Practice 10.

Applications of definite integral

1. Variation of a function

$$\begin{tikzcd}[column sep=huge] \begin{array}{c} s \\ \left(\begin{array}{c}\text{displacement} \\ \text{position}\end{array}\right)\end{array} \arrow[r, bend left, "\text{Diff}"] & \begin{array}{c} v \\ \left(\begin{array}{c}\text{velocity} \\ \text{speed}\end{array}\right)\end{array} \arrow[l, bend left, "\text{Int}"] \arrow[r, bend left, "\text{Diff}"] & \begin{array}{c} a \\ (\text{acceleration})\end{array} \arrow[l, bend left, "\text{Int}"] \end{tikzcd}$$

$$a(t) = v'(t) = s''(t) \qquad v(t) = \int a(t)\,dt$$

$$s(t) = \int v(t)\,dt$$

Problem 1. The acceleration of a moving body is described by the function $a(t) = t^3 - t^2 \left(\frac{m}{s^2}\right)$.

(a) Determine the functions of displacement $s(t)$ and velocity $v(t)$ if $s(0) = 1$ and $v(0) = 0$.

(b) How much is the average speed between $t=0$ and $t=1$?

(c) What is the displacement of the body between $t_1 = 0$ and $t_2 = 2$.

(d) How much is the total distance between $t=0$ and $t=2$?

(a) $v(t) = \int a(t)\,dt = \int t^3 - t^2\,dt = \frac{t^4}{4} - \frac{t^3}{3} + C$

$v(0) = 0 - 0 + C \stackrel{!}{=} 0 \Rightarrow C = 0 \Rightarrow \underline{\underline{v(t) = \frac{t^4}{4} - \frac{t^3}{3}}}$

$s(t) = \int v(t)\,dt = \int \frac{t^4}{4} - \frac{t^3}{3}\,dt = \frac{t^5}{20} - \frac{t^4}{12} + C$

$s(0) = 0 - 0 + C = 1 \Rightarrow C = 1 \Rightarrow \underline{\underline{s(t) = \frac{t^5}{20} - \frac{t^4}{12} + 1}}$

(b)
$$\bar{v}_{[0,1]} = \frac{\int_0^1 v(t)\,dt}{1-0} = \frac{[s(t)]_0^1}{1} = s(1) - s(0)$$

$$= \left(\frac{1}{20} - \frac{1}{12} + 1\right) - (0 - 0 + 1) = \frac{1}{20} - \frac{1}{12} = -\frac{1}{30} \; \left(\frac{m}{s}\right)$$

(Velocity can be negative!)

(c) $$[s(t)]_{t_1}^{t_2} = s(2) - s(0) = \left(\frac{2^5}{20} - \frac{2^4}{12} + 1\right) - (0 - 0 + 1) = \frac{4}{15} \quad (m)$$

(d) Total distance: $\int_0^2 |v(t)|\,dt = ?$ $\circledast$

$v(t) = 0$? $\quad \frac{t^4}{4} - \frac{t^3}{3} = 0 \iff t^3\left(\frac{t}{4} - \frac{1}{3}\right) = 0$

$\underline{t = 0}$ or $\frac{t}{4} = \frac{1}{3}$

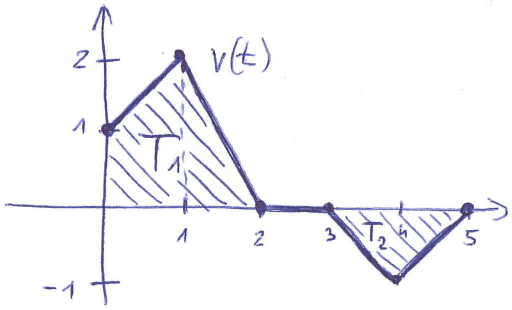
$\underline{t = 4/3}$

$$\circledast = \left|\int_0^{4/3} v(t)\,dt\right| + \left|\int_{4/3}^{2} v(t)\,dt\right| = \left|s\left(\tfrac{4}{3}\right) - s(0)\right| + \left|s(2) - s\left(\tfrac{4}{3}\right)\right|$$

$$= \left|\left(\frac{\left(\frac{4}{3}\right)^5}{20} - \frac{\left(\frac{4}{3}\right)^4}{12} + 1\right) - (0 - 0 + 1)\right| + \left|\left(\frac{32}{20} - \frac{16}{12} + 1\right) - \left(\frac{\left(\frac{4}{3}\right)^5}{20} - \frac{\left(\frac{4}{3}\right)^4}{12} + 1\right)\right|$$

$= \ldots$ It is not necessary to complete.

Problem 2. The following graph shows the velocity of a moving body.



(a) What is the initial direction of the body? Why?

(b) How much is the displacement between $t_1=0$ and $t_2=5$?

(c) What is the total distance (between $t=0$ and $t=5$)?

(a) Forward, because $v(0)=1>0$.

