Stability and periodicity for differential equations with delay

Outline of Ph.D. Thesis

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The thesis summarizes the results of Balázs and Krisztin [2, 1]. Article [2] is accepted for publication, an electronic version is available. Article [1] is submitted. The author has another paper [3], joint with van den Berg, Courtois, Dudás, Lessard, Vörös-Kiss, Williams and Yin, that is not presented here.

In papers [2, 1] and in the thesis we study two different types of differential equations with delay. As for the two equations different technical tools are developed, we consider them in separated chapters with slightly different notions.

The common in the two types of problems is that both are motivated by applications, and both require new, non-classical theoretical techniques. Another joint feature is that we solve open problems for both types of problems. In addition, we believe that the developed methods will turn out to be useful for a wide class of analogous models.

First we study a price model that was introduced by Erdélyi, Brunovský and Walther [5, 4, 19]. The model equation contains only one parameter, $a > 0$. The main result is that in case $0 < a < 1$ the zero solution is globally asymptotically stable. This gives an affirmative answer for a conjecture of Erdélyi, Brunovský and Walther. Earlier local stability was known for all $a \in (0, 1)$, see [5]. As linearization fails at zero, a center manifold reduction was used. Global attractivity was proven only for $a \in (0,\ 0.61)$ by Garab, Kovács and Krisztin [8]. Our proof is based on the key idea that it is possible to connect the problem with a different type of equations, namely with neutral functional differential equations, and in addition, that Lyapunov functionals can be constructed for the neutral type problems.

The second part of the thesis considers a system which is composed of a delay differential equation and two auxiliary equations defining the delay. The delay differential equation satisfies a negative feedback condition studied earlier in several fundamental papers [12, 13], leading to the development of topics of nonlinear functional analysis like fixed point theory in infinite dimensions. The studied particular system was introduced by Ranjan, La and Abed [17, 16] to model a rate control mechanism for a simple computer network. Mathematically, the difficulty arises from the particular form of the delay defined by the two auxiliary equations. The classical results for constant delays [7, 9], the recently developed methods for state-dependent delay [10, 18] do not seem to be applicable here. The first difficulty is to find a suitable phase space where the corresponding initial value problem has a unique maximal solution, and the solutions define a continuous semiflow. In fact, we develop two different frameworks to study the problem. These require different phase spaces and different definitions for solutions. It depends on the question which approach is more suitable. The second main result is that the rate control system of Ranjan et al. may lead to a slowly oscillating periodic rate around the optimal rate, provided that the stationary solution at the optimal rate is unstable. This answers affirmatively a conjecture of Ranjan and his coauthors [15, 14].

1 Global stability for price models with delay

Our primary aim is to prove the global stability conjecture for the price model of Erdélyi, Brunovský and Walther [5, 4, 19]

$$\dot{x}(t) = a[x(t) - x(t-1)] - \beta|x(t)|x(t), \tag{1.1}$$

where $a > 0, \beta > 0$. For $0 < a < 1$, the local asymptotic stability of $x = 0$ was shown by Erdélyi, Brunovský and Walther, and they conjectured global asymptotic stability. Recently, Garab, Kovács and Krisztin [8] obtained global asymptotic stability of $x = 0$ for equation (1.1) provided $a \in (0, 0.61)$. As $x = 0$ is non-hyperbolic, local stability is already nontrivial.

In the sequel, we always assume $r > 0$, $a > 0$, and

$$(\mathrm{H_g}) \qquad \begin{cases} g : \mathbb{R} \to \mathbb{R} \text{ is } C^1\text{-smooth, } ug(u) > 0 \text{ for } u \neq 0, \\ \int_0^s g(u)\, du \to \infty \text{ as } |s| \to \infty. \end{cases}$$

By using Stieltjes integrals, equation (1.1) can be written as

$$\dot{x}(t) = a \int_0^r x(t-s) d\eta(s) - g(x(t)) \tag{1.2}$$

with η satisfying

$$(\mathrm{H}_\eta) \qquad \begin{cases} \eta : [0, r] \to [0, \infty) \text{ is of bounded variation,} \\ \eta(0) = \eta(r) = 0, \ \int_0^r \eta(s)\, ds = 1. \end{cases}$$

