MULTIPLICATIONS AND ELEMENTARY OPERATORS IN THE BANACH SPACE SETTING

EERO SAKSMAN AND HANS-OLAV TYLLI

1. Introduction

This expository survey is mainly dedicated to structural properties of the *elementary operators*

$$\mathcal{E}_{A,B};\ S \mapsto \sum_{j=1}^{n} A_j S B_j, \tag{1.1}$$

where $A = (A_1, \ldots, A_n), B = (B_1, \ldots, B_n) \in L(X)^n$ are fixed n-tuples of bounded operators on X and X is a (classical) Banach space. The simplest operators contained in (1.1) are the left and right multiplications on $L(X)$ defined by $L_U : S \mapsto US$ and $R_U : S \mapsto SU$ for $S \in L(X)$, where $U \in L(X)$ is fixed. Thus the operators in (1.1) can be written as

$$\mathcal{E}_{A,B} = \sum_{j=1}^{n} L_{A_j} R_{B_j}.$$

This concrete class includes many important operators on spaces of operators, such as the two-sided multiplications $L_A R_B$, the commutators (or inner derivations) $L_A - R_A$, and the intertwining maps (or generalized derivations) $L_A - R_B$ for given $A, B \in L(X)$. Elementary operators also induce bounded operators between operator ideals, as well as between quotient algebras such as the Calkin algebra $L(X)/K(X)$, where $K(X)$ are the compact operators on X. Note also that definition (1.1) makes sense in the more general framework of Banach algebras.

Elementary operators first appeared in a series of notes by Sylvester [Sy1884] in the 1880's, in which he computed the eigenvalues of the matrix operators corresponding to $\mathcal{E}_{A,B}$ on the $n \times n$-matrices. The term *elementary operator* was coined by Lumer and Rosenblum [LR59] in the late 50's. The literature related to elementary operators is by now very large, and there are many excellent surveys and expositions of certain aspects. Elementary operators on C*-algebras were extensively treated by Ara and Mathieu in [AM03, Chapter 5]. Curto [Cu92] gives an exhaustive overview of spectral properties of elementary operators, Fialkow [Fi92] comprehensively discusses their structural properties (with an emphasis on Hilbert space aspects and methods), and Bhatia and Rosenthal [BR97] deals with their applications to operator equations and linear algebra. Mathieu [Ma01b], [Ma01a] surveys some recent topics in the computation of the norm of elementary operators, and elementary operators on the Calkin algebra. These references also contain a number of applications, and we also note the survey by Carl and Schiebold [CS00], where they describe an intriguing approach to certain nonlinear equations from soliton physics which involves some elementary operators (among many other tools).

This survey concentrates on aspects of the theory of elementary operators that, roughly speaking, involves "Banach space techniques". By such methods we mean e.g. basic sequence techniques applied in X or in $K(X)$, facts about the structure of complemented subspaces of classical Banach spaces X, as well as useful special properties of the space X (such as approximation properties or the Dunford-Pettis property). The topics and results covered here are chosen to complement the existing surveys, though some overlap will be unavoidable. Our main motivation is to draw attention to the usefulness of Banach space methods in this setting. In fact, it turns out that Banach space techniques are helpful also when X is a Hilbert space.

This survey will roughly be divided as follows. In Section 2 we discuss various qualitative properties such as (weak) compactness or strict singularity of the basic two-sided multiplications $S \mapsto ASB$ for $A, B \in L(X)$. In Section 3 we concentrate on questions related to the norms and spectra of elementary operators in various settings. We include a quite detailed proof, using only elementary concepts, of the known formula $\sigma(\mathcal{E}_{A,B}) = \sigma_T(A) \circ \sigma_T(B)$ for the spectrum of $\mathcal{E}_{A,B}$ in terms of the Taylor joint spectra of the n-tuples A and B. We also describe the state of the art in computing the norm of elementary operators. Section 4 discusses properties of the induced elementary operators on the Calkin algebra $L(X)/K(X)$, such as a solution to the Fong-Sourour conjecture in the case where the Banach space X has an unconditional basis, and various rigidity properties of these operators. The results included here demonstrate that elementary operators have nicer properties on the Calkin algebra. There is some parallel research about tensor product operators $A\widehat{\otimes}_\alpha B$ for various tensor norms α and fixed operators A, B, which may be more familiar to readers with a background in Banach space theory.

