Differential Equations

$f(t)$ is a time dependent process
- e.g. amount of bacteria or a drug in the blood

The process is described by its derivative (i.e. rate of change) and its initial value:

initial value problem

diff. eq.
$$f'(t) = g(t)$$
or
$$f'(t) = G(f(t))$$ ← our typical form
or
$$f'(t) = G(f(t), t)$$

AND

initial condition $\{ f(0) = c$

Def: Equilibrium (plural: equilibria) is a state of the process that doesn't change (i.e. constant in time)

↳ $f(t) \equiv \text{constant}$

↳ $f'(t) = 0$

↳ finding equilibria

⇕

finding zeros of the right hand side of the diff. eq.

Tangent field

Typically the right hand side is an equation in t and $f(t)$

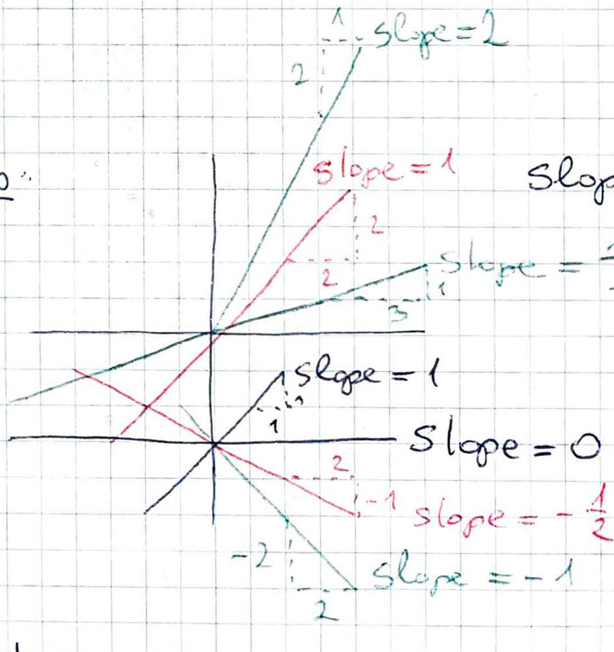
$\rightsquigarrow$ an equation in t and f

This equation gives the tangent vectors to the solutions at any given $(t, f(t))$ (time, initial value at that time) pairs.

„Slopes"

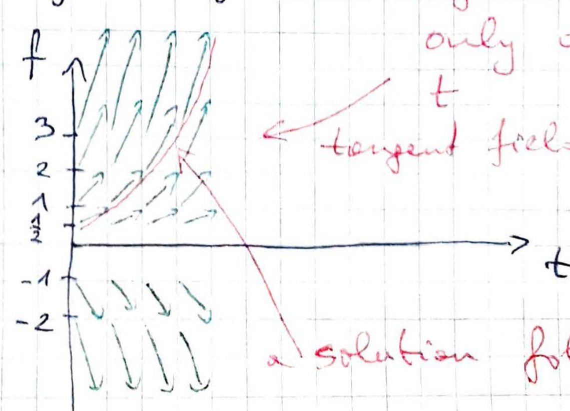
Reminder:

Slopes:



slope $\sim \frac{\Delta y}{\Delta x}$

Example: $f'(t) = f(t)$ ← right hand side depends only on $f(t)$ and not t



tangent field doesn't depend on t

a solution follows the tangent field

Stability of equilibria

Assume that $\boxed{c}$ is an equilibrium of $f'(t) = G(f(t))$

$\downarrow$

That is $G(c) = 0$

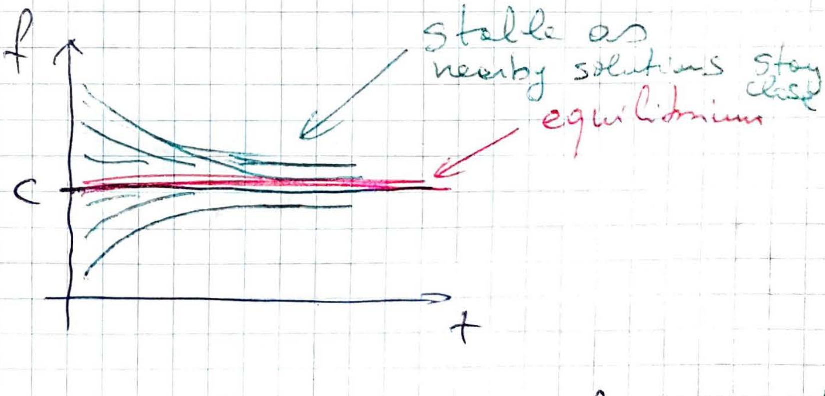
$\downarrow$

$$\begin{cases} f'(t) = G(f(t)) \\ f(0) = c \end{cases}$$

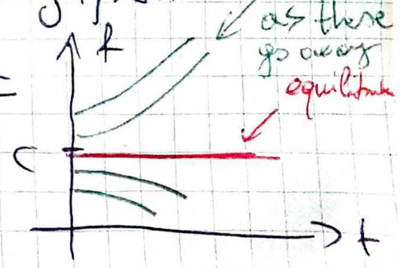
← the solution is $f(t) = c$ for all t as $f'(t) = 0$, hence, $f(t)$ is constant

1) Assume that we start the process close to $\boxed{c}$ but not at $\boxed{c}$

2) If the solutions stay close (or converge) to $\boxed{c}$, then $\boxed{c}$ is <u>stable</u>



3) If these solutions go away from $\boxed{c}$, then it is <u>unstable</u>

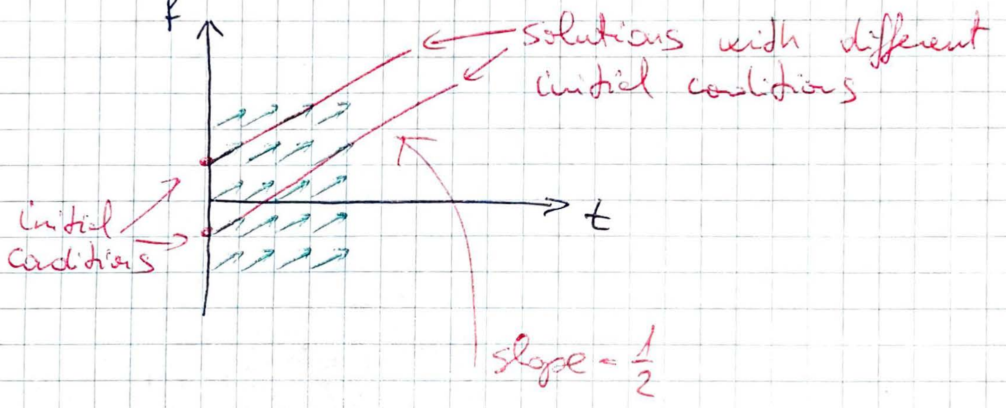


Examples

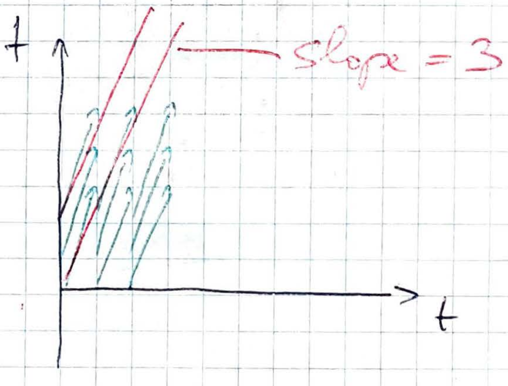
(1.) $f'(t) = \alpha \quad , \alpha \neq 0$