Following [5], $x(t)$ in equation (1.2) can represent the price of an asset at time t. Indeed, if $x : I \to \mathbb{R}$ is continuously differentiable on an interval containing $[t-r,t]$, then integrating the Stieltjes integral $\int_0^r x(t-s)\,d\eta(s)$ by parts, and using $\eta(0)=\eta(r)=0$, we find

$$\begin{aligned}\int_0^r x(t-s)\,d\eta(s) &= [x(t-s)\eta(s)]_{s=0}^{s=r} - \int_0^r \eta(s)\,d_s x(t-s)\\ &= -\int_0^r \eta(s)\tfrac{d}{ds}x(t-s)\,ds\\ &= \int_0^r \dot{x}(t-s)\,d_s\left(\int_0^s \eta\right).\end{aligned} \tag{1.3}$$

As η is nonnegative, the function $[0,r]\ni s\mapsto \int_0^s \eta \in \mathbb{R}$ is monotone nondecreasing. Then (1.3) shows that the term $\int_0^r x(t-s)\,d\eta(s)$ is zero if x is constant on $[t-r,t]$, and it is positive (negative) if $\dot{x}(s)>0$ (<0) for all $s\in[t-r,t]$.

Observe that if the function $s\mapsto \int_0^s \eta(u)du$ in the integral term $\int_0^r \dot{x}(t-s)\,d_s\left(\int_0^s \eta\right)$ in equality (1.3) is replaced by an arbitrary nondecreasing function $\mu:[0,r]\to\mathbb{R}$ of bounded variation, then the obtained integral term $\int_0^r \dot{x}(t-s)\,d\mu(s)$ can be still interpreted as the tendency of the price. This motivates to study the neutral type differential equation

$$\dot{y}(t) = a\int_0^r \dot{y}(t-s)d\mu(s) - g(y(t)), \tag{1.4}$$

as well as a price model provided $a>0$ and $\mu:[0,r]\to\mathbb{R}$ is of bounded variation and nondecreasing with an additional technical assumption.

There is another reason to study the neutral type equation (1.4). It plays a crucial role in the proof of the stability results for equations (1.1), (1.2). However, equation (1.2) and equation (1.4) are not equivalent. A solution of equation (1.2) satisfies equation (1.4) with $\mu(s)=\int_0^s \eta$ only for $t>r$. The phase spaces and the stability definitions are also different for equations (1.2) and (1.4).

Let $(c_n)_{n=0}^\infty$ be a sequence of nonnegative numbers with $\sum_{n=0}^\infty c_n \le 1$, and let $(r_n)_{n=0}^\infty$ be a sequence in $[0,r]$ such that $r_0=0$, and $r_n>0$ for all $n\in\mathbb{N}$. Let $H:[0,r]\to\mathbb{R}$ be given by $H(0)=0$, $H(s)=1$ for $s\in(0,r]$. Define $\sigma:[0,r]\to\mathbb{R}$ by

$$\sigma(s) = c_0H(s) + \sum_{n:\ r_n\le s} c_n, \quad s\in[-r,0],$$

and let a nondecreasing, absolutely continuous $\nu:[0,r]\to\mathbb{R}$ be given with $\nu(r)-\nu(s)\le 1$. Our hypothesis on μ is that it is nondecreasing without a singular part, that is,

$$(\mathrm{H}_\mu)\qquad \begin{cases}\mu:[0,r]\to\mathbb{R},\ \mu=\nu+\sigma,\\ \int_0^r d\mu = 1,\ \text{i.e., } \nu(r)-\nu(0)+\sum_{n=0}^\infty c_n = 1\end{cases}$$

holds. For $\varphi \in C([-r,0],\mathbb{R})$ let $\|\varphi\| = \max_{-r\le s\le 0}|\varphi(s)|$. Define the subset

$$Y = \left\{ \psi \in C^1([-r,0],\mathbb{R}) \;\middle|\; \dot\psi(0) = a\int_0^r \dot\psi(-s)\,d\mu(s) - g(\psi(0)) \right\}$$

of $C^1([-r,0],\mathbb{R})$ and let $\|\psi\|_Y = \left((\psi(0))^2 + \int_0^r (\dot\psi(-s))^2\,ds \right)^{1/2}$.

A solution of equation (1.4) with initial function $\psi \in Y$ is a continuously differentiable function $y = y^\psi : [-r, t_\psi) \to \mathbb{R}$ such that $y_0 = \psi$, and (1.4) holds for $t \in (0, t_\psi)$.