(b) $s(t)=\int v(t)\,dt$

$$[s(t)]_0^5 = \int_0^5 v(t) = T_1 + T_2 = \frac{5}{2}-1=\frac{3}{2}$$

$$T_1 = \frac{(1+2)\cdot 1}{2} + \frac{1\cdot 2}{2} = \frac{3}{2}+1=\frac{5}{2}$$

$$T_2 = -\frac{2\cdot 1}{2} = -1$$

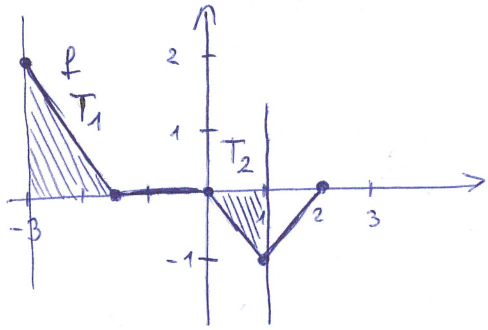
(c) $$\left|\int_0^2 v(t)\,dt\right| + \left|\int_3^5 v(t)\,dt\right| = |T_1|+|T_2| = \frac{5}{2}+1=\frac{7}{2}$$

② Integral mean – Average of the integral of a function.

$$\boxed{\overline{f}_{[a,b]} = \frac{\int_a^b f(x)\,dx}{b-a}}$$

← Integral mean of the function $f(x)$ on the interval $[a,b]$.

Problem 3.



$$\overline{f}_{[-3,1]} = ?$$

$$T_1 = \frac{2\cdot\left(\frac{1}{2}+1\right)}{2} = \frac{2\cdot\frac{3}{2}}{2} = \frac{3}{2} \qquad T_2 = -\frac{1\cdot 1}{2} = -\frac{1}{2}$$

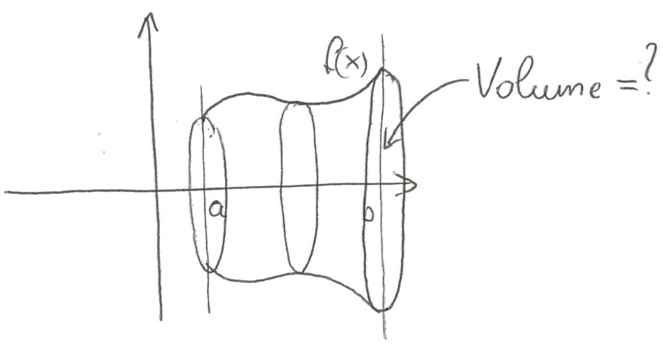
$$\overline{f}_{[-3,1]} = \frac{T_1+T_2}{1-(-3)} = \frac{\frac{3}{2}-\frac{1}{2}}{4} = \boxed{\frac{1}{4}}$$

Problem 4. The amount of a drug in the blood is described by the function $m(t) = e^{-t} - e^{-2t}$. How much is the average amount between $t_1 = 0$ and $t_2 = 2$?

$$\overline{m}_{[0,2]} = \frac{\int_0^2 e^{-t} - e^{-2t}\,dt}{2-0} = \frac{1}{2}\cdot\left[\frac{e^{-t}}{-1} - \frac{e^{-2t}}{-2}\right]_0^2 = \frac{1}{2}\cdot\left[\left(\frac{e^{-2}}{-1} - \frac{e^{-4}}{-2}\right) - \left(\frac{1}{-1} + \frac{1}{2}\right)\right]$$

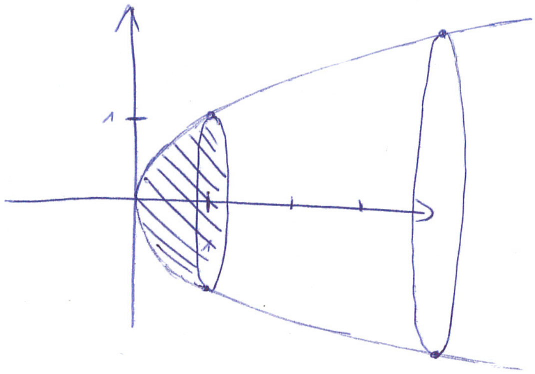
$$= \frac{1}{2}\left(\frac{e^{-4}}{2} - e^{-2}\right) + \frac{1}{4}$$

③ Volume of solids of revolution



$$V = \pi \cdot \int_a^b f^2(x)\,dx$$

Problem 5. Determine the volume of the solid obtained by rotating the region bounded by $f(x)=\sqrt{x}$, $x=0$ $x=1$ and the x axis.



$$V=\pi\cdot\int_0^1\left(\sqrt{x}\right)^2dx=\pi\cdot\int_0^1 x\,dx$$

$$=\pi\cdot\left[\frac{x^2}{2}\right]_0^1=\pi\cdot\left(\frac{1}{2}-0\right)=\frac{\pi}{2}$$