- there is no equilibrium as $\alpha \neq 0$

<u>example</u>: $f'(t) = \frac{1}{2}$



<u>example</u>: $f'(t) = 3$

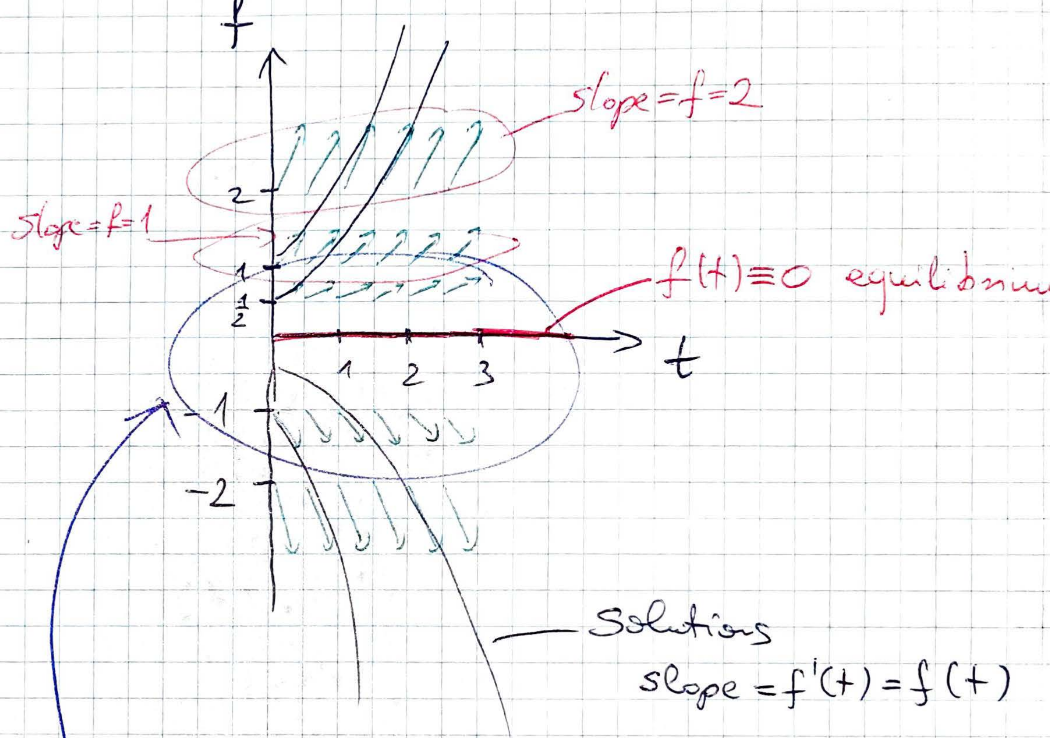


② $f'(t) = \alpha f(t) \qquad \alpha \neq 0$

- equilibrium: $f' = \alpha f = 0 \iff f = 0$