From $g(0) = 0$, $y = 0$ is a solution of (1.4), and by $(\mathrm{H_g})$ it is the only equilibrium solution. The solution $y = 0$ is called stable if for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that, for each $\psi \in Y$ with $\|\psi\|_Y < \delta(\varepsilon)$, the solution y^ψ exists on $[-r,\infty)$ and $\|y_t^\psi\|_Y < \varepsilon$ for all $t \ge 0$; and globally asymptotically stable if it is stable and for each $\psi \in Y$ the solution y^ψ exists on $[-r,\infty)$ and $\|y_t^\psi\|_Y \to 0$ as $t \to \infty$.

Theorem 1.1. *Assume Hypotheses* $(\mathrm{H_g})$, $(\mathrm{H_\mu})$ *hold, and* $a \in (0,1)$. *Then for each* $\psi \in Y$ *the unique maximal solution* y^ψ *of equation* (1.4) *is defined on* $[-r,\infty)$, *and the zero solution of* (1.4) *is globally asymptotically stable.*

The proof of Theorem 1.1 is based on a Lyapunov functional which has been inspired by the book of Kolmanovskii and Myshkis [11, Chapter 9, p. 374].

The natural phase space for equation (1.2) is $C([-r,0],\mathbb{R})$. A maximal solution of (1.2) with initial function $\varphi \in C([-r,0],\mathbb{R})$ is a continuous function $x = x^\varphi : [-r, t_\varphi) \to \mathbb{R}$ with $t_\varphi > 0$ so that $x|_{[-r,0]} = \varphi$, x is differentiable on $(0, t_\varphi)$, (1.2) holds on $(0, t_\varphi)$, and any other solution with the same initial function is a restriction of x^φ.

Theorem 1.2. *Assume Hypotheses* $(\mathrm{H_g})$, $(\mathrm{H_\eta})$ *hold, and* $a \in (0,1)$. *Then the zero solution of equation* (1.2) *is globally asymptotically stable.*

Corollary 1.3. *If* $a \in (0,1)$ *then the zero solution of equation* (1.1) *is globally asymptotically stable.*

2 A differential equation with a state-dependent queueing delay

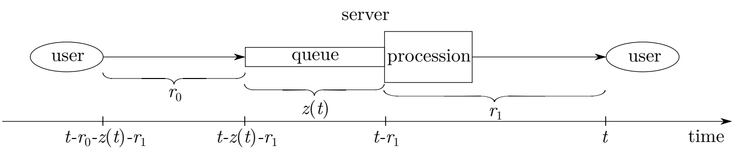
The particular model, that motivated our study, was introduced by Ranjan, La and Abed in [17, 16]. It is a fluid model of a network containing a single user and a single server. The user sends data by rate $x(t) \in [a, b]$ to the server, $0 < a < b$. The server processes the incoming data by the capacity $c \in (a, b)$. The data arriving at the server may form a queue with length $y(t) \in [0, q]$ before procession, $q > 0$. Let $r_0 \geq 0$ be the transfer time from the user to the server, $z(t)$ be the waiting time in the queue, and $r_1 > 0$ be the sum of the procession time and the transmission time from the server to the user, see the figure. This model can be described by the rate control system

$$\begin{aligned} \dot{x}(t) = \kappa\big[&x(t)U'(x(t)) \\ &- x(t - r_0 - z(t) - r_1)p(x(t - z(t) - r_1))\big] \end{aligned} \tag{2.1}$$

$$\dot{y}(t) = \begin{cases} x(t - r_0) - c & \text{if } 0 < y(t) < q \\ [x(t - r_0) - c]^+ & \text{if } y(t) = 0 \\ -[x(t - r_0) - c]^- & \text{if } y(t) = q \end{cases} \tag{2.2}$$

$$z(t) = \frac{1}{c} y(t - z(t) - r_1) \tag{2.3}$$

where $\kappa > 0$, U is the utility, p is the price per unit flow, (2.2) is required to hold almost everywhere, $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$.



The primary aim is to find a suitable framework to study the above types of rate control systems. Neither the classical results for equations with constant delay nor the recently developed results for state-dependent delay do not work here.

The secondary aim is to apply the developed framework, and to show that the rate control system (2.1), (2.2), (2.3) may lead to a slowly oscillating periodic rate around the optimal rate x_*, provided that the stationary solution $x = x_*$, $y = 0$, $z = 0$ is unstable and $r_0 = 0$.

This answers affirmatively a conjecture of Ranjan and his coauthors [15, 14].

