C_0-semigroups of holomorphic Carathéodory isometries in reflexive TRO

L.L. STACHÓ

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Abstract. We refine earlier results concerning the structure of strongly continuous one-parameter semigroups (C_0-SGR) of holomorphic Carathéodory isometries of the unit ball in infinite dimensional reflexive TROs (ternary rings of operators) We achieve finite algebraic formulas for them in terms of joint boundary fied points and Möbius charts.

1. Introduction, results

Throughout this work let $\mathbf{H}_1, \mathbf{H}_2$ denote complex Hilbert spaces and

$$(1.1)\qquad \mathbf{E} := \mathcal{L}(\mathbf{H}_1, \mathbf{H}_2) = \big\{\text{bded lin. } \mathbf{H}_1 \leftarrow \mathbf{H}_2 \text{ operators}\big\}, \quad \dim(\mathbf{H}_2) < \infty.$$

We shall write $\langle x|y\rangle$ (being linear in x and antilinear in y) for the scalar product in any case without danger of confusion. The chief object of our interest will be the open unit ball $\mathbf{B}$ of $\mathbf{E}$ with its (unique) holomorphic invariant metric $d_{\mathbf{B}}$ such that $d_{\mathbf{B}}(0, x) = \operatorname{artanh} \|x\|$ called the Carathéodory distance of $\mathbf{B}$ along with a C_0-semigroup (strongly continuous one-parameter semigroup, abbreviated as C_0-SGR in the sequel)

$$[\Phi^t : t \in \mathbb{R}] \subset \mathrm{Iso}(d_{\mathbf{B}}) := \big\{\text{holomorphic } d_{\mathbf{B}}\text{-isometries}\big\}.$$

Our title indicates a wider context: TROs i.e. ternary rings of operators are isometric copies of closed subspaces of C^*-algebras being invariant under the operation $(x, y, z) \mapsto xy^*z$. Even with reflexivity, they provide a natural mathematical setting for several theories in quantum physics. Reflexive TROs are finite L^∞-direct sums (cf. [7]) of spaces of the type

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(1.1) and, according to a recent refinement [16] of considerations concerning the facial structure of JB*-triples (Banach spaces with holomorphically symmetric unit balls) due to Peralta and Apazoglou [1,2], the holomorphic Carathéodory isometries of a reflexive JB*-triple reduce to the direct sum of their factor restrictions. In particular, we can confine ourselves to the case (1.1) when describing the algebraic structure of a C_0-SGR of holomorphic Carathéodory isometries of the unit ball in a reflexive TRO. We finished our previous work [16] with the following related result:

Theorem A. *Assume* $0 \in \mathrm{dom}(\Phi') = \{X \in \mathbf{B} : t \mapsto \Phi^t(X) \text{ is differentiable}\}$ *and* $E \in \partial\mathbf{B}$ *is a common fixed point of the continuous extensions of the maps* Φ^t *to ther close unit ball. Then for all* $X \in \mathbf{B}$ *we have*

$$\Phi^t(X) = E + W^t(X-E)\Big[\int_0^t S^{t-h} b^* W^h(X-E)\, dh + S^t\Big]^{-1}$$

for some $b \in \mathbf{E}$ *and* $[W^t : t \in \mathbb{R}_+] \subset \mathcal{L}(\mathbf{H}_1)$, $[S^t : t \in \mathbb{R}_+] \subset \mathcal{L}(\mathbf{H}_2)$ *are suitable* C_0*-SGRs with generators of the form* $U' - Eb^*$ *and* $V' + b^*E$*, respectively where* U' *is a possibly unbounded maximal skew-symmetric linear operator in* $\mathbf{H}_1$ *being the generator of a* C_0*-SGR of linear isometries while* $V' \in \mathcal{L}(\mathbf{H}_1)$ *is a (finite dimensional) skew-selfadjoint operator.*

Theorem A summarizes an alternative of the results of Vesentini [18] and Khatskevich-Reich-Shoikhet [10] improving them with adjusted continuity arguments by means of fixed points for the underlying projective linear representations. However there remains an inconvenience in Theorem A from the view point of applications: the construction of the linear C_0-SGR $[U^t : t \in \mathbb{R}_+]$. At first sight this problem may seem rather harmless: we should determine a C_0-SGR of bounded linear operators being a rank-one perturbation of a C_0-SGR of Hilbert space isometries. All the standard procedures lead to infinite series or limits with convergence which is hard to control in practice. In an earlier work [15] we established finite algebraic formulas for the case of $\dim(\mathbf{H}_2) = 1$ (with $\mathbf{E} \simeq \mathbf{H}_1$)) by exploiting the fact that all the boundary points of a Hilbert ball are tripotents of the associated Jordan triple structure. In this paper we are going to extend the technique of [15]. The difficulty we have to face relies upon the fact that, in case of (1.1), the associated Jordan triple product is

$$\{XY^*Z\} = \frac{1}{2}XY^*Z + \frac{1}{2}ZY^*X \tag{1.2}$$

giving rise to tripotents with $E = \{EE^*E\} = EE^*E$ thus being partial isometries and hence no generic points of the boundary of the unit ball (unlike if $\dim(\mathbf{H}_2) = 1$). As the basis of our main result we establish the following observation.

