Portmanteau theorem for unbounded measures

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Abstract

We prove an analogue of the portmanteau theorem on weak convergence of probability measures allowing measures which are unbounded on an underlying metric space but finite on the complement of any Borel neighbourhood of a fixed element.

1 Introduction

Weak convergence of probability measures on a metric space has a very important role in probability theory. The well known *portmanteau theorem* due to A. D. Alexandroff (see for example Theorem 11.1.1 in Dudley [1]) provides useful conditions equivalent to weak convergence of probability measures; any of them could serve as the definition of weak convergence. Proposition 1.2.13 in the book of Meerschaert and Scheffler [3] gives an analogue of the portmanteau theorem for bounded measures on $\mathbb{R}^d$. Moreover, Proposition 1.2.19 in [3] gives an analogue for special unbounded measures on $\mathbb{R}^d$, more precisely, for extended real valued measures which are finite on the complement of any Borel neighbourhood of $0 \in \mathbb{R}^d$.

By giving counterexamples we show that the equivalences of (c) and (d) in Propositions 1.2.13 and 1.2.19 in [3] are not valid (see our Remarks 2.2 and 2.3). We reformulate Proposition 1.2.19 in [3] in a more detailed form adding new equivalent assertions to it (see Theorem 2.1). Moreover, we note that Theorem 2.1 generalizes the equivalence of (a) and (b) in Theorem 11.3.3 of [1] in two aspects. On the one hand, the equivalence is extended allowing not necessarily finite measures which are finite on the complement of any Borel neighbourhood of a fixed element of an underlying metric space. On the other hand, we do not assume the separability of the underlying metric space to prove the equivalence. But we mention that this latter possibility is hiddenly contained in Problem 3, p. 312 in [1]. For completeness we give a detailed proof of Theorem 2.1. Our proof goes along the lines of the proof of the original portmanteau theorem and differs from the proof of Proposition 1.2.19 in [3].

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To shed some light on the sense of a portmanteau theorem for unbounded measures, let us consider the question of weak convergence of infinitely divisible probability measures μ_n, $n \in \mathbb{N}$ towards an infinitely divisible probability measure μ_0 in case of the real line $\mathbb{R}$. Theorem VII.2.9 in Jacod and Shiryayev [2] gives equivalent conditions for weak convergence $\mu_n \xrightarrow{w} \mu_0$. Among these conditions we have

$$\int_{\mathbb{R}} f \, \mathrm{d}\eta_n \to \int_{\mathbb{R}} f \, \mathrm{d}\eta_0 \qquad \text{for all} \quad f \in \mathcal{C}_2(\mathbb{R}), \tag{1.1}$$

where η_n, $n \in \mathbb{Z}_+$ are nonnegative, extended real valued measures on $\mathbb{R}$ with $\eta_n(\{0\}) = 0$ and $\int_{\mathbb{R}} (x^2 \wedge 1) \, \mathrm{d}\eta_n(x) < \infty$ (i.e., Lévy measures on $\mathbb{R}$) corresponding to μ_n, and $\mathcal{C}_2(\mathbb{R})$ is the set of all real valued bounded continuous functions f on $\mathbb{R}$ vanishing on some Borel neighbourhood of 0 and having a limit at infinity. Theorem 2.1 is about equivalent reformulations of (1.1) when it holds for all real valued bounded continuous functions on $\mathbb{R}$ vanishing on some Borel neighbourhood of 0.

2 An analogue of the portmanteau theorem

Let $\mathbb{N}$ and $\mathbb{Z}_+$ be the set of positive and nonnegative integers, respectively. Let (X, d) be a metric space and x_0 be a fixed element of X. Let $\mathcal{B}(X)$ denote the σ-algebra of Borel subsets of X. A Borel neighbourhood U of x_0 is an element of $\mathcal{B}(X)$ for which there exists an open subset $\widetilde{U}$ of X such that $x_0 \in \widetilde{U} \subset U$. Let $\mathcal{N}_{x_0}$ denote the set of all Borel neighbourhoods of x_0, and the set of bounded measures on X is denoted by $\mathcal{M}^b(X)$. The expression "a measure μ on X" means a measure μ on the σ-algebra $\mathcal{B}(X)$.

