Practice 1

Repetition of high school mathematics

① Operations with fractions

e.g. • $\frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \frac{1}{6}$ (substraction; Extension)

• $2 - \frac{3}{4} = \frac{8}{4} - \frac{3}{4} = \frac{5}{4}$

• $\frac{1}{5} \cdot \left(\frac{1}{3} - \frac{2}{5}\right) = \frac{1}{5} \cdot \left(\frac{5}{15} - \frac{6}{15}\right) = \frac{1}{5} \cdot \left[-\left(\frac{1}{15}\right)\right] = -\frac{1}{75}$ (multiplication)

• $\frac{2}{5}\left(\frac{2}{3} - 4\right) = \frac{2}{5} \cdot \left(\frac{2}{3} - \frac{12}{3}\right) = \frac{2}{5} \left(\frac{-10}{3}\right) = -\frac{4}{3}$ (Reduction.)

② Power

Definition: $a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}} \quad (n \in \mathbb{N}) \qquad a^0 = 1 \quad (a \neq 0)$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \quad (a > 0) \qquad a^{-n} = \frac{1}{a^n} \quad (a \neq 0)$$

Rules:

- $a^n \cdot a^m = a^{n+m}$
- $\frac{a^m}{a^n} = a^{m-n}$
- $(a^n)^m = a^{n \cdot m}$
- $(ab)^n = a^n \cdot b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Eg. $\frac{x^2 \sqrt{y}\, z^{-1}}{\sqrt[3]{x^2} \cdot y^{-2} \cdot z^2} = \frac{x^2}{x^{2/3}} \cdot \frac{y^{1/2}}{y^{-2}} \cdot \frac{z^{-1}}{z^2} = x^{2-\frac{2}{3}} \cdot y^{\frac{1}{2}-(-2)} \cdot z^{-1-2} = x^{4/3}\, y^{5/2}\, z^{-3}$

$$\frac{x^2 - x}{x} = \frac{x^2}{x} - \frac{x}{x} = x - 1$$

③ Exponential and Logarithmical expressions

I. $a^{\log_a x} = x$ **II.** $\log_a a^x = x$

Notation: $\lg x = \log_{10} x$

The $\log_a x$ (logarithm of x to base a) is the exponent by which a must be raised to yield x.

Eg. for case I:

- $2^{\log_2 5} = \underline{5}$
- $4^{\log_2 3} = (2^2)^{\log_2 3} = (2^{\log_2 3})^2 = 3^2 = \underline{9}$
- $10^{\lg 6} - 10^{\lg 2} = \frac{10^{\lg 6}}{10^{\lg 2}} = \frac{6}{2} = \underline{3}$

Eg. for case II:

- $\log_2 8 = \log_2 2^3 = \underline{3}$
- $\log_3 \frac{1}{\sqrt{3}} = \log_3 3^{-1/2} = \underline{-\frac{1}{2}}$
- $\log_4 2 = \log_4 \sqrt{4} = \log_4 4^{1/2} = \underline{1/2}$
- $\lg 0.01 = \lg 10^{-2} = \underline{-2}$

④ Logarithmic identities

- $\log_a(xy) = \log_a x + \log_a y$
- $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
- $\log_a x^b = b \cdot \log_a x$

Eg.

- $\lg(100x^2) = \lg 100 + \lg x^2$
  $= \lg 10^2 + 2\lg x = 2 + 2\lg x$
- $\lg 4 + 2\lg 5 = \lg 4 + \lg 5^2 =$
  $= \lg 4 \cdot 5^2 = \lg 100 = \underline{2}$

⑤ Solving equations

(a) Linear equation. $(ax + b = 0)$

Eg: $8x + 3 = 6$

$8x - 3 = 0$

$8x = 3$

$x = \frac{8}{3}$

(b) Quadratic equation $(ax^2+bx+c=0)$

Quadratic formula

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Eg. $x^2-x-2=0$

$a=1$
$b=-1$
$c=-2$

$$x_{1,2}=\frac{1\pm\sqrt{1-4\cdot1\cdot(-2)}}{2}=\frac{1\pm\sqrt{1+8}}{2}=\frac{1\pm3}{2}$$

$\frac{1+3}{2}=2$

$\frac{1-3}{2}=-1$

Eg. $x^2-5=0$

Solution 1: $a=1$, $b=0$, $c=-5$ ⇒ Quadr. formula.

Solution 2: $x^2=5$)!

$|x|=\sqrt{5}$

$x=\sqrt{5}$ $\quad x=-\sqrt{5}$

$\sqrt{x^2}=|x|$ → But: $x^3=8$) No abs. value!

$x=2$

Eg. $x^5-x^3=0$

$x^3(x^2-1)=0$) Factorization

A product is zero, if some of its part is zero.

$x^3=0$ or $x^2-1=0$

$x=0$

$x^2=1$

$|x|=1$

$x=1$

$x=-1$

Eg. $x^2\cdot3^x-5x\cdot3^x=6\cdot3^x$

$3^x(x^2-5x-6)=0$

$3^x=0$ or $x^2-5x-6=0$

$x_{1,2}=6$, -1

↑ contradiction

3^x is always positive

Mixture problems

SOLUTION = SALT + SOLVENT

= ACID + SOLVENT

$$\text{Salinity / Acidity } (\%) = \frac{\text{Amount of salt}}{\text{Amount of solution}} \cdot 100 = \frac{m_a}{m_{sol}} \cdot 100$$

Eg. How much acid is in 100 mg of 20% acid solution?

~~(What is the amount of the solution?)~~

$$\left.\begin{array}{l} m_{sol} = 100 \\ c = 20 \\ \hline m_a = ? \end{array}\right\} \rightarrow c = \frac{m_a}{m_{sol}} \cdot 100 \rightarrow 20 = \frac{m_a}{100} \cdot 100$$

$m_a = 20$ mg

Eg. A 25% acid solution contains 100mg of acid. What is the amount of the solution?

$$\left.\begin{array}{l} m_a = 100 \\ c = 25 \\ \hline m_{sol} = ? \end{array}\right\} \rightarrow c = \frac{m_a}{m_{sol}} \cdot 100 \rightarrow 25 = \frac{100}{m_{sol}} \cdot 100$$

$$0.25 = \frac{100}{m_{sol}}$$

$$m_{sol} = \frac{100}{0.25} = 400 \text{ mg}$$

Mixturing solutions

	Amount	Acidity
solution 1	x	$p\%$
solution 2	y	$q\%$
Mixture	$x+y$	$c\%$

$$c = \frac{px + qy}{x + y}$$

IMPORTANT: SALT → 100%

SOLVENT → 0%
water

Eg. You need 20 liters of 20% acid solution. You have jugs of 10% solution and 25% solution. How many liters of each should you combine to get the needed solution?

	Amount	Acidity
sol 1	x	10
sol 2	y	25
Mix	20 ℓ	20%

$x+y=20$

$y=20-x \rightarrow 10x+25y=20\cdot 20$

$10x+25(20-x)=400$

$10x+500-25x=400$

$400=15x$

$x=\frac{100}{15}$

sol 1: $\frac{100}{15}\ \ell$

sol 2: $20-\frac{100}{15}=\frac{300}{15}-\frac{100}{15}=\frac{200}{15}\ \ell$

Eg. How many kilograms of pure water is to be added to 100 kilograms of a 30% saline solution to make it a 10% saline solution?

(solution of sodium chloride)

water → sol 1	x	0%	
sol 2	100	30%	
Mix	$100+x$	10%	

$0\cdot x+30\cdot 100=10(100+x)$

$3000=1000+10x$

$2000=10x$

$x=200$ kg water