Set $r = r_0 + r_1 + q/c > 0$ as an upper bound for the delays. For a Lipschitz continuous $\varphi : I \to \mathbb{R}$, let

$$\mathrm{lip}(\varphi) = \sup_{s\in I,\, t\in I,\, s<t} \left| \frac{\varphi(t)-\varphi(s)}{t-s} \right| \in [0,\infty) \quad \text{and}$$
$$\mathrm{slope}(\varphi) = \left\{ \frac{\varphi(t)-\varphi(s)}{t-s} \,:\, s \in I,\ t \in I,\ s \neq t \right\} \subseteq \mathbb{R}.$$

First we consider a slightly more general system of equation

$$\dot{x}(t) = F(x_t, y_t) \tag{2.4}$$

together with (2.2). An upper bound $K > 0$ for the absolute value of the right hand side of equation (2.4) comes from the nature of the problem. Then the subsets

$$X = \left\{ \varphi \in C_{[-r,0]} \;\middle|\; \varphi([-r,0]) \subseteq [a,b],\ \mathrm{lip}(\varphi) \le K \right\} \quad \text{and}$$
$$Y = \left\{ \psi \in C_{[-r,0]} \;\middle|\; \psi([-r,0]) \subseteq [0,q],\ \mathrm{slope}(\psi) \subseteq [a-c, b-c] \right\}$$

of $C_{[-r,0]}$ will contain all possible segments x_t and y_t. On $X \subset C_{[-r,0]}$, $Y \subset C_{[-r,0]}$, $X \times Y \subset C_{[-r,0]} \times C_{[-r,0]}$ we use the induced subspace topologies and the corresponding norms. By the Arzelà–Ascoli theorem, X, Y and $X \times Y$ are compact subsets of $C_{[-r,0]}$ and $C_{[-r,0]} \times C_{[-r,0]}$, respectively. Assume that the map $F : X \times Y \to \mathbb{R}$ has the following properties:

(H1) there exists $L > 0$ such that, for all $\varphi^1, \varphi^2 \in X$, $\psi^1, \psi^2 \in Y$,

$$\left| F(\varphi^1, \psi^1) - F(\varphi^2, \psi^2) \right| \le L \left\| (\varphi^1, \psi^1) - (\varphi^2, \psi^2) \right\|;$$

(H2) $\max_{(\varphi,\psi)\in X\times Y} |F(\varphi,\psi)| \le K$;

(H3) there exists $r_2 \in (0, r_1]$ such that $F(\varphi, \psi^1) = F(\varphi, \psi^2)$ provided $\varphi \in X$, $\psi^1 \in Y$, $\psi^2 \in Y$, and $\psi^1|_{[-r,-r_2]} = \psi^2|_{[-r,-r_2]}$;

(H4) $F(\varphi, \psi) > 0$ if $\varphi \in X$, $\psi \in Y$, $\varphi(0) = a$, and $F(\varphi, \psi) < 0$ if $\varphi \in X$, $\psi \in Y$ and $\varphi(0) = b$.

A solution of system (2.4), (2.2) *in the phase space* $X \times Y$ with initial condition $x_0 = \varphi \in X$, $y_0 = \psi \in Y$ is a pair $x = x^{\varphi,\psi} : [-r, \omega) \to \mathbb{R}$

and $y = y^{\varphi,\psi} : [-r,\omega) \to \mathbb{R}$ such that $0 < \omega \le \infty$, $x_t \in X$ for all $t \in [0,\omega)$, $x_0 = \varphi$; x is differentiable on $(0,\omega)$; $y_t \in Y$ for all $t \in [0,\omega)$, $y_0 = \psi$; equation (2.4) holds on $(0,\omega)$; and equation (2.2) holds almost everywhere in $(0,\omega)$. The solution is called maximal if any other solution $(\widehat{x},\widehat{y})$ with $\widehat{x}_0 = \varphi$, $\widehat{y}_0 = \psi$ is a restriction of (x,y).

Theorem 2.1. *For each $(\varphi,\psi) \in X \times Y$ there exists a unique solution $x^{\varphi,\psi} : [-r,\infty) \to \mathbb{R}$, $y^{\varphi,\psi} : [-r,\infty) \to \mathbb{R}$ of system (2.4), (2.2) on $[-r,\infty)$ satisfying the initial condition $x_0 = \varphi$, $y_0 = \psi$. The mapping*

$$\Phi : [0,\infty) \times X \times Y \ni (t,\varphi,\psi) \mapsto \left(x_t^{\varphi,\psi}, y_t^{\varphi,\psi}\right) \in X \times Y$$

defines a continuous semiflow on $X \times Y$. In addition, the solution operator $\Phi(t,\cdot,\cdot)$ is Lipschitz continuous for all $t \ge 0$.

