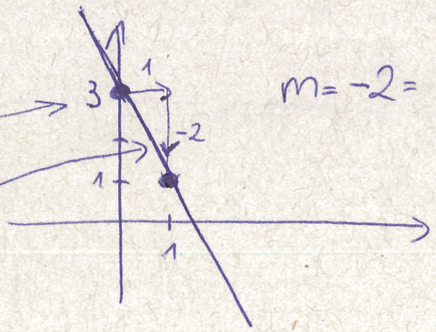
$$y-2 = -\frac{3}{2}(x+3)$$

$$y = -\frac{3}{2}(x+3)+2$$

$$y = -\frac{3}{2}x - \frac{9}{2} + 2 = -\frac{3}{2}x - \frac{5}{2}$$

③ Given: slope and y-intersection ($y = mx+b$)

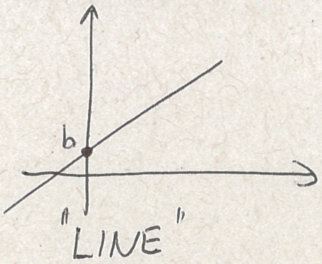
Eg. Given: $y = -2x+3$. Plot it.



(↕)
$$m = -2 = \frac{\Delta y}{\Delta x} = \frac{-2}{1}$$
(↔)

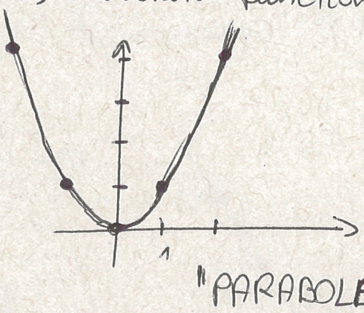
Basic functions

1.) Linear function $y = ax + b$



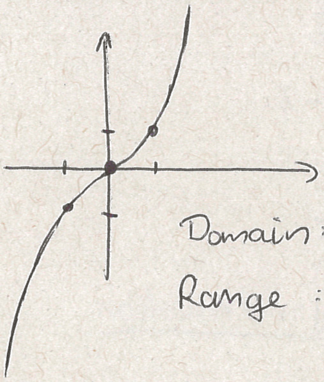
Domain: $\mathbb{R}$
Range: $\mathbb{R}$

2.) Quadratic function: $f(x) = x^2$



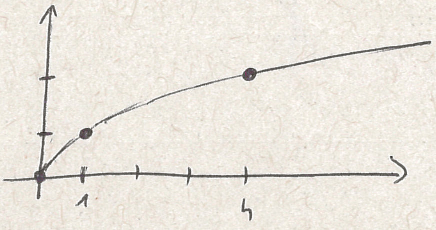
Domain: $\mathbb{R}$
Range: $\mathbb{R}^+_0$
$y \geq 0$

3.) Cube function: $f(x) = x^3$



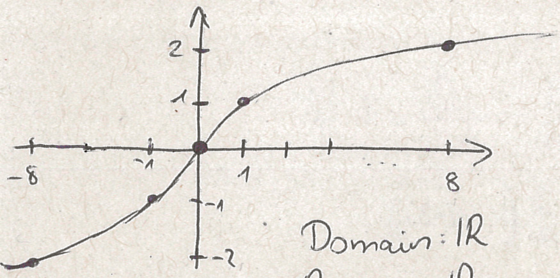
Domain: $\mathbb{R}$
Range: $\mathbb{R}$