Proposition 1.3. *There is a Möbius shift*

$$M_A(X) := [1 - AA^*]^{-1/2}(X + A)(1 + A^*X)^{-1}[1 - A^*A]^{1/2}$$

with a suitable point $A \in \mathbf{B}$ along with a partial isometry $E \in \mathbf{E}$ such that the orbit $t \mapsto M_{-A} \circ \Phi^t \circ M_A(0)$ is differentiable and E is a common fixed point of the maps $M_{-A} \circ \overline{\Phi^t} \circ M_A$ $(t \in \mathbb{R}_+)$ with the continuous extensions $\overline{\Phi^t}$ of Φ^t to the closed unit ball.

Recall [18] that C_0-SGRs of holomorphic Carathéodory isometries of the unit ball admit a holomorphic extension to the closed unit ball (even in the much wider setting of JB*-triples [9]) and these extension admit common fixed points in the case (1.1). The term *Möbius transformation* refers to surjective biholomorphic Carathéodory isometry of the unit ball (in a JB*-triple in general). In particular, in our setting (1.1), every Möbius transformation $\Theta \in \mathrm{Iso}(d_{\mathbf{B}})$ has the form

(1.4) $\Theta(X) = M_C\big(UXV^*\big)$ with linear isometries $U \in \mathcal{L}(\mathbf{H}_1)$, $V \in \mathcal{L}(\mathbf{H}_2)$.

Notice that the operator V above is necessarily unitary since $\dim(\mathbf{H}_2) < \infty$. In (1.4) we have $C = \Theta(0)$ unambiguously, while the isometries U, V are determined only up to a common scalar of absolute value 1. We say that two C_0-SGRs $[\Phi^t : t \in \mathbb{R}_+]$, $[\Psi^t : t \in \mathbb{R}_+]$ are Möbius equivalent if $\Psi^t = \Theta \circ \Theta^{-1}$ $(t \in \mathbb{R}_+)$ for some Möbius transformation. Recall also [18] that in case of the existence of a common fixed point for the maps Φ^t $(t \in \mathbb{R}_+)$ inside the open unit ball $\mathbf{B}$, the C_0-SGR $[\Phi^t : t \in \mathbb{R}_+]$ is Möbius equivalent to a C_0-SGR of linear isometries. Our main results reads now as follows.

Theorem 1.5. *Let $\mathbf{\Psi} = [\Psi^t : t \in \mathbb{R}_+] \subset \mathrm{Iso}(d_{\mathbf{B}})$ be a C_0-SGR not being Möbius equivalent to a C_0-SGR of linear isometries. Then there is a Möbius equivalent C_0-SGR $\mathbf{\Phi} = [\Phi^t : t \in \mathbb{R}_+] \subset \mathrm{Iso}(d_{\mathbf{B}})$ to $\mathbf{\Psi}$ of the form described in Theorem A involving terms such that $E (\neq 0)$ is a partial isometry with* $\mathrm{ran}(E) \subset \mathrm{dom}(U')$, *and the C_0-SGR $[W^t : t \in \mathbb{R}_+] \subset \mathcal{L}(\mathbf{H}_1)$ has the form*

$$W^t X = W_0^t PX + W_1^t QX + \int_0^t W_1^{t-h}\big[Q(U' - Eb^*)P\big]W_0^h PX\, dh$$

where $P := EE^*$ denotes the orthogonal projection onto $\mathrm{ran}(E)$, *$Q := 1 - P$, $W_0^t = \exp\big(tP[U' - Eb^*]P\big)$, $[W_1^t : t \in \mathbb{R}_+] \subset \mathcal{L}\big(\mathbf{H}_1 \ominus \mathrm{ran}(E)\big)$ is the (well-defined) C_0-SGR of isometries with generator $QU'|[\mathbf{H}_1 \ominus \mathrm{ran}(E)]$.*

The advantage of the above refinement relies upon the fact that $\mathrm{ran}(E)$ is only finite dimensional, thus the algebraic irregularities will be settled by two finite dimensional linear C_0-groups (namely $\big[\exp\big(t[V'-E^*b]\big) : t\in\mathbb{R}\big]$ generated by $[W_0^t : t\in\mathbb{R}_+]$ and $\big[\exp\big(tP[U'-E^*b]P\big) : t\in\mathbb{R}\big]$ generated by $[S^t : t\in\mathbb{R}_+]$). The complementary infinite dimensional parts are controlled by the C_0-SGR $[W_1^t : t\in\mathbb{R}_+]$ of linear isometries. We can use the Stone type spectral integral formula of its Deddens type C_0-group dilation to achieve the following generalization of [15, Cor.2.10]:

Corollary 1.6. *Any C_0-SGR $[\Psi^t : t\in\mathbb{R}_+]\subset \mathrm{Iso}(d_{\mathbf{B}})$ an be regarded as the restriction $\Psi^t=\widehat{\Psi}^t$ of some C_0-group $[\widehat{\Psi}^t : t\in\mathbb{R}]$ of surjective Carathéodory isometries of the unit ball of a TRO $\widehat{E}=\mathcal{L}(\widehat{\mathbf{H}}_1,\mathbf{H}_2)$ with some covering Hilbert space $\widehat{\mathbf{H}}_1\supset\mathbf{H}_1$.*

2. Preliminaries: projective representation, Kaup type vector fields

For the sake a possibly most self-contained presentation, we recall some further results from [16] based on works by Vesentini, Kaup, Khatskevich-Reich-Shoikhet and Peralta [18,9,10,APPER,12] with minor extensions concerning the structure of a generic C_0-SGR $\mathbf{\Phi}=[\Phi^t : t\in\mathbb{R}_+]\subset\mathrm{Iso}(d_{\mathbf{B}})$. In particular we know already that all the $d_{\mathbf{B}}$-isometries Φ^t admit a holomorphic extension to the ball of radius $\|\Phi^t(0)\|^{-1}$ centered at the origin, and these extensions have a common fixed point within the closed unit ball. Analogously to the features of linear C_0-SGRs in Hille-Yosida theory, the (infinitesimal) generator $\Phi' : X\mapsto \frac{d}{dt}\big|_{t=0+}\Phi^t(X)$ is defined on the intersection of a dense affine submanifold with the unit ball, it determines $\mathbf{\Phi}$ unambiguously and it domain consists of the starting points of the differentiable orbits $t\mapsto\Phi^t(X)$. It is convenient to study $\mathbf{\Phi}$ under the hypothesis (which may be assumed up to Möbius equivalence)

$$(2.1)\qquad 0\in\mathrm{dom}(\Phi')\quad\text{i.e.}\quad t\mapsto a(t):=\Phi^t(0)\ \text{is differentiable}$$

by means of the projective representation

$$\mathfrak{P}\big[\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\big] : X\mapsto (AX+B)(CX+D)^{-1}$$

for $A\in\mathcal{L}(\mathbf{H}_1)$, $B\in\mathcal{L}(\mathbf{H}_1,\mathbf{H}_2)=\mathbf{E}$, $C\in\mathcal{L}(\mathbf{H}_2,\mathbf{H}_1)=\mathbf{E}^*$, $D\in\mathcal{L}(\mathbf{H}_2)$ satisfying the identity $\mathfrak{P}(\mathcal{A}\mathcal{B})=\mathfrak{P}(\mathcal{A})\,\mathfrak{P}(\mathcal{B})$. Namely can write

$$(2.2)\quad \Phi^t=\mathfrak{P}(\mathcal{A}_t),\quad \mathcal{A}^t=\begin{bmatrix}A_t&B_t\\C_t&D_t\end{bmatrix},\quad [\mathcal{A}^t]^*\mathcal{J}\mathcal{A}_t=\mathcal{J}=\begin{bmatrix}\mathrm{Id}_{\mathbf{H}_1}&0\\0&-\mathrm{Id}_{\mathbf{H}_2}\end{bmatrix}$$

for some C_0-SGR $[\mathcal{A}^t : t \in \mathbb{R}_+] \subset \mathcal{L}(\mathbf{H}_1 \oplus \mathbf{H}_2)$. This fact was used by Vesentini [18] already (going back to Hirzebruch's ideas [5] in finite dimensions) without sufficient existence proof for the infinite dimensional case, which is settled with adjusted continuity arguments in [16, Prop.7.5]. Möbius shifts have particular interest in the case (2.1). With the standard notation

$$\mathcal{M}_a := \begin{bmatrix} (1-aa^*)^{-1/2} & 0 \\ 0 & (1-a^*a)^{-1/2} \end{bmatrix} \begin{bmatrix} 1 & a \\ a^* & 1 \end{bmatrix} \qquad (a \in \mathbf{B})$$

we have

$$\mathcal{A}^t = \mathcal{M}_{a(t)} \mathcal{U}_t, \quad a(t) := \Phi^t(0),\ \mathcal{U}_t := \begin{bmatrix} U_t & 0 \\ 0 & V_t \end{bmatrix} \tag{2.3}$$

where $U_t^* U_t = U_0 = \mathrm{Id}_{\mathbf{H}_1}$ and $V_t^* V_t = V_t V_t^* = V_0 = \mathrm{Id}_{\mathbf{H}_2}$. It is worth to mention a remarkable fact concerning the factorization (2.3) contained implicitly in [18]: as a consequence of the identities $\mathcal{M}_a^* \mathcal{J} \mathcal{M}_a = \mathcal{J}$ and $\mathrm{diag}(U,V)\mathcal{M}_a = \mathcal{M}_{UaV} \mathrm{diag}(U,V)$ with isometries U, V, the operators of the form $\mathcal{M}_a \mathrm{diag}(U,V)$ with $U^*U = \mathrm{Id}_{\mathbf{H}_1}, V^*V = VV^* = \mathrm{Id}_{\mathbf{H}_2}$ form a multiplication semigroup. Under (2.1), the orbit $t \mapsto \mathcal{U}_t$ is strongly differentiable and for the infinitesimal generator we have