The ideas and arguments will be sketched for a number of results that we highlight here, and several open problems will be stated. The topics selected for discussion have clearly been influenced by our personal preferences and it is not possible to attempt any exhaustive record of Banach space aspects of the theory of elementary operators in this exposition. Further interesting results and references can be found in the original papers and the surveys mentioned above.

Elementary operators occur in many circumstances, and they can be approached using several different techniques. This survey is also intended for non-experts in Banach space theory, and we have accordingly tried to ensure that it is as widely readable as possible by recalling many basic concepts. Our notation will normally follow the references [LT77] and [Wo91], and we just recall a few basic ones here. We put $B_X = \{x \in X : \|x\| \le 1\}$ and $S_X = \{x \in X : \|x\| = 1\}$ for the Banach space X. If $A \subset X$ is a given subset, then $[A]$ denotes the closed linear span of A in X. Moreover, $L(X, Y)$ will be the space of bounded linear operators $X \to Y$ and $K(X, Y)$, respectively, $W(X, Y)$ the closed subspaces of $L(X, Y)$ consisting of the compact, respectively the weakly compact operators. The class of finite rank operators $X \to Y$ is denoted by $\mathcal{F}(X, Y)$. We refer e.g. to [LT77], [DJT95], [JL03] or [Wo91] for more background and any unexplained terminology related to Banach spaces.

2. Qualitative aspects

In this section we focus on concrete qualitative properties of the basic two-sided multiplication operators $L_A R_B$ for fixed $A, B \in L(X)$, where

$$L_A R_B(S) = ASB \tag{2.1}$$

for $S \in L(X)$ and X is a Banach space. The first qualitative result for $L_A R_B$ is probably due to Vala [Va64], who characterized the compact multiplication operators on the space of bounded operators. Recall that X has the approximation property if for every compact subset $D \subset X$ and $\varepsilon > 0$ there is a finite rank operator $U : X \to X$ so that $\sup_{x \in D} \|x - Ux\| < \varepsilon$.

Theorem 2.1. *Suppose that $A, B \in L(X)$ are non-zero bounded operators. Then $L_A R_B$ is a compact operator $L(X) \to L(X)$ if and only if $A \in K(X)$ and $B \in K(X)$.*

Proof. The necessity-part is a simple general fact which we postpone for a moment (see part (i) of Proposition 2.3). The following straightforward idea for the sufficiency-part comes from the proof of [ST94, Thm. 2], which dealt with weak compactness (see also [DiF76] for a similar idea for the ϵ-tensor product).

Suppose that $A \in K(X)$ and $B \in K(X)$. We first consider the situation where X has the approximation property. In this case there is a sequence $(A_n) \subset \mathcal{F}(X)$ of finite-rank operators satisfying $\|A - A_n\| \to 0$ as $n \to \infty$. Clearly

$$\|L_{A_n}R_B - L_A R_B\| \leq \|A_n - A\| \cdot \|B\| \to 0$$

so that $L_{A_n}R_B \to L_A R_B$ as $n \to \infty$, whence it is enough to prove the claim assuming that A is a rank-1 operator, that is, $A = x^* \otimes y$ for fixed $x^* \in X^*$ and $y \in X$. In this case one gets for $S \in L(X)$ and $z \in X$ that

$$(x^* \otimes y) \circ S \circ Bz = \langle x^*, SBz\rangle y = \langle B^*S^*x^*, z\rangle y,$$

that is, $L_A R_B(S) = B^*S^*x^* \otimes y$. Hence

$$L_A R_B(B_{L(X)}) \subset \Phi \circ B^*(B_{X^*}), \tag{2.2}$$

where $\Phi : X^* \to L(X,Y)$ is the bounded linear operator $\Phi(z^*) = \|x^*\|z^* \otimes y$ for $z^* \in X^*$. Since $\Phi \circ B^*$ is compact by assumption, the claim follows.