4.) Squareroot function: $f(x) = \sqrt{x}$



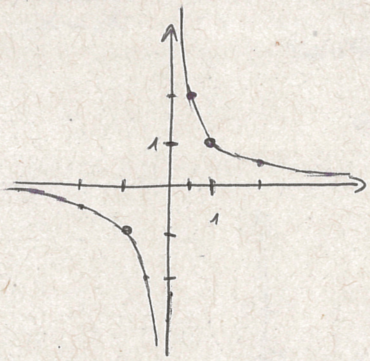
Domain: $x \geq 0$, Range: $y \geq 0$

5.) Cube root function: $f(x) = \sqrt[3]{x}$



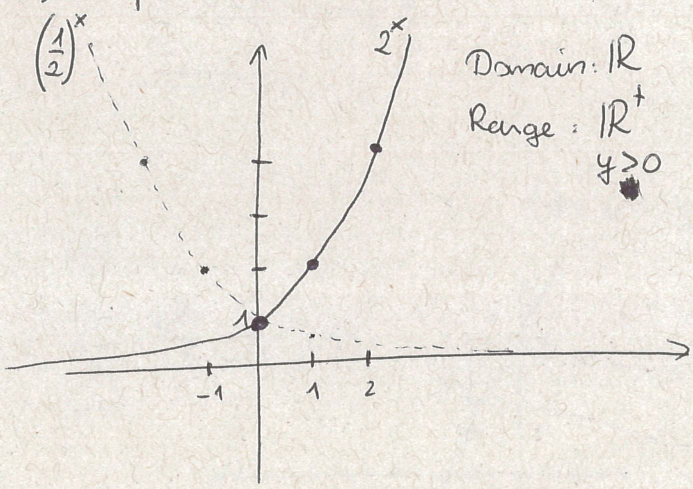
Domain: $\mathbb{R}$
Range: $\mathbb{R}$

6.) Reciprocal function: $f(x) = \frac{1}{x}$



Domain: $x \neq 0$
Range: $y \neq 0$

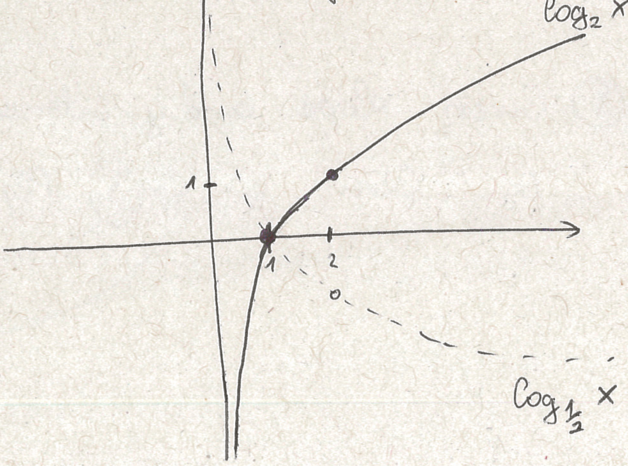
7.) Exponential function: $f(x) = 2^x$



Domain: $\mathbb{R}$
Range: $\mathbb{R}^+$
$y > 0$

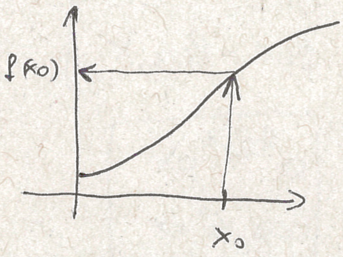
8.) Logarithmic function: $f(x) = \log_2 x$

Domain: $x > 0$
Range: $\mathbb{R}$



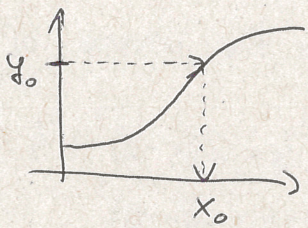
Inverse of one-to-one functions

General problem: x_0 is given, $f(x_0) = ?$



$y_0 = f(x_0)$

Inverse problem: $f(x_0)$ is given, $x_0 = ?$



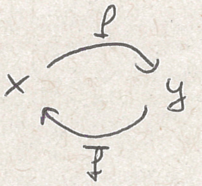
$x_0 = \bar{f}(y_0)$

Originally $f(x)$ is a function with variable x, and y depends on x. We want to find an $\bar{f}(y)$ function, that has variable y, x depends on y, and $y = f(x)$.