$$\mathcal{A}' = \begin{bmatrix} U' & b \\ b^* & V' \end{bmatrix} \quad \text{with } b := \frac{d}{dt}\Big|_{t=0+} a(t),\ U' := \frac{d}{dt}\Big|_{t=0+} U_t,\ V' := \frac{d}{dt}\Big|_{t=0+} V_t, \tag{2.4}$$

where $\mathrm{dom}(U') = \{x \in \mathbf{H}_1 : t \mapsto U_t x \text{ is differentiable}\}$, $\mathrm{dom}(V') = \mathbf{H}_2$. We know [16] that U' is the infinitesimal generator of a C_0-SGR of $\mathbf{H}_1$-isometries and

$$\mathbf{J} := \{X \in \mathbf{E} : \mathrm{ran}(X) \subset \mathrm{dom}(U')\} \tag{2.5}$$

is a is a dense Jordan subtriple with respect to the triple product (1.2). Therefore, since the Möbius shift $M_{a(t)} = \mathfrak{P}\mathcal{M}_{a(t)}$ well-defined and holomorphic on the ball $\|a(t)\|^{-1}\mathbf{B}$, the nonlinear infinitesimal generator has then the form

$$\Phi'(X) = b - Xb^*X + U'X - XV', \quad \mathrm{dom}(\Phi') = \mathbf{B} \cap \mathbf{J}. \tag{2.6}$$

In particular, $\mathbf{J}$ contains the common fixed points of the continuous extensions $\overline{\Phi^t}$ in the closed unit ball. The finite dimensional operator V' is necessarily skew-symmetric (i.e. i[selfadjoint]).

As for a historical remark: in the setting of (locally) uniformly continuous groups of holomorphic automorphisms of the unit ball of a JB*-triple (Banach space with holomorphically symmetric unit ball) W. Kaup

[9] established the formula $\Phi'(X) = b - \{Xb^*X\} + iAX$ $(X \in \mathbf{B})$ with Banach-hermitian operators A. His arguments were based on Lie theoretical considerations with a specific topology found by H. Upmeier on the group of holomorphic automorphisms. We shall call such vector fields with possibly unbounded generators of C_0-SGR of isometries in place of iA *Kaup type vector fields.* It seems that the physically interesting case of C_0-SGRs cannot be treated with Kaups tools: If 2.1) does not hold, the generator Φ' cannot be of Kaup type. In [15] we gave a formula free of Möbius equivalence for the case of $\dim(\mathbf{H}_2) = 1$ of Hilbert balls in terms of a joint fixed point. One can find Möbius free formulas in the setting of generic TRO factors in [10], however, they seem to be hard to transform into closed algebraic formulas (like those in our main results) unless we understood better the use of common fixed points. In particular there seems to be no proof in the literature for the plausible fact that unbounded Kaup type vector fields in JB*-triples are generators of C_0-SGRs of holomorphic Caratéodory isometries of the unit ball. In the presence of a linear representation (like $\mathfrak{P}$ in our case below or for spin factors) we can proceed as follows.

Lemma 2.7. *Let* $\Omega(X) := b-\{Xb^*X\}+\mathrm{A}X$ $(X \in \mathbf{D})$ *be a Kaup type vector field in* $\mathbf{E}$ *such that* $\mathrm{A} = \frac{d}{dt}\big|_{t=0+}\mathrm{U}^t$ *is the infinitesimal generator of a* C_0*-SGR* $[\mathrm{U}^t : t\in\mathbb{R}_+]$ *of linear* $\mathbf{E}$*-isometries with domain* $\mathbf{D}$. *Then there exists a (unique)* C_0*-SGR* $[\Phi^t : t \in \mathbb{R}_+] \subset \mathrm{Iso}(d_{\mathbf{B}})$ *such that* $\Phi' = \Omega|\mathbf{B}$.

Proof. Let $\mathcal{B} := \left[\begin{smallmatrix} 0 & b \\ b^* & 0\end{smallmatrix}\right]$ and let $[\mathcal{U}^t : t \in \mathbb{R}_+]$ be a C_0-SGR with $\mathfrak{P}\mathcal{U}^t = \mathrm{U}^t$ $(t \in \mathbb{R}_+)$. Define $\mathcal{U} := \frac{d}{dt}\big|_{t=0+}\mathcal{U}^t$. Actually we can write $\mathcal{U} = \left[\begin{smallmatrix} U & 0 \\ 0 & V\end{smallmatrix}\right]$ with suitable skew-symmetric linear operators $U : \mathrm{dom}(U) \to \mathbf{H}_1$, $V \in \mathcal{L}(\mathbf{H}_2)$. It is well-known ([9, Section 2] for Jordan pairs) that we have