Let $\mathcal{C}(X)$, $\mathcal{C}_{x_0}(X)$ and $\mathrm{BL}_{x_0}(X)$ denote the spaces of all real valued bounded continuous functions on X, the set of all elements of $\mathcal{C}(X)$ vanishing on some Borel neighbourhood of x_0, and the set of all real valued bounded Lipschitz functions vanishing on some Borel neighbourhood of x_0, respectively.

For a measure η on X and for a Borel subset $B \in \mathcal{B}(X)$, let $\eta|_B$ denote the restriction of η onto B, i.e., $\eta|_B(A) := \eta(B \cap A)$ for all $A \in \mathcal{B}(X)$.

Let μ_n, $n \in \mathbb{Z}_+$ be bounded measures on X. We write $\mu_n \xrightarrow{w} \mu$ if $\mu_n(A) \to \mu(A)$ for all $A \in \mathcal{B}(X)$ with $\mu(\partial A) = 0$. This is called *weak convergence of bounded measures* on X.

Now we formulate a portmanteau theorem for unbounded measures.

Theorem 2.1 *Let (X, d) be a metric space and x_0 be a fixed element of X. Let η_n, $n \in \mathbb{Z}_+$, be measures on X such that $\eta_n(X \setminus U) < \infty$ for all $U \in \mathcal{N}_{x_0}$ and for all $n \in \mathbb{Z}_+$. Then the following assertions are equivalent:*

(i) $\int_{X \setminus U} f \, \mathrm{d}\eta_n \to \int_{X \setminus U} f \, \mathrm{d}\eta_0$ *for all* $f \in \mathcal{C}(X)$, $U \in \mathcal{N}_{x_0}$ *with* $\eta_0(\partial U) = 0$,

(ii) $\eta_n|_{X \setminus U} \xrightarrow{w} \eta_0|_{X \setminus U}$ *for all* $U \in \mathcal{N}_{x_0}$ *with* $\eta_0(\partial U) = 0$,

(iii) $\eta_n(X \setminus U) \to \eta_0(X \setminus U)$ *for all* $U \in \mathcal{N}_{x_0}$ *with* $\eta_0(\partial U) = 0$,

(iv) $\int_X f \, \mathrm{d}\eta_n \to \int_X f \, \mathrm{d}\eta_0$ *for all* $f \in \mathcal{C}_{x_0}(X)$,

(v) $\int_X f \, \mathrm{d}\eta_n \to \int_X f \, \mathrm{d}\eta_0$ *for all* $f \in \mathrm{BL}_{x_0}(X)$,

(vi) *the following inequalities hold:*
 (a) $\limsup_{n\to\infty} \eta_n(X \setminus U) \leqslant \eta_0(X \setminus U)$ *for all open neighbourhoods* U *of* x_0,
 (b) $\liminf_{n\to\infty} \eta_n(X \setminus V) \geqslant \eta_0(X \setminus V)$ *for all closed neighbourhoods* V *of* x_0.

Proof. (i)⇒(ii): Let U be an element of $\mathcal{N}_{x_0}$ with $\eta_0(\partial U) = 0$. Note $\eta_n|_{X\setminus U} \in \mathcal{M}^b(X)$, $n \in \mathbb{Z}_+$. By the equivalence of (a) and (b) in Proposition 1.2.13 in [3], to prove $\eta_n|_{X\setminus U} \xrightarrow{w} \eta_0|_{X\setminus U}$ it is enough to check $\int_X f \, \mathrm{d}\eta_n|_{X\setminus U} \to \int_X f \, \mathrm{d}\eta_0|_{X\setminus U}$ for all $f \in \mathcal{C}(X)$. For this it suffices to show that for all real valued bounded measurable functions h on X, for all $A \in \mathcal{B}(X)$ and for all $n \in \mathbb{Z}_+$ we have

$$\int_X h \, \mathrm{d}\eta_n|_A = \int_A h \, \mathrm{d}\eta_n. \tag{2.1}$$

By Beppo-Levi's theorem, a standard measure-theoretic argument implies (2.1).