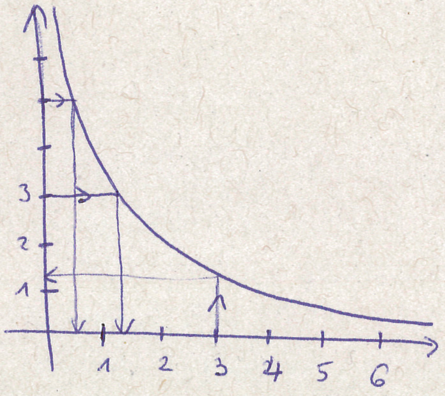
$\bar{f}(y) = x$ $\quad$ Inverse

$f: x \longmapsto y$

$\bar{f}: y \longmapsto x$



Eg.



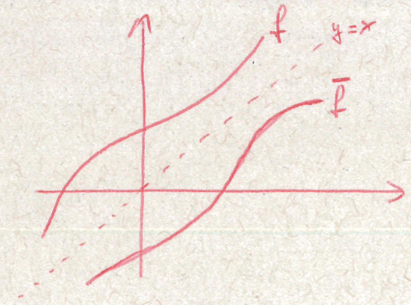
Estimate $f(3)$, $\bar{f}(3)$, $\bar{f}(5)$.

$f(3) \approx \frac{1}{3}$ $\qquad$ $\bar{f}(5) \approx 1/2$

$\bar{f}(3) \approx 1/3$

Connection between f and $\bar{f}$: $\boxed{f(\bar{f}(x)) = x, \quad \bar{f}(f(x)) = x}$

Graphical connection:



The graph of $\bar{f}$ is a reflection of the graph of f respect to the line $y = x$.

Important inverse pairs:

(1) $f(x) = a^x$

$\bar{f}(x) = \log_a x$

$$\boxed{\begin{aligned} a^{\log_a x} &= x \\ \log_a a^x &= x \end{aligned}}$$

(2) $f(x) = \sqrt{x}$

$\bar{f}(x) = x^2$

If $x \geq 0$!!

$(\sqrt{x})^2 = x$

$\sqrt{x^2} = |x| = x$

(3) $f(x) = x^3$

$\bar{f}(x) = \sqrt[3]{x}$

$(\sqrt[3]{x})^3 = x$

$\sqrt[3]{x^3} = x$

Eg. Give the inverse of the function $f(x) = 2(x-3)^3 + 4$.

Write: $y = 2(x-3)^3 + 4 \longrightarrow$ We have to solve it to x.

$$y - 4 = 2(x-3)^3$$

$$\frac{y-4}{2} = (x-3)^3$$

$$\sqrt[3]{\frac{y-4}{2}} = x - 3$$

$$x = \sqrt[3]{\frac{y-4}{2}} + 3 \longrightarrow \bar{f}(y) = \sqrt[3]{\frac{y-4}{2}} + 3$$

$$\bar{f}(x) = \sqrt[3]{\frac{x-4}{2}} + 3$$

Eg. $f(x) = 3 - 2 \cdot 10^{4x-5}$, $\bar{f}(x) = ?$

$$y = 3 - 2 \cdot 10^{4x-5}$$

$$y - 3 = -2 \cdot 10^{4x-5}$$

$$\frac{y-3}{-2} = 10^{4x-5}$$

$$\log_{10} \frac{y-3}{-2} = 4x - 5$$

$$5 + \lg \frac{y-3}{-2} = 4x$$

$$x = \frac{5 + \lg \frac{y-3}{-2}}{4} \longrightarrow \bar{f}(y) = \frac{5 + \lg \frac{y-3}{-2}}{4} \longrightarrow \bar{f}(x) = \frac{5 + \lg \frac{x-3}{-2}}{4}$$