$$\exp(t\mathcal{B}) = \mathcal{M}_{a(t)}, \quad \mathfrak{P}\mathcal{M}_a(t) = \mathrm{M}_{a(t)} \quad (t \in \mathbb{R})$$

for some C_0-group $[\mathrm{M}_{a(t)} : t \in \mathbb{R}]$ of Möbius shifts. According to the Bounded Perturbation Theorem [3], there is a C_0-SGR $[\mathcal{F}^t : t \in \mathbb{R}_+]$ of linear operators with generator $\mathcal{F}' = \frac{d}{dt}\big|_{t=0+}\mathcal{U}^t = \mathcal{B} + \mathcal{U}$. Recall [3, p.230 Ex. 3.11] that pointwise we have

$$\mathcal{F}^t = \lim_{n\to\infty}\left[\exp\left(\frac{t}{n}\mathcal{B}\right)\mathcal{U}^{t/n}\right]^n = \lim_{n\to\infty}\left[\mathcal{M}_{a(t/n)}\mathcal{U}^{t/n}\right]^n \quad (t \in \mathbb{R}_+).$$

As mentioned, the operator matrices of the form $\mathcal{M}_c\mathcal{V}$ with $c \in \mathbf{B}$ and $\mathcal{V} = u \oplus v$ with isometries u, v form a multiplication semigroup. Therefore $\mathcal{F}^t = \lim_{n\to\infty}\mathcal{F}_{t,n}$ pointwise (i.e. strongly) where $\mathcal{F}_{t,n} := \mathcal{M}_{t,c_n}\mathcal{V}_{t,n}$ with suitable $c_{t,n} \in \mathbf{B}$ and $\mathcal{V}_{t,n}$ such that $\mathfrak{P}\mathcal{V}_{t,n}$ is a linear $\mathbf{E}$-isometry. Thus the maps $\Phi_{t,n} := \mathfrak{P}\mathcal{F}_{t,n}$ are holomorphic $d_{\mathbf{B}}$-isometries. Recall also [18] that the represented object $\mathfrak{P}\mathcal{F}$ by an operator matrix $\mathcal{F}$ is a holomorphic $d_{\mathbf{B}}$-isometry if and only if $\mathcal{F}^*\mathcal{J}\mathcal{F} = \mathcal{J} = \mathrm{diag}(\mathrm{Id}_{\mathbf{H}_1}, \mathrm{Id}_{\mathbf{H}_2})$. It is easy to see that this relation is preserved under stong convergence. $\big($Proof: If $\mathcal{F}_n x \to \mathcal{F}x$ for all x and $\mathcal{F}_n^*\mathcal{J}\mathcal{F}_n = \mathcal{J}$ then we have $\langle \mathcal{J}x|y\rangle = \langle \mathcal{J}\mathcal{F}_n x|\mathcal{F}_n y\rangle \to \langle \mathcal{J}\mathcal{F}x|\mathcal{F}y\rangle$ implying $\langle \mathcal{J}x|y\rangle = \langle \mathcal{F}^*\mathcal{J}\mathcal{F}x|y\rangle$ for all $x, y\big)$. Therefore the maps $\Phi^t := \mathfrak{P}\mathcal{F}^t$ are holomorphic $d_{\mathbf{B}}$-isometries which completes the proof.

3. Proof of Proposition 1.3

Lemma 3.1. *Assume* $\mathbf{\Phi}$ *has a Kaup type generator* (2.6) *and let* F *be a common fixed point of the continuous extensions* $\overline{\Phi}^t$ *being an inner point of the* (*necessarily finite dimensional*) *face* $\mathbf{F}$ *of* $\overline{\mathbf{B}}$ *with respect to its relative topology. Then the middle point* E *of* $\mathbf{F}$ *is a tripotent belonging to the subtriple* $\mathbf{J}$ *in* (2.6).

Proof. We know [1,2] that

$$\mathbf{F} = E + \big[\mathbf{B} \cap E^{\perp\mathrm{Jordan}}\big] = \{E + A : E \perp^{\mathrm{Jordan}} A,\ \|A\| < 1\}$$

where the middle point E of $\mathbf{F}$ is a tripotent $E = \{EE^*E\} = EE^*E \neq 0$ for the triple product (1.2). Recall that, by definition, $E \perp^{\mathrm{Jordan}} A$ if and only if $\{EA^*X\} = \{AE^*X\} = 0$ for all $X \in \mathbf{E}$. In case of the triple product (1.2) this relation means simply that $\mathrm{ran}(E) \perp \mathrm{ran}(A)$ and $\ker(E) \perp \ker(A)$ with orthogonality in $\mathbf{H}_1$ and $\mathbf{H}_2$, respectively. It is also well-known that $E^{\perp\mathrm{Jordan}}$ is a finite, say N-dimensional subtriple of $\mathbf{E}$ with $N \leq \dim(\mathbf{H}_2)$ and

$$F = E + A \quad \text{where} \quad A = \sum_{k=1}^{m} \lambda_k E_k, \quad 0 < \lambda_1 < \cdots < \lambda_m < 1, \quad m \leq N$$