(ii)⇒(iii): Let U be an element of $\mathcal{N}_{x_0}$ with $\eta_0(\partial U) = 0$. By (ii), we have $\eta_n|_{X\setminus U} \xrightarrow{w} \eta_0|_{X\setminus U}$. Since $\eta_0|_{X\setminus U}(\partial X) = \eta_0|_{X\setminus U}(\emptyset) = 0$, we get $\eta_n(X\setminus U) = \eta_n|_{X\setminus U}(X) \to \eta_0|_{X\setminus U}(X) = \eta_0(X \setminus U)$, as desired.

(iii)⇒(ii): Let U be an element of $\mathcal{N}_{x_0}$ with $\eta_0(\partial U) = 0$ and let $B \in \mathcal{B}(X)$ be such that $\eta_0|_{X\setminus U}(\partial B) = 0$. We have to show $\eta_n|_{X\setminus U}(B) \to \eta_0|_{X\setminus U}(B)$.

Since $B \cap (X \setminus U) = X \setminus [X \setminus (B \cap (X \setminus U))]$ and $\eta_n|_{X\setminus U}(B) = \eta_n(B \cap (X \setminus U))$, $n \in \mathbb{Z}_+$, by (iii), it is enough to check $\eta_0\big(\partial\big(X \setminus (B \cap (X \setminus U))\big)\big) = 0$. First we show

$$\partial\big(B \cap (X\setminus U)\big) \subset \big(\partial B \cap (X\setminus U)\big) \cup \partial U \quad \text{for all subsets } B, U \text{ of } X. \tag{2.2}$$

Let x be an element of $\partial\big(B \cap (X \setminus U)\big)$ and $(y_n)_{n\geqslant 1}$, $(z_n)_{n\geqslant 1}$ be two sequences such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = x$ and $y_n \in B \cap (X \setminus U)$, $z_n \in X \setminus (B \cap (X \setminus U))$, $n \in \mathbb{N}$. Then for all $n \in \mathbb{N}$ we have one or two of the following possibilities:

- $y_n \in B$, $y_n \in X \setminus U$ and $z_n \in X \setminus B$,
- $y_n \in B$, $y_n \in X \setminus U$ and $z_n \in U$.

Then we get $x \in \big(\partial B \cap ((X \setminus U) \cup \partial U)\big) \cup \big(\partial U \cap (B \cup \partial B)\big) \cup \big(\partial B \cap \partial U\big)$. Since $\partial B \cap ((X \setminus U) \cup \partial U) \subset (\partial B \cap (X \setminus U)) \cup \partial U$, we have $x \in \big(\partial B \cap (X \setminus U)\big) \cup \partial U$, as desired.

Using (2.2) we get $\eta_0\big(\partial\big(X\setminus(B\cap(X\setminus U))\big)\big) \leqslant \eta_0\big(\partial B\cap(X\setminus U)\big) + \eta_0(\partial U) = 0$. Indeed, by the assumptions $\eta_0\big(\partial B\cap(X\setminus U)\big) = 0$ and $\eta_0(\partial U) = 0$. Hence $\eta_0\big(\partial\big(X\setminus(B\cap(X\setminus U))\big)\big) = 0$.

(ii)⇒(i): Using again the equivalence of (a) and (b) in Proposition 1.2.13 in [3] and (2.1) we obtain (i).