How should one proceed in the general situation? The main problem compared to the preceding argument is that it may not be possible to approximate the operator A by finite dimensional ones. For that end consider first the case where A is replaced by any rank-1 operator $C : X \to Y$ and Y is an arbitrary Banach space. Exactly the same argument as above applies and we obtain that $L_C R_B : L(X) \to L(X,Y)$ is compact. Also, if Y has the approximation property, then we *may* approximate any $C \in K(X,Y)$ by finite-rank operators and deduce that $L_C R_B$ is compact. Next let $J : X \to \ell^\infty(B_{X^*})$ be the isometric embedding defined by

$$Jx = (x^*(x))_{x^* \in B_{X^*}}, \quad x \in X,$$

and recall that $\ell^\infty(B_{X^*})$ has the approximation property. By choosing $C = JA$ in the previous reasoning we get that $L_C R_B$ is a compact operator. Finally, observe that $L_C R_B = L_J \circ (L_A R_B)$, where $L_J : L(X) \to L(X, \ell^\infty(B_{X^*}))$ is an isometric embedding, which forces $L_A R_B$ to be compact. □

Vala's argument in [Va64] was quite different. He applied an Ascoli-Arzela type characterization of compact sets of compact operators, which was inspired by a symmetric version of the Ascoli-Arzela theorem used by Kakutani.

It is less straightforward to formulate satisfactory characterizations of arbitrary compact elementary operators $\mathcal{E}_{A,B}$, because of the lack of uniqueness in the representations of these operators (for instance, $L_{A-\lambda} - R_{A-\lambda} = L_A - R_A$ for every $A \in L(X)$ and every scalar λ). One possibility is to assume some linear independence among the representing operators. We next state a generalization of Theorem 2.1 of this type, due to Fong and Sourour [FS79] (cf. also [Fi92, Thm. 5.1]).

Theorem 2.2. *Let $A = (A_1, \ldots, A_n), B = (B_1, \ldots, B_n) \in L(X)^n$, where X is any Banach space. Then $\mathcal{E}_{A,B}$ is a compact operator $L(X) \to L(X)$ if and only if there are $r \in \mathbb{N}$ and compact operators $C_1, \ldots, C_r$ in the linear span of $\{A_1, \ldots, A_n\}$ and compact operators $D_1, \ldots, D_r$ in the linear span of $\{B_1, \ldots, B_n\}$ so that*

$$\mathcal{E}_{A,B} = \sum_{j=1}^{r} L_{C_j} R_{D_j}.$$

Vala's result (Theorem 2.1) raised the problem when the basic maps $L_A R_B$ are weakly compact, that is, when $L_A R_B(B_{L(X)})$ is a relatively weakly compact set. It is difficult in general to characterize the weakly compact subsets of $L(X)$, and the picture for weak compactness is much more complicated compared to Theorem 2.1. As the first step Akemann and Wright [AW80] characterized the weakly compact multipliers $L_A R_B$ in the case of Hilbert spaces H as follows: *$L_A R_B$ is weakly compact $L(H) \to L(H)$ if and only if $A \in K(H)$ or $B \in K(H)$* (see Example 2.6 below). The weakly compact one-sided multipliers L_A and R_A on $L(H)$ were identified much earlier by Ogasawara [Og54] (cf. also [Y75] for some additional information).

Subsequently the weak compactness of $L_A R_B$ was studied more systematically by the authors [ST92], Racher [Ra92], and in a more general setting by Lindström and Schlüchtermann [LSch99]. The following basic general facts were noticed in [ST92]. The original proof of part (ii) in [ST92] is somewhat cumbersome, and easier alternative arguments were given in [Ra92] and [ST94] (see also [LSch99]). The argument included below is arguably the simplest one conceptually.