$\implies$ 0 is the only equilibrium

<u>example</u>: $f'(t) = f(t)$

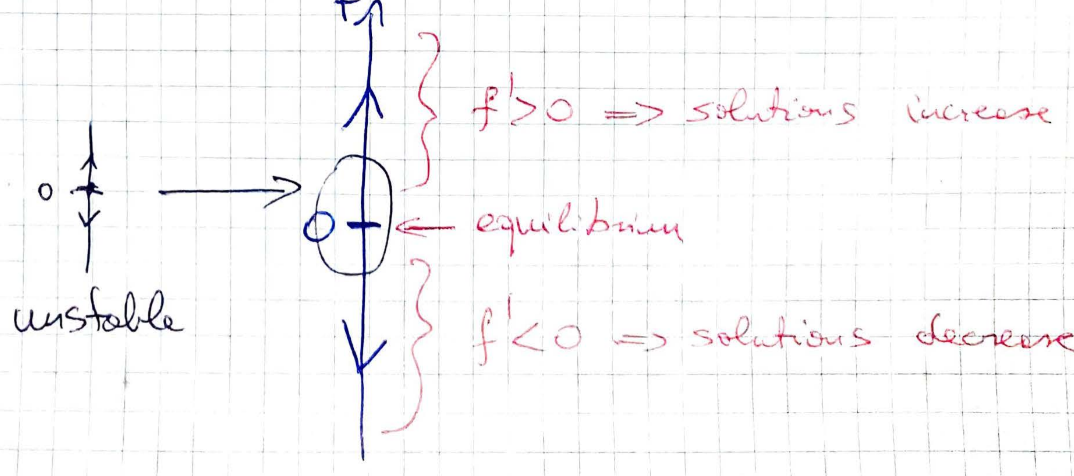


solutions leave from 0

⇓

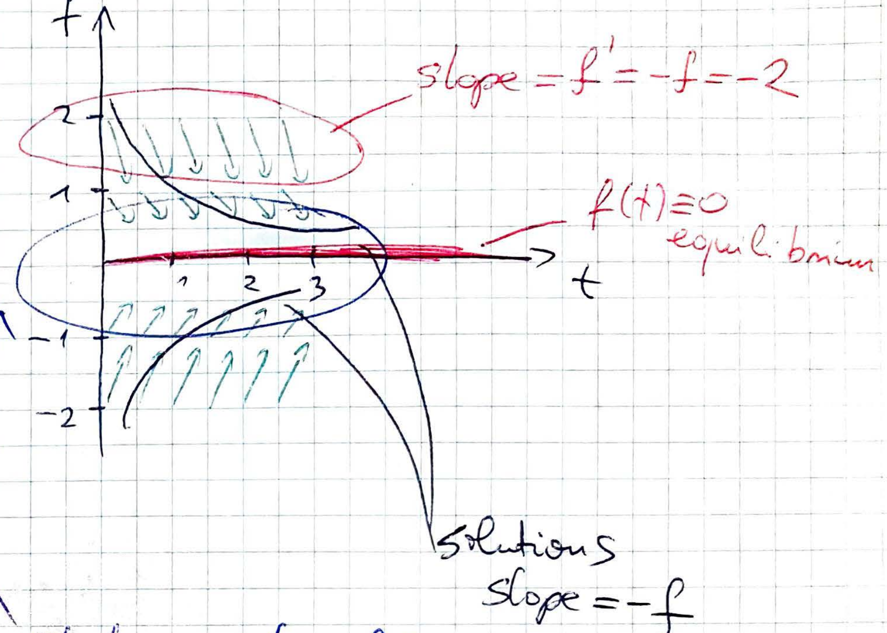
0 is <u>unstable</u>

We can demonstrate this on the f-axis:



example:

$$f'(t) = -f(t)$$

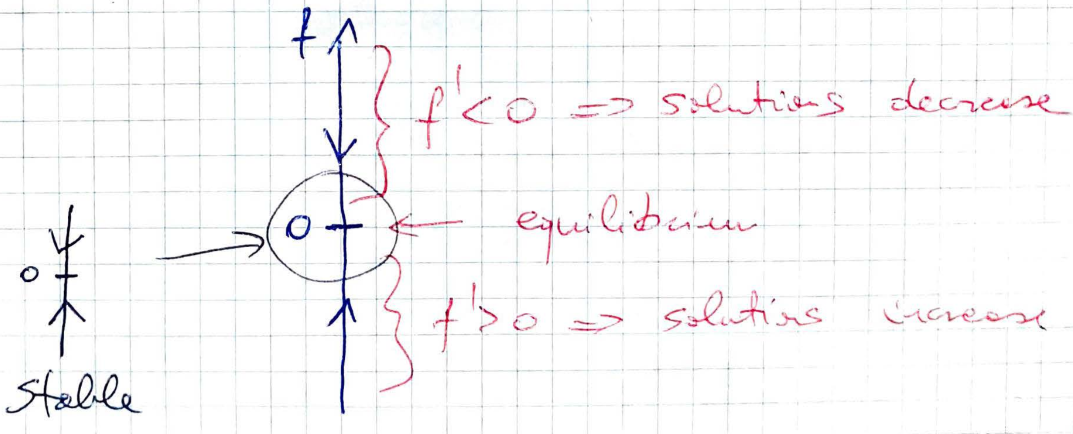


Solutions stay close, and tend towards 0

$\Downarrow$

0 is stable

We can demonstrate this on the f-axis



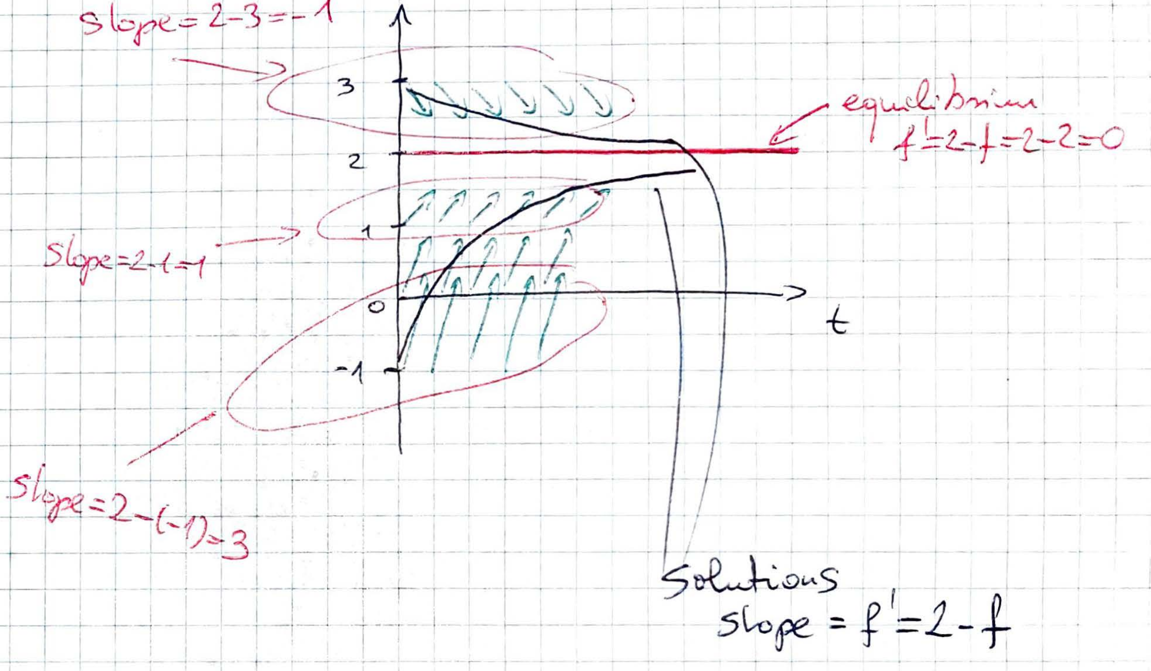
③

$$f'(t) = \alpha(\beta - f(t)) \qquad \alpha \neq 0$$

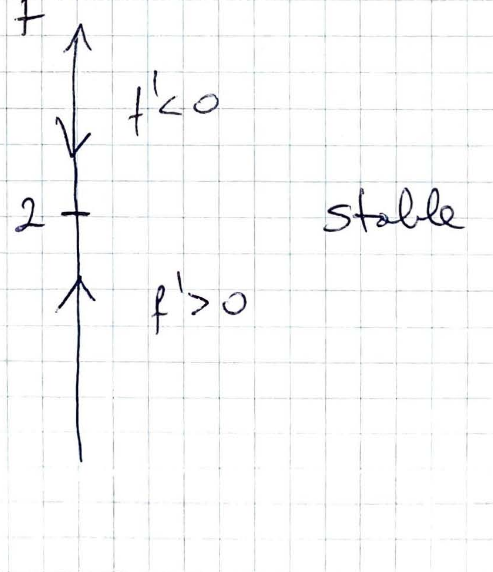
equilibrium: $f' = 0 \iff f(t) \equiv \beta$

<u>example</u>:

$$f'(t) = 2 - f(t)$$



2 is <u>stable</u>



4.