In order to sketch the main steps of the proof, let $(\varphi,\psi) \in X \times Y$ be given. By (H3), a standard contraction argument yields $T \in (0,r_2]$ and a unique $x : [-r,T] \to \mathbb{R}$ so that equation (2.4) holds on $(0,T)$, for arbitrary extension of ψ to $[-r,T]$. Next we redefine $y : [-r,T] \to \mathbb{R}$ on $(0,T]$ such that equation (2.2) holds almost everywhere on $[0,T]$. We extend the right hand side of (2.2) to an upper semicontinuous multivalued map, and apply a standard result from [6] for differential inclusions. By the method of steps the solution can be uniquely extended to a maximal solution on $[-r,\infty)$.

Introduce $Z = [0,q/c] \subset \mathbb{R}$ as a state space for the variable $z(t)$.

There is a unique Lipschitz continuous map $\sigma : Y \to Z$ satisfying

$$\sigma(\psi) = \frac{1}{c}\psi(-\sigma(\psi) - r_1). \tag{2.5}$$

Assume that a map $G : X \times Z \to \mathbb{R}$ is given such that, with the particular choice $F : X \times Y \ni (\varphi,\psi) \mapsto G(\varphi,\sigma(\psi)) \in \mathbb{R}$, Hypotheses (H1)–(H4) hold. Consider the system composed of the equations

$$\dot{x}(t) = G(x_t, z(t)), \tag{2.6}$$

and (2.2)–(2.3). Then, in the phase space $X \times Y$, for each $(\varphi,\psi) \in X \times Y$, system (2.6), (2.2), (2.3) has the unique solution $x^{\varphi,\psi}[-r,\infty) \to \mathbb{R}$, $y^{\varphi,\psi} : [-r,\infty) \to \mathbb{R}$, $z^{\varphi,\psi} : [0,\infty) \to \mathbb{R}$ where $\left(x^{\varphi,\psi}, y^{\varphi,\psi}\right)$ is the solution of system (2.4), (2.2) with the above choice of F, and $z^{\varphi,\psi}(t) = \sigma(y_t^{\varphi,\psi})$, $t \ge 0$.

There is a unique Lipschitz continuous map $\gamma : X \times Z \to Y$ so that $\psi = \gamma(\varphi, \zeta)$ is constant on $[-r, -\zeta - r_1]$ and satisfies equation (2.2) on $[-\zeta - r_1, 0]$.

The meaning of the existence of γ is that from the past of the rate and from the present waiting time it is possible to recover the past of the length of the queue. This allows to use $X \times Z$ a suitable phase space as well, although it requires a different definition of solutions.

Theorem 2.2. *For each $(\varphi, \zeta) \in X \times Z$ there exists a unique pair of functions $x^{\varphi,\zeta} : [-r, \infty) \to \mathbb{R}$, $z^{\varphi,\zeta} : [0, \infty) \to \mathbb{R}$ such that (x, z) is a solution of system* (2.6), (2.2), (2.3) *in the phase space $X \times Z$ satisfying the initial condition $x_0 = \varphi$, $z(0) = \zeta$. The mapping*

$$\Psi : [0, \infty) \times X \times Z \ni (t, \varphi, \zeta) \mapsto \left(x_t^{\varphi,\zeta}, z^{\varphi,\zeta}(t)\right) \in X \times Z$$

defines a continuous semiflow on $X \times Z$. In addition, the solution operator $\Psi(t, \cdot, \cdot)$ is Lipschitz continuous for all $t \geq 0$.

Suppose that there exists $x_* \in (a, c)$ serving as a stationary solution of the rate control equation. Defining $v(t) = x(t) - x_*$ and $d = c - x_* > 0$, we obtain the system

$$\dot{v}(t) = -f(v(t)) - g(v(t - z(t) - 1)) \tag{2.7}$$

$$\dot{y}(t) = \begin{cases} v(t) - d & \text{if } 0 < y(t) < q \\ [v(t) - d]^+ & \text{if } y(t) = 0 \\ -[v(t) - d]^- & \text{if } y(t) = q \end{cases} \tag{2.8}$$

$$z(t) = \frac{1}{c} y(t - z(t) - 1) \tag{2.9}$$

A solution (v, z) is called *slowly oscillatory* if for any two zeros $t_1 < t_2$ of v the inequality $z(t_2) + 1 < t_2 - t_1$ holds.