Proposition 2.3. *Let X be any Banach space and let $A, B \in L(X)$.*
(i) If $L_A R_B$ is a weakly compact operator $L(X) \to L(X)$, and $A \neq 0 \neq B$, then $A \in W(X)$ and $B \in W(X)$.
(ii) If $A \in K(X)$ and $B \in W(X)$, or if $A \in W(X)$ and $B \in K(X)$, then $L_A R_B$ is weakly compact $L(X) \to L(X)$.

Proof. (i) The identity $L_A R_B(x^* \otimes x) = B^* x^* \otimes Ax$ for $x^* \in X^*, x \in X$, is the starting point. Fix $x \in S_X$ with $Ax \neq 0$, and note that

$$B^*(B_{X^*}) \otimes Ax \subset \{L_A R_B(x^* \otimes x) : x^* \in B_{X^*}\} \subset L_A R_B(B_{L(X)}),$$

where the right-hand set is relatively weakly compact in $L(X)$ by assumption. It follows that the adjoint B^* (and consequently also B) is a weakly compact operator. The fact that $A \in W(X)$ is seen analogously.

(ii) Suppose first that $A \in K(X)$ and $B \in W(X)$. In this situation the proof is quite analogous to the corresponding one of Theorem 2.1: if X has the approximation property then one notes that it is enough to consider the case where $A = x^* \otimes y$ for some $x^* \in X^*$ and $y \in X$. Here the inclusion (2.2) again yields the weak compactness of $L_A R_B$, since B^* is also weakly compact by Gantmacher's theorem. In the general case one again picks an isometric embedding $J : X \to \ell^\infty(B_{X^*})$ and one applies the approximation property of $\ell^\infty(B_{X^*})$ to obtain that

$$L_{JA} R_B = L_J \circ (L_A R_B) : L(X) \to L(X, Y)$$

is weakly compact. Recall next a useful fact: (relative) weak (non-)compactness is unchanged under isometries. Hence, as $L_J : L(X) \to L(X, Y)$ is an isometric embedding we obtain that $L_A R_B$ is weakly compact.

Next consider the case $A \in W(X)$ and $B \in K(X)$. From the preceding case applied to $L_{B^*} R_{A^*}$ we get that $G \equiv \{U^* : U \in L_A R_B(B_{L(X)})\}$ is a relatively weakly compact set, since obviously

$$G \subset L_{B^*} R_{A^*}(B_{L(X^*)}).$$

This implies that $L_A R_B(B_{L(X)})$ is also a relatively weakly compact set, since the map $U \to U^*$ is an isometric embedding $L(X) \to L(X^*)$. □

For any given Banach space X the exact conditions for the weak compactness of $L_A R_B$ on $L(X)$ fall between the extremal conditions contained in (i) and (ii) of Proposition 2.3, and examples demonstrating a wide variety of different behaviour were included in [ST92] and [Ra92]. To get our hands on these examples we will need more efficient criteria for the weak compactness of $L_A R_B$, which can be obtained by restricting attention to suitable classes of Banach spaces. Before that we also observe that in the study of the maps $S \mapsto ASB$ one is naturally lead to consider (possibly different) Banach spaces X_1, X_2, X_3, X_4 and compatible operators $A \in L(X_3, X_4)$ and $B \in L(X_1, X_2)$. In this case (2.1) still defines a bounded linear operator $L_A R_B \colon L(X_2, X_3) \to L(X_1, X_4)$ (strictly speaking $L_A R_B$ is now a composition operator, but by a minor abuse of language we will still talk about multiplication operators). Above we restricted attention to the case $X_1 = X_2 = X_3 = X_4 = X$ for notational simplicity.