(iii)⇒(iv): Let f be an element of $\mathcal{C}_{x_0}(X)$. Then there exists $A \in \mathcal{N}_{x_0}$ such that $f(x) = 0$ for all $x \in A$ and $\eta_0(\partial A) = 0$. Indeed, the function $t \mapsto \eta_0\big(\{x \in X :$

$d(x, x_0) \geqslant t\}\big)$ from $(0, +\infty)$ into $\mathbb{R}$ is monotone decreasing, hence the set $\big\{t \in (0, +\infty) : \eta_0(\{x \in X : d(x, x_0) = t\}) > 0\big\}$ of its discontinuities is at most countable. Consequently, for all $\widetilde{U} \in \mathcal{N}_{x_0}$ there exists some $t > 0$ such that $U := \{x \in X : d(x, x_0) < t\} \in \mathcal{N}_{x_0}$, $U \subset \widetilde{U}$ and $\eta_0(\partial U) = 0$. (At this step we use that an element $\widetilde{U}$ of $\mathcal{N}_{x_0}$ contains an open subset of X containing x_0.) This implies the existence of A. We show that the set $D := \big\{t \in \mathbb{R} : \eta_0\big(\{x \in X : f(x) = t\}\big) > 0\big\}$ is at most countable. The function $F : \mathbb{R} \to [0, \eta_0(X \setminus A)]$, defined by

$$F(t) := \eta_0\big(\{x \in X \setminus A : f(x) < t\}\big), \quad t \in \mathbb{R},$$

is monotone increasing and left continuous. (Note that $\eta_0(X \setminus A) < \infty$, by the assumption on η_0.) Hence it has at most countably many discontinuity points, and $t_0 \in \mathbb{R}$ is a discontinuity point of F if and only if $F(t_0 + 0) > F(t_0)$, i.e., $\eta_0\big(\{x \in X \setminus A : f(x) = t_0\}\big) > 0$. If $t_0 \neq 0$, then $\{x \in X : f(x) = t_0\} = \{x \in X \setminus A : f(x) = t_0\}$, thus $t_0 \neq 0$ is a discontinuity point of F if and only if $\eta_0(\{x \in X : f(x) = t_0\}) > 0$. Hence if $t \in D$ then $t = 0$ or t is a discontinuity point of F, consequently D is at most countable. Since f is bounded and D is at most countable, there exists a real number $M > 0$ such that $-M, M \notin D$ and $|f(x)| < M$ for $x \in X$. Let $\varepsilon > 0$. Choose real numbers t_i, $i = 0, \ldots, k$ such that $-M = t_0 < t_1 < \cdots < t_k = M$, $t_i \notin D$, $i = 0, \ldots, k$ and $\max_{0 \leqslant i \leqslant k-1}(t_{i+1} - t_i) < \varepsilon$. The countability of D implies the existence of t_i, $i = 0, \ldots, k$. Let

$$B_i := f^{-1}\big([t_i, t_{i+1})\big) \cap (X \setminus A) = \Big\{x \in X \setminus A : t_i \leqslant f(x) < t_{i+1}\Big\}$$

for all $i = 0, \ldots, k-1$. Then B_i, $i = 0, \ldots, k-1$, are pairwise disjoint Borel sets and $X \setminus A = \bigcup_{i=0}^{k-1} B_i$. Since f is continuous, the boundary $\partial(f^{-1}(H))$ of the set $f^{-1}(H)$ is a subset of the set $f^{-1}(\partial H)$ for all subsets H of $\mathbb{R}$. Using (2.2) this implies $\partial(X \setminus B_i) = \partial B_i \subset f^{-1}(\{t_i\}) \cup f^{-1}(\{t_{i+1}\}) \cup \partial A$ for all $i = 0, \ldots, k-1$. Since $t_i \notin D$, $i = 0, \ldots, k$, $\eta_0(\partial A) = 0$ and

$$\eta_0(\partial(X \setminus B_i)) \leqslant \eta_0\big(\{x \in X : f(x) = t_i\}\big) + \eta_0\big(\{x \in X : f(x) = t_{i+1}\}\big) + \eta_0(\partial A),$$

we get $\eta_0(\partial(X \setminus B_i)) = 0$, $i = 0, \ldots, k-1$. Since $A \subset X \setminus B_i$, we have $X \setminus B_i \in \mathcal{N}_{x_0}$ for all $i = 0, \ldots, k-1$. Hence condition (iii) implies that $\eta_n(B_i) \to \eta_0(B_i)$ as $n \to \infty$, $i = 0, \ldots, k-1$. By the triangle inequality