with some Jordan-orthogonal family of tripotents $E_1, \ldots, E_m$ in $E^{\perp\mathrm{Jordan}}$. Thus, by setting $E_0 := E$ and $\lambda_0 := 1$, the Jordan spectral decomposition of F is simply $F = \sum_{k=0}^m \lambda_k E_k$ because the values λ_k are pairwise different. Therefore the Jordan subtriple $\mathbf{J}_F$ generated by F (i.e. the minimal subspace containing F and being closed under the triple product) has the form $\oplus_{k=0}^m \mathbb{C}E_k$. In particular, $E = E_0 \in \mathbf{J}_F$. However, as being a joint fixed point, we have $F \in \mathbf{J}$ and hence also $\mathbf{J}_F \subset \mathbf{J}$ because $\mathbf{J}$ is a (not necessarily closed) subtriple.

Corollary 3.2. (End of the proof for 1.3). *If* $\mathbf{E} = \mathcal{L}(\mathbf{H}_1, \mathbf{H}_2)$ *with* $r = \dim(\mathbf{H}_2) < \infty$ *and* $[\Psi^t : t \in \mathbb{R}_+]$ *is a* C_0*-SGR in* $\mathrm{Iso}(d_\mathbf{B})$ *then there is a* C_0*-SGR* $[\Phi^t : t \in \mathbb{R}_+]$ *in* $\mathrm{Iso}(d_\mathbf{B})$ *being Möbius equivalent to* $[\Psi^t : t \in \mathbb{R}_+]$ *such that its generator is of Kaup type and whose continuous extensions to the closed unit ball admit a common fixed point which is a tripotent.*

Proof. As mentioned in Section 2, any C_0-SGR in $\mathrm{Iso}(\mathbf{E})$ whose 0-orbit is differentiable has a Kaup type generator (whose domain is the intersection of a not necessarily closed Jordan subtriple with the unit ball) and the continuous extensions of its members admit a common fixed point in the closed unit ball. Recall also [2] that the boundary of the unit ball $\mathbf{B}$ is a union of finite (at most r) dimensional faces. In accordance with Lemma 3.1, let $F = E + A$ be a common fixed point of $[\overline{\Psi}^t : t \in \mathbb{R}_+]$ where E is a tripotent and $A \perp^{\mathrm{Jordan}} E$ with $\|A\| < 1$. Consider the Möbius equivalent C_0-SGR $[\Phi^t : t \in \mathbb{R}_+]$ with $\Phi^t := M_{-A} \circ \Psi^t \circ M_A$. In course the proof of the lemma we established that $A \in \mathbf{B} \cap \mathbf{J}_F = \mathbf{B} \cap \oplus_{k=1}^m \mathbb{C}E_k$, whence

$$\pm A \in \mathbf{B} \cap \mathbf{J} = \mathrm{dom}(\Psi') = \big\{X : t \mapsto \Psi^t(X) \text{ is differentiable}\big\}.$$

In particular the orbits $t \mapsto \Psi^t(A)$ and $t \mapsto \Phi^t(0) = M_{-A}\big(\Psi^t(M_A(0))\big) = M_{-A}\big(\Psi^t(A)\big)$ are differentiable. That is $0 \in \mathrm{dom}(\Phi')$, which implies that Φ' is of Kaup type. Also we have

$$\overline{\Phi}^t\big(M_{-A}(F)\big) = M_{-A}\big(\overline{\Psi}^t(F)\big) = M_{-A}(F) \qquad (t \in \mathbb{R}_+)$$

that is the point $M_{-A}(F)$ is a common fixed point for $[\overline{\Phi}^t : t \in \mathbb{R}_+]$. To complete the proof, we have to establish that $M_{-A}(F) = E$. We prove this relation by means of Kaup's celebrated Möbius formula [9, Section 2]

$$M_C(X) = C + B(C)^{1/2}[1 + L(X, C)]^{-1}X \quad \big(C \in \mathbf{B},\ \|X\| < \|C\|^{-1}\big)$$

in terms of the so-called linear resp. quadratic representation operators $L(X, Y) : Z \mapsto \{XY^*Z\}$ resp. $Q(X, Y) : Z \mapsto \{XZ^*Y\}$ and the Bergman operator $B(C) := \mathrm{Id} - 2L(C, C) + Q(C, C)^2$. Since $E \perp^{\mathrm{Jordan}} A$, we have $L(E, A) = L(A, E) = Q(A)E = 0$ implying also $B(A)^{1/2}E = E$. Hence

$$\begin{aligned} M_{-A}(F) = M_{-A}(E + A) &= -A + B(A)^{1/2}[1 - L(E + A, A)]^{-1}(E + A) = \\ &= -A + B(A)^{1/2}[1 - L(A, A)]^{-1}(E + A) = \\ &= -A + B(A)^{1/2}[1 - L(A, A)]^{-1}E + B(A)^{1/2}[1 - L(A, A)]^{-1}A = \\ &= -A + B(A)^{1/2}E + B(A)^{1/2}[1 - L(A, A)]^{-1}A = \\ &= -A + E + B(A)^{1/2}[1 - L(A, A)]^{-1}A = \\ &= M_{-A}(A) + E = 0 + E = E. \end{aligned}$$