$$f'(t) = \alpha(\beta - f(t))(\gamma - f(t)) \qquad \alpha \neq 0$$

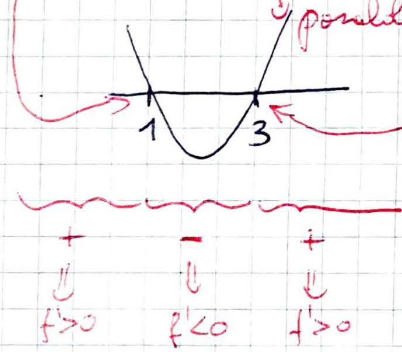
- equilibrium: $f' = 0 \iff f(t) \equiv \beta$ or $f(t) \equiv \gamma$

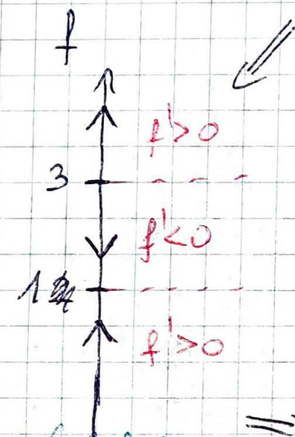
<u>example</u>:

$$f'(t) = 2(1-f(t))(3-f(t))$$

$$2(1-f)(3-f) = 2f^2 - 8f + 6$$

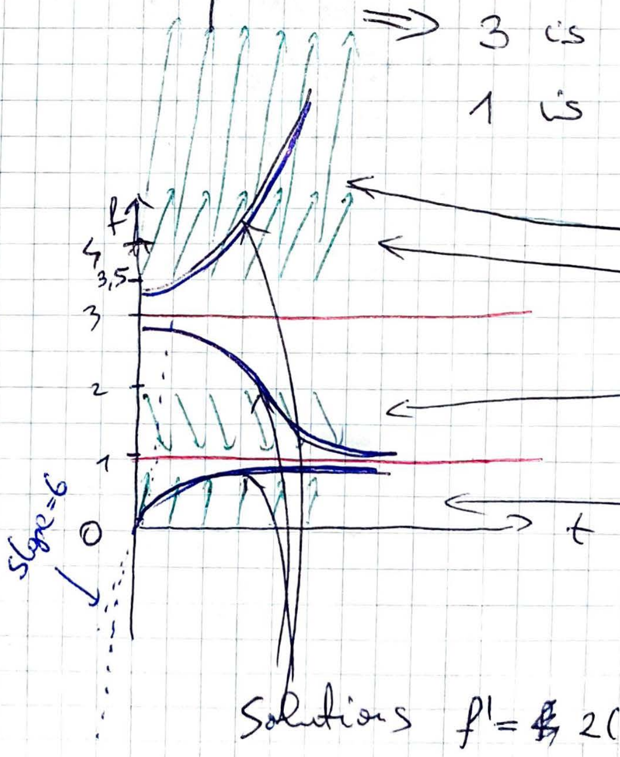
parabola up





$\Rightarrow$ 3 is <u>unstable</u>

1 is <u>stable</u>



$f(4) = 2(-3)(-1) = 6$

$f(3{,}5) = 2(-2{,}5)(-0{,}5) = 2{,}5$

$f(2) = 2(-1)(1) = -2$

$f(0) = 2(1)(3) = 6$

Solutions $f' = 2(1-f)(3-f)$

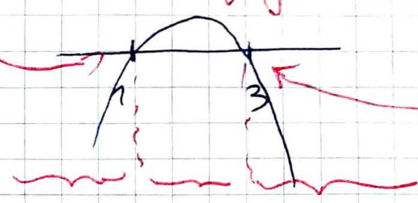
We can see that 3 is unstable, 1 is stable exomple:

$$f'(t) = -\frac{1}{4}(1-f(t))(3-f(t))$$

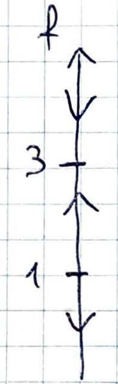
S

$$-\frac{1}{4}f^2 + f - \frac{3}{4}$$

parabola down

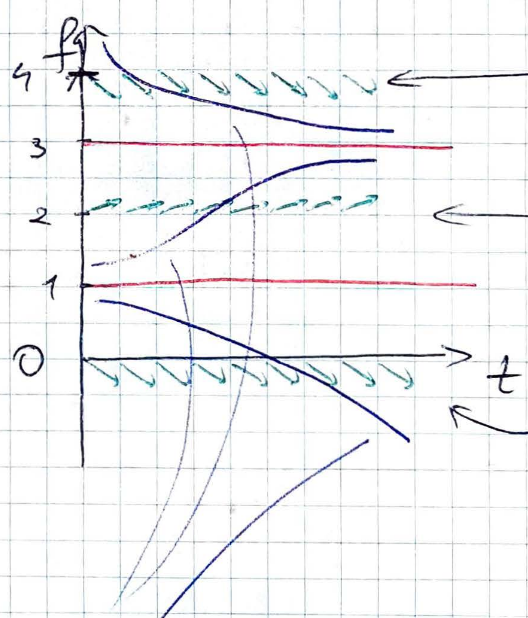


$-$ $\Downarrow$ $f'<0$ $\quad$ $+$ $\Downarrow$ $f'>0$ $\quad$ $-$ $\Downarrow$ $f'<0$

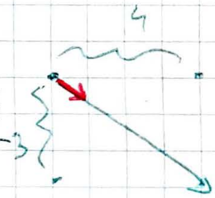


$\Rightarrow$ 3 is stable

1 is unstable



$f(4) = -\frac{3}{4}$



$f(2) = -\frac{1}{4}\cdot(-1)\cdot(1) = \frac{1}{4}$

$f(0) = -\frac{1}{4}\cdot 1 \cdot 3 = -\frac{3}{4}$

Solutions slope $= f' = -\frac{1}{4}(1-f)(3-f)$

We can see that 3 is stable, 1 is unstable.

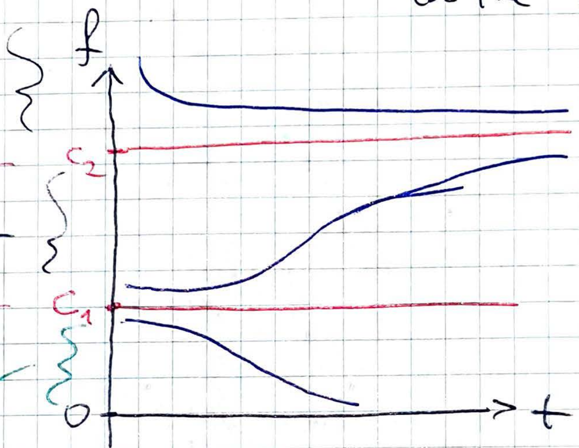
Typical biological / natural process

$f(t)$ is measuring the amount of something

$\hookrightarrow f(t) \geq 0$

(e.g. - how much drug is there in the blood

- how much bacteria is there in the blood)



typical picture

processes starting from $(0, c_1)$ converge to 0

c_1 is an equilibrium (unstable)

processes starting from (c_1, c_2) converge to c_2

c_2 is an equilibrium (stable)

processes starting from (c_2, ∞) converge to c_2

$\Longrightarrow$ "Under c_1, the process dies out"

"Over c_1, the process survives"

"c_1 is the critical value that separates dying and nondying solutions"

Final excercise

$$f'(t) = \underbrace{\underbrace{\underbrace{\frac{1}{2} f(t)(3-f(t)) - 1}_{-\frac{1}{2} f^2(t) + \frac{3}{2} f(t) - 1}}_{-\frac{1}{2}\left[f^2(t) - 3f(t) + 2\right]}}$$

$$-\frac{1}{2}\left[\underbrace{f^2(t) - 3f(t) + 2}_{(f(t)-1)(f(t)-2)}\right]$$

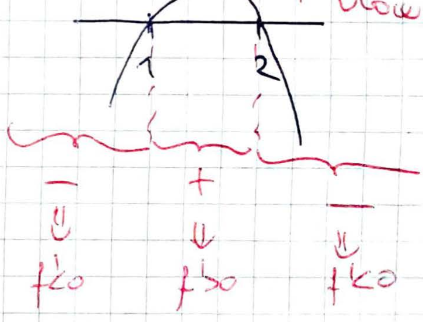
if you don't see this, use:

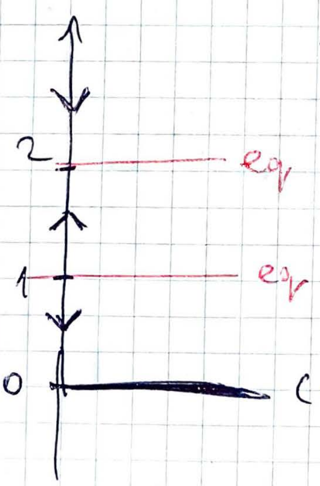
$$f^2 - 3f + 2 = 0$$

$$f_{1,2} = \frac{3 \pm \sqrt{9 - 2 \cdot 4}}{2} = \frac{3 \pm 1}{2} \begin{cases} 2 \\ 1 \end{cases}$$

So: $f'(t) = -\frac{1}{2} f^2(t) + \frac{3}{2} f(t) - 1$

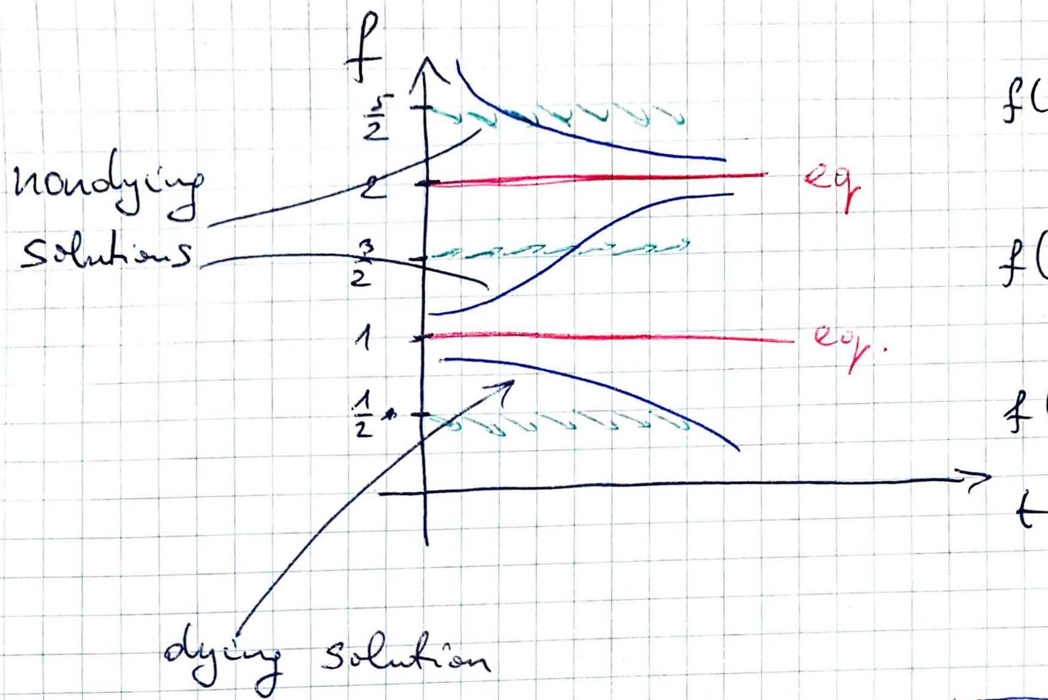
parabola down





(0 is just the limit as $f \geq 0$)

1 is unstable, 2 is stable, (0,1) dies out
[1, ∞) survives, 1 is the critical value



$f(\frac{5}{2}) = -\frac{3}{8}$

$f(\frac{3}{2}) = \frac{1}{8}$

$f(\frac{1}{2}) = -\frac{3}{8}$