Remark 2.4. Theorem 2.1 and Proposition 2.3 remain valid in the general setting, with purely notational changes in the proofs. For instance, part (ii) of Proposition 2.3 can be stated as follows: *If $A \in K(X_3, X_4)$ and $B \in W(X_1, X_2)$, or if $A \in W(X_3, X_4)$ and $B \in K(X_1, X_2)$, then $L_A R_B$ is a weakly compact operator $L(X_2, X_3) \to L(X_1, X_4)$.*

The first type of examples belong to the class of Banach spaces, where the duals of the spaces $K(X, Y)$ of compact operators can be described using trace-duality. We refer to [DiU77] for the definition and the properties of the Radon-Nikodym property (RNP). It suffices to recall here that X has the RNP if X is reflexive or if X is a separable dual space, while X fails the RNP if X contains a linear isomorphic copy of c_0 or $L^1(0,1)$. The following concrete range-inclusion criterion for the weak compactness of $L_A R_B$ is quite efficient. We again restrict our attention to the case $X_1 = \ldots = X_4 = X$, and the formulation below is far from optimal.

Proposition 2.5. *Let X be a Banach space having the approximation problem, and suppose that X^* or X^{**} has the RNP. Let $A, B \in L(X)$ be non-zero operators. Then $L_A R_B$ is weakly compact $L(X) \to L(X)$ if and only if $A, B \in W(X)$ and*

$$L_{A^{**}} R_{B^{**}}(L(X^{**})) \subset K(X^{**}). \tag{2.3}$$

If X is reflexive, then (2.3) *reduces to*

$$\{ASB : S \in L(X)\} \subset K(X). \tag{2.4}$$

Condition (2.3) is based on the trace duality identifications $K(X)^* = N(X^*)$ and $N(X^*)^* = L(X^{**})$, where the first identification requires suitable approximation properties and the RNP conditions (see e.g. section 2 of the survey [Ru84]). Recall here that $T = \sum_{n=1}^\infty y_n^* \otimes y_n$ is a nuclear operator on the Banach space Y, denoted $T \in N(Y)$, if the sequences $(y_n^*) \subset Y^*$ and $(y_n) \subset Y$ satisfy $\sum_{n=1}^\infty \|y_n^*\| \cdot \|y_n\| < \infty$. The nuclear norm $\|T\|_N$ is the infimum of $\sum_{n=1}^\infty \|y_n^*\| \cdot \|y_n\|$ over all such representations of T. Recall that trace duality is defined by

$$\langle T, S \rangle \equiv trace(T^* S) = \sum_{n=1}^\infty S x_n^{**}(x_n^*). \tag{2.5}$$

for $S \in L(X^{**})$ and $T = \sum_{n=1}^\infty x_n^{**} \otimes x_n^* \in N(X^*)$. For $S \in K(X)$ and $T \in N(X^*)$ one has analogously that $\langle T, S \rangle = \sum_{n=1}^\infty S^{**} x_n^{**}(x_n^*)$. Thus one has $K(X)^{**} = L(X^{**})$ in this setting, where the canonical embedding $K(X) \subset K(X)^{**}$ coincides with the natural isometry $S \mapsto S^{**}$ from $K(X)$ into $L(X^{**})$. One easily checks on the rank-1 operators that the (pre)adjoints of $L_A R_B$ satisfy

$$(L_A R_B : K(X) \to K(X))^* = L_{A^*} R_{B^*} : N(X^*) \to N(X^*), \tag{2.6}$$

$$(L_{A^*} R_{B^*} : N(X^*) \to N(X^*))^* = L_{A^{**}} R_{B^{**}} : L(X^{**}) \to L(X^{**}). \tag{2.7}$$

Hence $(L_A R_{B|K(X)})^{**} = L_{A^{**}} R_{B^{**}}$, where $L_A R_{B|K(X)}$ is the restricted operator $K(X) \to K(X)$, so that (2.3) reduces to a well-known general criterion (see [Wo91, Thm. 2.C.6(c)]) for the weak compactness of bounded operators.

The following examples from [ST92] demonstrate some typical applications of (2.3). For $p = 2$ this is the result of Akemann and Wright [AW80] cited above.