4. Proofs for main results

According to Corollary 3.2, there is a C_0-SGR $\mathbf{\Phi} = [\Phi^t : t \in \mathbb{R}_+]$ being Möbius equivalent tp $\mathbf{\Psi}$ which satisfies the hypothesis of Theorem A with a common fixed point $E \in \mathrm{Fix}(\overline{\mathbf{\Phi}})$ such that $0 \neq E = \{EE^*E\} = EE^*E$. We have to see only that the C_0-SGR $[W^t : t \in \mathbb{R}_+] \subset \mathcal{L}(\mathbf{H}_1)$ with infinitesimal generator

$$W' = U' - Eb^*$$

can be written in the form stated in Theorem 1.5. Throughout this section we use the notations and assumptions of Theorem 1.5, furthermore we let

$$\mathbf{H}_{1,1} := \mathrm{ran}(E) = \mathrm{ran}(P), \quad \mathbf{H}_{1,2} := \mathbf{H}_1 \ominus \mathbf{H}_{1,1}, \quad \mathbf{D} := \mathrm{dom}(U').$$

Lemma 4.1. *The projections P, Q map $\mathbf{D}$ into itself,* $\mathrm{dom}(W') = \mathbf{D}$ *and*

$$PW'Q = 0 \quad \text{with} \quad \mathrm{dom} W'Q = Q\mathbf{D}\,\text{dense} \subset \mathbf{H}_{1,2}. \tag{4.2}$$

Proof. We have $\mathbf{D} = \mathrm{dom}(U') = \mathrm{dom}(W')$ since $W' = U' - Eb^*$ is a bounded perturbation of U'. Since $E \in \mathrm{Fix}(\overline{\mathbf{\Phi}})$ is a common fixed point of the holomorphic extensions of the maps Φ^t to balls containing the closed unit ball, as mentioned in Section 2, we have

$$0 = \frac{d}{dt}\Big|_{t=0+} E = \frac{d}{dt}\Big|_{t=0+} \overline{\phi^t}(E) = b - Eb^*E + U'E - EV' \tag{4.3}$$

with $E \in \mathbf{J} = \mathbb{C}\mathrm{dom}(\Phi')$. By (2.5) it follows $\mathrm{ran}(X) \subset \mathrm{dom}(U')$ that is $\mathbf{H}_{1,1} = \mathrm{ran}(P) \subset \mathbf{D}$. In particular the projection P maps $\mathbf{D}$ into itself. Since $\mathbf{D}$ is a linear submanifold in $\mathbf{H}_1$, for any vector $y \in \mathbf{D}$ we have $y = Py + Qy$ with $Py \in \mathbf{D}$ and $Qy = y - Py \in \mathbf{D}$. That is also $Q\mathbf{D} \subset \mathbf{D}$ and $\mathbf{H}_1$ is the orthogonal sum of its linear submanifolds $\mathbf{H}_{1,1} = P\mathbf{D}$ and $Q\mathbf{D} = \mathbf{D} \cap Q\mathbf{H}_1 = \mathbf{D} \cap \mathbf{H}_{1,2}$. Finally we deduce (4.5) from $(4,3)$ as follows. Since E is a partial isometry, $QE = (1 - EE^*E) = E - EE^*E = 0$. Thus we have

$$0 = Q(b - Eb^*E + U'E - EV') = Q(b + U'E)$$

Since the possibly unbounded operator U' is skew-symmetric, by passing to adjoints and then multiplying with E from the left, we get

$$\begin{aligned} 0 &= E(b^* + E^*[U']^*)Q = E(b^* - E^*U')Q = \\ &= (Eb^* - EE^*U')Q = (EE^*Eb^* - EE^*U')Q = \\ &= EE^*(Eb^* - U')Q = P(Eb^* - U')Q. \end{aligned}$$

4.4. Proof of Theorem 1.5.

In terms of $\mathbf{H}_{1,1}\oplus\mathbf{H}_{1,2}$ matrices, we can write W' in the triangular form

$$W'y = \begin{bmatrix} P(U'-Eb^*)P & 0 \\ Q(U'-Eb^*)P & QU'Q \end{bmatrix} \begin{bmatrix} Py \\ Qy \end{bmatrix} \qquad (y\in\mathbf{D})$$