Set $A = a - x_*$, $B = b - x_*$, and assume the following conditions:

(S1) $f, g \in C^1([A, B], \mathbb{R})$;

(S2) $f(\xi)\xi \geq 0$ and $g(\xi)\xi > 0$ for all $\xi \in [A, B] \setminus \{0\}$, $g'(0) > 0$;

(S3) $g([A, B]) \in (-f(B), -f(A))$;

(S4) the map $\mathbb{C} \ni \lambda \mapsto \lambda + f'(0) + g'(0)e^{-\lambda} \in \mathbb{C}$ has a zero with positive real part.

Define the continuous functions $\widetilde{f}, \widetilde{g} : [A, B] \to \mathbb{R}$ as follows:

$$\widetilde{f}(\xi) = \begin{cases} \frac{f(\xi)}{\xi} & \text{if } \xi \neq 0, \\ f'(0) & \text{if } \xi = 0, \end{cases} \qquad \widetilde{g}(\xi) = \begin{cases} \frac{g(\xi)}{\xi} & \text{if } \xi \neq 0, \\ g'(0) & \text{if } \xi = 0. \end{cases}$$

There are constants $f_1 \geq 0$, $g_1 > g_0 > 0$ such that $\widetilde{f}([A, B]) \subseteq [0, f_1]$, and $\widetilde{g}([A, B]) \subseteq [g_0, g_1]$. Let $K_0 = (f_1 + g_1) \max\{-A, B\}$, $r = 1 + q/c$, $K_1 = rK_0$ and

$$\mathcal{X} = \left\{\varphi \in C_{[-r,0]} \;\middle|\; \varphi([-r, 0]) \subseteq [A, B],\ \operatorname{lip}(\varphi) \leq K_1\right\}.$$

The solutions of system (2.7), (2.8), (2.9) define a continuous semiflow by

$$(v_t^{\varphi,\zeta}, z^{\varphi,\zeta}(t)) = \Psi(t, \varphi + x_*, \zeta) - (x_*, 0)$$

on $\mathcal{X} \times Z$, and the same Lipschitz continuity holds for the semiflow as for Ψ in Theorem 2.2.

By using (S4) it is easy to see that $(0, 0) \in \mathcal{X} \times Z$ is an unstable stationary solution.

Define the sets

$$\begin{aligned} W = \Big\{(\varphi, \zeta) \in \mathcal{X} \times Z \;\Big|\; & \varphi\big|_{[-r,-\zeta-1]} \equiv 0, \\ & s \mapsto \varphi(s)e^{f_1 s} \text{ is nondecreasing},\ \varphi(0) > 0\Big\}, \quad \text{and} \\ W_0 = W \cup \{(0, 0)\}&. \end{aligned}$$

For $(\varphi, \zeta) \in W$, $v = v^{\varphi,\zeta}$, $z = z^{\varphi,\zeta}$, the negative feedback condition on g in (S2) gives the existence of

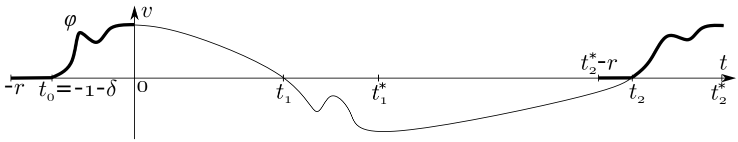
$$t_1 = \min\{t > 0 \mid v(t) = 0\}, \quad t_2 = \min\{t > t_1 \mid v(t) = 0\}$$

and t_2^*, such that $t_2^* - z(t_2^*) - 1 = t_2$, depending continuously on (φ, ζ). In addition, there is a uniform upper bound on t_2^* by (S4). This allows us to define a return map $P : W_0 \to \mathcal{X} \times Z$ by

$$P(\varphi, \zeta) = \begin{cases} (0, 0) & \text{if } (\varphi, \zeta) = (0, 0) \\ \left(\widehat{v_{t_2^*}}, z(t_2^*)\right) & \text{otherwise} \end{cases}$$

where $\widehat{v_{t_2^*}} \in \mathcal{X}$ is determined by $\widehat{v_{t_2^*}}(s) = v(t_2^* + s)$ for $s \in [t_2 - t_2^*, 0]$, and $\widehat{v_{t_2^*}}(s) = 0$ for $s \in [-r, t_2 - t_2^*]$.