$$\begin{aligned}\Big|\int_X f \,\mathsf{d}\eta_n - \int_X f \,\mathsf{d}\eta_0\Big| &= \Big|\int_{X \setminus A} f \,\mathsf{d}\eta_n - \int_{X \setminus A} f \,\mathsf{d}\eta_0\Big| \\ &\leqslant 2 \max_{0 \leqslant i \leqslant k-1}(t_{i+1} - t_i) + \Big|\sum_{i=0}^{k-1} t_i\big(\eta_n(B_i) - \eta_0(B_i)\big)\Big|.\end{aligned}$$

Hence $\limsup_{n \to \infty} \big|\int_X f \,\mathsf{d}\eta_n - \int_X f \,\mathsf{d}\eta_0\big| \leqslant 2 \max_{0 \leqslant i \leqslant k-1}(t_{i+1} - t_i) < 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, (iv) holds.

(iv)⇒(v): It is trivial, since $\mathrm{BL}_{x_0}(X) \subset \mathcal{C}_{x_0}(X)$.

(v)⇒(vi): First let U be an open neighbourhood of x_0. Let $\varepsilon > 0$. We show the existence of a closed neighbourhood U_ε of x_0 such that $U_\varepsilon \subset U$ and $\eta_0(U \setminus U_\varepsilon) < \varepsilon$, and of a function $f \in \mathrm{BL}_{x_0}(X)$ such that $f(x) = 0$ for $x \in U_\varepsilon$, $f(x) = 1$ for $x \in X \setminus U$ and $0 \leqslant f(x) \leqslant 1$ for $x \in X$.

For all $B \in \mathcal{B}(X)$ and for all $\lambda > 0$ we use notation $B^\lambda := \big\{x \in X : d(x, B) < \lambda\big\}$, where $d(x, B) := \inf\{d(x, z) : z \in B\}$. Since U is open, we get $U = \bigcup_{n=1}^\infty F_n$, where $F_n := X \setminus (X \setminus U)^{1/n}$, $n \in \mathbb{N}$. Then $F_n \subset F_{n+1}$, $n \in \mathbb{N}$, F_n is a closed subset of X for all $n \in \mathbb{N}$ and $\bigcap_{n=1}^\infty (X \setminus F_n) = X \setminus U$. We also have $\eta_0(X \setminus F_N) < \infty$ for some sufficiently large $N \in \mathbb{N}$ and $X \setminus F_n \supset X \setminus F_{n+1}$ for all $n \in \mathbb{N}$, and hence the continuity of the measure η_0 implies that $\lim_{n\to\infty} \eta_0(X \setminus F_n) = \eta_0(X \setminus U)$. Since $\eta_0(X \setminus U) < \infty$, there exists some $n_0 \in \mathbb{N}$ such that $\eta_0(X \setminus F_{n_0}) - \eta_0(X \setminus U) < \varepsilon$. Set $U_\varepsilon := F_{n_0}$. Since $\eta_0(X \setminus F_{n_0}) - \eta_0(X \setminus U) = \eta_0\big((X \setminus F_{n_0}) \setminus (X \setminus U)\big) = \eta_0(U \setminus F_{n_0})$, the set U_ε is a closed neighborhood of x_0, $U_\varepsilon \subset U$ and $\eta_0(U \setminus U_\varepsilon) < \varepsilon$.

We show that the function $f : X \to \mathbb{R}$, defined by $f(x) := \min(1, n_0 d(x, U_\varepsilon))$, $x \in X$, is an element of $\mathrm{BL}_{x_0}(X)$, $f(x) = 0$ for $x \in U_\varepsilon$, $f(x) = 1$ for $x \in X \setminus U$ and $0 \leqslant f(x) \leqslant 1$ for $x \in X$.