since $QE = 0$ and hence $Q(U'–Eb^*)Q = QU'Q$ by the previous lemma. Observe that the skew-symmetric operator $U_1' := QU'Q$ is a bounded perturbation of $U'-Eb^*$ which is the infinitesimal generator of the C_0-SGR of linear $\mathbf{H}_1$-operators. Hence U_1' itself is also the generator of some C_0-SGR of linear $\mathbf{H}_{1,2}$-operators being isometries due to the skew-symmetry of $U\prime$. Hence the restriction $W_1' := U_1'\|\mathbf{D}_1$ to the range section $\mathbf{D}_1 := Q\mathbf{D} = \mathbf{D}\cap\mathbf{H}_{1,2}$ of Q is the generator of a (unique) C_0-SGR $[W_1^t : t\in\mathbb{R}_+]$ of linear $\mathbf{H}_{1,2}$-isometries. On the other hand, as being a finite dimensional operator, $W_0' := P(U'-Eb^*)|\mathbf{H}_{1,1} = P(U'-Eb^*)|\mathrm{ran}(P)$ is the infinitesimal generator of the C_0-group $[W_1^t : t\in\mathbb{R}]$, $W_1^t = \exp\big(tP(U'-Eb^*)|\mathbf{H}_{1,1}\big)$. Therefore, according to the triangularization lemma [15, Lemma 3.8], we have

$$W^t = \begin{bmatrix} W_0^t & 0 \\ \int_0^t W_0^{t-s}Q(U'-Eb^*)PW_1^s\,ds & W_1^t \end{bmatrix} \tag{4.5}$$

whence the statement of Theorem 1.5 is immediate.

4.6. Proof of Corollary 1.6.

According to the Deddens type dilation lemma [15, Lemma 5.1], for the infinite dimensional corner in (4.5) we can write

$$W_0^t = \widehat{W_0^t}\big|\mathbf{H}_{1,2} \quad (t\in\mathbb{R}_+)$$

by means of some C_0-group $[\widehat{W_0^t} : t\in\mathbb{R}]$ of surjective linear isometries of a suitable Hilbert space $\widehat{\mathbf{H}}_{1,2}$ containing $\mathbf{H}_{1,2}$ as a subspace. Introduce the extension $\widehat{\mathbf{H}}_1 := \mathbf{H}_{1,1}\oplus\widehat{\mathbf{H}}_{1,2}$ of the space $\mathbf{H}_1 = \mathbf{H}_{1,1}\oplus\mathbf{H}_{1,2}$ and let Π denote the canonical orthogonal projection $\widehat{\mathbf{H}}_{1,2}\to\mathbf{H}_{1,2}$. Consider the vector field

$$\widehat{\Omega}(\widehat{X}) := b - Eb^*E + \widehat{W_0'}\widehat{X} - \widehat{X}W_1' \qquad \big(\widehat{X}\in\widehat{\mathbf{E}}\big)$$

where $\widehat{\mathbf{E}} := \mathcal{L}(\widehat{\mathbf{H}}_1,\mathbf{H}_2)$, and $\widehat{W_0'}$ resp. W_1' stand for the infinitesimal generators of $[\widehat{W_0^t} : t\in\mathbb{R}]$ reap, $[W_0^t : t\in\mathbb{R}]$. Notice that $\widehat{W_0'}$ is a *self-adjoint* $\widehat{\mathbf{H}}_1$-operator. By definition, $\widehat{\Omega}$ is of Kaup type. Moreover even its time-reversed ${}^{\mathrm{T}}\widehat{\Omega} : \widehat{X}\mapsto b-Eb^*E-\widehat{W_0'}\widehat{X}+\widehat{X}V'$ is of Kaup type since also $-\widehat{W_0'}$ resp. $-V'$ are infinitesimal generators of linear C_0-SGR (even C_0-groups).

Lemma 2.7 guarantees that $\widehat{\Omega}$ is the non-linear infinitesimal generator of some C_0-subgroup $\widehat{\Phi}^t : t \in \mathbb{R}]$ of holomorphic Carathéodory isometries of the unit ball $\widehat{\mathbf{B}}$ of $\widehat{\mathbf{E}}$. Since $\widehat{\Omega}$ is a holomorphic extension of Ω from $\mathbf{E}$ to $\widehat{\mathbf{E}}$, also the maps $\widehat{\Phi}^t$ $t \in \mathbb{R}_+)$ extend holomorphically the maps Φ^t with the same time index from $\mathbf{B}$ to $\widehat{\mathbf{B}}$ which completes the proof.

Remark 4.7. By Stene's Theorem, there is an orthogonal $\widehat{\mathbf{H}}_1$-projection valued measure $\Lambda \mapsto S(\Lambda)$ defined the family of all Borelian subsets of $\mathbb{R}$ giving rise to the spectral resolution $\widehat{W_0^t} = \int_{\lambda \in \mathbb{R}} \exp(it\lambda) dS(\lambda)$ $(t \in \mathbb{R})$. Unfortunately, the the finite dimensional linear group $[V^t : t \in \mathbb{R}]$ is not necessarily of toroidal type, thus it admits only a direct exponential resolution in terms of Jordan block matrices. Nevertheless, analogous (but clearly more sophisticated) explicit formulas like those in [15, Thm.2.1] are available. A finer discussion like that in [15, Thm.2.6] seems to be an open problem for the moment.

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L.L. Stachó, University of Szeged, Bolyai Institute, Aradi vértanúk tere 1, Szeged, Hungary 6720; *e-mail*: stacho@math.u-szeged.hu