P is continuous, and $P(W_0) \subseteq W_0$, $P(W) \subseteq W$. It is a crucial result that $P(\varphi, \zeta)$ cannot decay too fast: there are constants $\theta > 0$,



$\rho > 0$ with $v^{\varphi,\zeta}(t_2^*) \geq \theta\,(\varphi(0))^\rho$ for all $(\varphi,\zeta) \in W$. This fact allows to construct a C^2-function α on $[0, q/c]$ such that $\alpha(0) = 0$, $\alpha' > 0$, $\alpha'' > 0$ on $(0, q/c]$, $\alpha(q/c)$ is small enough, and the delayed inequality $\alpha(\xi - d/c) \geq \theta\,(\alpha(\xi))^\rho$ holds for $\xi \in [d/c, q/c]$. Defining the compact subsets

$$W_{\alpha,K_1} = \left\{(\varphi,\zeta) \in W_0 \;\middle|\; \varphi(0) \geq \alpha(\zeta)\right\},$$
$$W_{\alpha,K_0} = \left\{(\varphi,\zeta) \in W_{\alpha,K_1} \;\middle|\; \mathrm{lip}(\varphi \leq K_0)\right\}$$

of $\mathcal{X} \times Z$, the inclusion $P(W_{\alpha,K_1}) \subseteq W_{\alpha,K_0}$ can be shown. This is the hardest part of the proof toward to the existence of periodic solutions. However, W_{α,K_1} and W_{α,K_0} are not convex. The subset

$$V_{\alpha,K_1} = \Big\{(\psi,\zeta) \in C_{[-1,0]} \times Z \,\Big|\, \psi([-1,0]) \subseteq [0,B],\ \mathrm{lip}(\psi) \leq K_1, \\ [-1,0] \ni s \mapsto \psi(s)e^{f_1 r s} \in \mathbb{R} \text{ is nondecreasing}, \\ \psi(-1) = 0,\ \psi(0) \geq \alpha(\zeta)\Big\}$$

of $C_{[-1,0]} \times \mathbb{R}$ is compact and convex. Set V_{α,K_1} can be mapped into W_{α,K_1} by the streching map Q given by $Q(\psi,\zeta) = (\varphi,\zeta)$ with $\varphi(s) = \psi(s/(\zeta+1))$, $s \in [-\zeta-1, 0]$, and $\varphi|_{[-r,-\zeta-1]} \equiv 0$. The squeezing map R, defined by $R(\varphi,\zeta) = (\psi,\zeta)$ with $\psi(s) = \varphi((\zeta+1)s)$, $s \in [-1,0]$, maps W_{α,K_0} into V_{α,K_1}. Browder's theorem can be applied to find a non-ejective fixed point of the map

$$\Pi : V_{\alpha,K_1} \in (\psi,\zeta) \mapsto R \circ P \circ Q(\psi,\zeta) \in V_{\alpha,K_1}$$

This yields a non-ejective fixed point of P in W_{α,K_1} that is nontrivial provided $(0,0) \in W_{\alpha,K_1}$ is ejective. Ejectivity of $(0,0) \in W_{\alpha,K_1}$ follows in a standard way from that of the zero solution of the constant delay equation $\dot{v}(t) = -f(v(t)) - g(v(t-1))$.

Theorem 2.3. *Assume that Conditions (S1)–(S4) hold. Then system* (2.7), (2.8), (2.9) *has a slowly oscillatory periodic solution.*

Publication list

[1] I. Balázs and T. Krisztin. “A Differential Equation with a State-Dependent Queueing Delay”. *SIAM Journal on Mathematical Analysis* (submitted for publication).

[2] I. Balázs and T. Krisztin. “Global Stability for Price Models with Delay”. *Journal of Dynamics and Differential Equations* (accepted, electronically available).

[3] I. Balázs, J. B. van den Berg, J. Courtois, J. Dudás, J.-P. Lessard, A. Vörös-Kiss, JF Williams, and X. Y. Yin. “Computer-Assisted Proofs for Radially Symmetric Solutions of PDEs”. *Journal of Computational Dynamics* 5.1&2 (2018), pp. 61–80.

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