Note that if $x \in U_\varepsilon$ then $d(x, U_\varepsilon) = 0$, hence $f(x) = 0$. And if $x \in X \setminus U$ then $d(x, U_\varepsilon) \geqslant d(X \setminus U, U_\varepsilon) \geqslant 1/n_0$, hence $f(x) = 1$. The fact that $0 \leqslant f(x) \leqslant 1$, $x \in X$ is obvious. To prove that f is Lipschitz, we check that

$$|f(x) - f(y)| \leqslant n_0 d(x, y) \qquad \text{for all } x, y \in X.$$

If $x, y \in X$ with $d(x, y) \geqslant 1/n_0$ then $|f(x) - f(y)| \leqslant 1 \leqslant n_0 d(x, y)$. If $x, y \in X$ with $d(x, y) < 1/n_0$ then we have to consider the following four cases apart from changing the role of x and y: $x \in X \setminus U$, $y \in U \setminus U_\varepsilon$; $x \in U_\varepsilon$, $y \in U \setminus U_\varepsilon$; $x, y \in U \setminus U_\varepsilon$ and the case $x, y \in U_\varepsilon$ or $x, y \in X \setminus U$.

Let us consider the case when $x, y \in U \setminus U_\varepsilon$ and $f(x) = 1$, $f(y) = n_0 d(y, U_\varepsilon)$. Then $d(x, U_\varepsilon) \geqslant 1/n_0$, $d(y, U_\varepsilon) \leqslant 1/n_0$ and we get $|f(x) - f(y)| = 1 - n_0 d(y, U_\varepsilon) \leqslant n_0 d(x, y)$. Indeed, $1/n_0 \leqslant d(x, U_\varepsilon) \leqslant d(x, y) + d(y, U_\varepsilon)$. The case $x, y \in U \setminus U_\varepsilon$ and $f(y) = 1$ $f(x) = n_0 d(x, U_\varepsilon)$ can be handled similarly. If $x, y \in U \setminus U_\varepsilon$ and $f(x) = n_0 d(x, U_\varepsilon)$, $f(y) = n_0 d(y, U_\varepsilon)$ then

$$|f(x) - f(y)| = n_0 |d(x, U_\varepsilon) - d(y, U_\varepsilon)| \leqslant n_0 d(x, y).$$

Indeed, since U_ε is closed, we have $|d(x, U_\varepsilon) - d(y, U_\varepsilon)| \leqslant d(x, y)$. If $x, y \in U \setminus U_\varepsilon$ and $f(x) = f(y) = 1$ then $|f(x) - f(y)| = 0 \leqslant n_0 d(x, y)$.

The other cases can be handled similarly. Hence $f \in \mathrm{BL}_{x_0}(X)$ and we get

$$\int_X f \,\mathsf{d}\eta_0 = \int_{X \setminus U_\varepsilon} f \,\mathsf{d}\eta_0 \leqslant \eta_0(X \setminus U_\varepsilon) = \eta_0(X \setminus U) + \eta_0(U \setminus U_\varepsilon) < \eta_0(X \setminus U) + \varepsilon,$$

and $\int_X f \,\mathsf{d}\eta_n \geqslant \int_{X \setminus U} f \,\mathsf{d}\eta_n = \eta_n(X \setminus U)$. Hence by condition (v) we have

$$\limsup_{n\to\infty} \eta_n(X \setminus U) \leqslant \limsup_{n\to\infty} \int_X f \,\mathsf{d}\eta_n = \int_X f \,\mathsf{d}\eta_0 < \eta_0(X \setminus U) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get (a).

Now let V be a closed neighbourhood of x_0. Let $\varepsilon > 0$. As in case of an open neighbourhood of x_0, one can show that there exist an open neighbourhood V_ε of x_0 such that $V \subset V_\varepsilon$ and $\eta_0(V_\varepsilon \setminus V) < \varepsilon$ and a function $f \in \mathrm{BL}_{x_0}(X)$ such that $f(x) = 0$ for $x \in V$, $f(x) = 1$ for $x \in X \setminus V_\varepsilon$ and $0 \leqslant f(x) \leqslant 1$ for $x \in X$. Then we get

$$\begin{aligned}\int_X f \, \mathsf{d}\eta_0 = \int_{X\setminus V} f \, \mathsf{d}\eta_0 &= \eta_0(X \setminus V_\varepsilon) + \int_{V_\varepsilon \setminus V} f \, \mathsf{d}\eta_0 \\ &\geqslant \eta_0(X \setminus V) - \eta_0(V_\varepsilon \setminus V) > \eta_0(X \setminus V) - \varepsilon,\end{aligned}$$

and $\int_X f \, \mathsf{d}\eta_n = \int_{X\setminus V} f \, \mathsf{d}\eta_n \leqslant \eta_n(X \setminus V)$. Hence by condition (v) we have

$$\liminf_{n\to\infty} \eta_n(X \setminus V) \geqslant \liminf_{n\to\infty} \int_X f \, \mathsf{d}\eta_n = \int_X f \, \mathsf{d}\eta_0 > \eta_0(X \setminus V) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain (b).

(vi)$\Rightarrow$ (iii): The proof can be carried out similarly to the proof of the corresponding part of Theorem 11.1.1 in Dudley [1]. □

Remark 2.1 Assertion (v) in Theorem 2.1 can be replaced by

$$\int_X f \, \mathsf{d}\eta_n \to \int_X f \, \mathsf{d}\eta_0 \qquad \text{for all } f \in \mathcal{C}^u_{x_0}(X),$$

where $\mathcal{C}^u_{x_0}(X)$ denotes the set of all uniformly continuous functions in $\mathcal{C}_{x_0}(X)$.

Remark 2.2 By giving a counterexample we show that (a) and (b) in condition (vi) of Theorem 2.1 are not equivalent. For all $n \in \mathbb{N}$ let η_n be the Dirac measure δ_2 on $\mathbb{R}$ concentrated on 2 and let η_0 be the Dirac measure δ_0 on $\mathbb{R}$ concentrated on 0. Then $\eta_0(\mathbb{R} \setminus V) = 0$ for all closed neighbourhoods V of 0, hence (b) in condition (vi) of Theorem 2.1 holds. But (a) in condition (vi) of Theorem 2.1 is not satisfied. Indeed, $U := (-1, 1)$ is an open neighbourhood of 0, $\eta_0(\mathbb{R} \setminus U) = 0$, but

$$\eta_n(\mathbb{R} \setminus U) = \eta_n\big((-\infty, -1] \cup [1, +\infty)\big) = 1, \quad n \in \mathbb{N},$$

hence $\limsup_{n\to\infty} \eta_n(\mathbb{R} \setminus U) = 1$. This counterexample also implies that the equivalence of (c) and (d) in Proposition 1.2.19 in [3] is not valid.

Remark 2.3 By giving a counterexample we show that the equivalence of (c) and (d) in Proposition 1.2.13 in [3] is not valid. For all $n \in \mathbb{N}$ let μ_n be the measure $2\delta_{1/n}$ on $\mathbb{R}$ and μ be the Dirac measure δ_0 on $\mathbb{R}$. We have $\mu(A) \leqslant \liminf_{n\to\infty} \mu_n(A)$ for all open subsets A of $\mathbb{R}$ but there exists some closed subset F of $\mathbb{R}$ such that $\limsup_{n\to\infty} \mu_n(F) > \mu(F)$. If A is an open subset of $\mathbb{R}$ such that $0 \in A$ then $\mu(A) = 1$ and $\mu_n(A) = 2$ for all sufficiently large n, which implies $\mu(A) \leqslant \liminf_{n\to\infty} \mu_n(A)$. If A is an open subset of $\mathbb{R}$ such that $0 \notin A$ then $\mu(A) = 0$, hence $\mu(A) \leqslant \liminf_{n\to\infty} \mu_n(A)$ is valid. Let F be the closed interval $[-1, 1]$. Then $\mu(F) = 1$ and $\mu_n(F) = 2$, $n \in \mathbb{N}$, which yields $\limsup_{n\to\infty} \mu_n(F) = 2$. Hence $\limsup_{n\to\infty} \mu_n(F) > \mu(F)$.

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