Example 2.6. *Let $1 < p < \infty$ and $A, B \in L(\ell^p)$. Then $L_A R_B$ is weakly compact on $L(\ell^p)$ if and only if $A \in K(\ell^p)$ or $B \in K(\ell^p)$.*

Proof. The implication "$\Leftarrow$" is included in Proposition 2.3.(ii). For the converse implication we first look at the simplest case where $p = 2$. Note that to apply (2.4) we must exhibit, for any given pair $A, B \notin K(\ell^2)$, an operator $S \in L(\ell^2)$ so that $ASB \notin K(\ell^2)$. This is easy to achieve. Indeed, by assumption there are closed infinite-dimensional subspaces $M_1, M_2 \subset \ell^2$, so that the restrictions $A_{|M_2}$ and $B_{|M_1}$ are bounded below. Pick constants $c_1, c_2 > 0$ such that

$$\|Bx\| \geq c_1\|x\| \text{ for } x \in M_1, \quad \|Ax\| \geq c_2\|x\| \text{ for } x \in M_2.$$

Define the bounded operator S on $\ell^2 = B(M_1) \oplus (BM_1)^\perp$ by requiring that S is an isometry from $B(M_1)$ onto M_2 and $S = 0$ on $(BM_1)^\perp$. Thus $\|ASBy\| \geq c_1\|S(By)\| \geq c_1c_2\|y\|$ for $x \in M_1$, whence $ASB \notin K(\ell^2)$.

To argue as above for $p \neq 2$ one needs for any $U \notin K(\ell^p)$ to find a subspace $M \subset \ell^p$ so that M is isomorphic to ℓ^p, M and $U(M)$ are complemented in ℓ^p, and $U_{|M}$ is bounded below. This basic sequence argument is familiar to Banach space theorists, and we refer e.g. to [LT,Prop. 2.a.1 and 1.a.9] or [Pi80, 5.1.3] for the details. □

A more serious refinement of these ideas yields the exact distance formula

$$dist(L_AR_B, W(L(\ell^p))) = dist(A, K(\ell^p)) \cdot dist(B, K(\ell^p))$$

for $A, B \in L(\ell^p)$ and $1 < p < \infty$, see [ST94, Thm. 2.(ii)].

Condition (2.4) gives, after some additional efforts, the following identification of the weakly compact maps L_AR_B for the direct sum $X = \ell^p \oplus \ell^q$, see [ST92, Prop. 3.5]. Here we represent operators S on $\ell^p \oplus \ell^q$ as an operator matrix

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

where $S_{11} \in L(\ell^p), S_{12} \in L(\ell^q, \ell^p)$, $S_{21} \in L(\ell^p, \ell^q)$ and $S_{22} \in L(\ell^q)$.

Example 2.7. *Let $1 < p < q < \infty$ and $A, B \in L(\ell^p \oplus \ell^q)$. Then L_AR_B is weakly compact on $L(\ell^p \oplus \ell^q)$ if and only if $A \in K(\ell^p \oplus \ell^q)$, or $B \in K(\ell^p \oplus \ell^q)$, or*

$$A \in \begin{pmatrix} K(\ell^p) & L(\ell^q, \ell^p) \\ L(\ell^p, \ell^q) & L(\ell^q) \end{pmatrix} \quad \text{and} \quad B \in \begin{pmatrix} L(\ell^p) & L(\ell^q, \ell^p) \\ L(\ell^p, \ell^q) & K(\ell^q) \end{pmatrix}. \tag{2.8}$$

Examples 2.6 and 2.7 provide ample motivation to consider the more delicate case $X = L^p(0,1)$ for $1 < p < \infty$ (recall that ℓ^p and $\ell^p \oplus \ell^2$ embed as complemented subspaces of $L^p(0,1)$). According to Proposition 2.3.(ii) the operator L_AR_B is weakly compact on $L(L^p(0,1))$ if $A \in K(L^p(0,1))$ or $B \in K(L^p(0,1))$, but these conditions are far from being necessary. For this recall that $U \in L(X,Y)$ is a *strictly singular* operator, denoted $U \in S(X,Y)$, if the restriction $U_{|M}$ does not define an isomorphism $M \to U(M)$ for any closed infinite-dimensional subspaces $M \subset X$. It is known that $UV \in K(L^p(0,1))$ whenever $U, V \in S(L^p(0,1))$, see [Mi70, Teor. 7]. Hence condition (2.4) immediately yields the following fact:

- *If $A, B \in S(L^p(0,1))$, then L_AR_B is weakly compact on $L(L^p(0,1))$.*

The preceding cases do not yet exhaust all the possibilities. In fact, note that (2.8) allows weakly compact multiplications L_AR_B arising from non-strictly singular operators A, B on $\ell^p \oplus \ell^2$, and this example easily transfers to $L^p(0,1)$ by complementation. The following question remains unresolved, and it is also conceivable that there is no satisfactory answer.

Problem 2.8. *Let $1 < p < \infty$ and $p \neq 2$. Characterize those $A, B \in L(L^p(0,1))$ for which L_AR_B is weakly compact on $L(L^p(0,1))$.*

For our second class of examples recall that X has the *Dunford-Pettis property* (DPP) if $\|Ux_n\| \to 0$ as $n \to \infty$ for any $U \in W(X,Y)$, any weak-null sequence (x_n) of X and any Banach space Y. For instance, ℓ^1, $L^1(0,1)$, c_0, $C(0,1)$ and $L^\infty(0,1)$ have the DPP (in fact, more generally any $\mathcal{L}^1$- and $\mathcal{L}^\infty$- space X has the DPP). We refer to the survey [Di80] for further information about this property. Note that if X has the DPP, then $UV \in K(Y,Z)$ for all weakly compact operators $U \in W(X,Z)$ and $V \in W(Y,X)$. This fact and Proposition 2.5 suggest that there might be an analogue of Vala's theorem for weakly compact multipliers L_AR_B on $L(X)$ if X has the DPP. It is an elegant result of Racher [Ra92] that this is indeed so (a more restricted version was contained in [ST92]). This fact provides plenty of non-trivial examples of weakly compact multiplications. The formulation included here of Racher's result is not the most comprehensive one.

Theorem 2.9. *Let X be a Banach space having the DPP, and suppose that $A, B \in L(X)$ are non-zero operators. Then L_AR_B is weakly compact $L(X) \to L(X)$ if and only if $A \in W(X)$ and $B \in W(X)$.*

The proof of Racher's theorem is based on the following useful auxiliary fact.

Lemma 2.10. *Let X_1, X_2, X_3, X_4 be Banach spaces and suppose that $A = A_1 \circ A_2 \in W(X_3, X_4)$, $B = B_1 \circ B_2 \in W(X_1, X_2)$ factor through the reflexive spaces Z_1, respectively Z_2, so that*

$$L_{A_2^{**}} R_{B_1}(L(X_2, X_3^{**})) \subset K(Z_2, Z_1). \tag{2.9}$$

Then L_AR_B is weakly compact $L(X_2, X_3) \to L(X_1, X_4)$.

Proof. Let $(T_\gamma) \subset B_{L(X_2,X_3)}$ be an arbitrary net. It follows from Tychonoff's theorem and the w^*-compactness of $B_{X_3^{**}}$ that there is a subnet, still denoted by (T_γ), and an operator $S \in L(X_2, X_3^{**})$ so that

$$\langle y^*, K_3T_\gamma x\rangle \to \langle y^*, Sx\rangle \quad \text{for all } x \in X_2, y^* \in X_3^*.$$

Here K_3 denotes the natural map $X_3 \to X_3^{**}$. Thus

$$\langle z^*, A_2^{**}K_3T_\gamma B_1(z)\rangle \to \langle z^*, A_2^{**}SB_1(z)\rangle \quad \text{for all } z \in Z_2, z^* \in Z_1^*,$$

where $A_2^{**}SB_1 \in K(Z_2, Z_1)$ and $A_2^{**}K_3T_\gamma B_1 \in K(Z_2, Z_1)$ for all γ by the assumption. A fundamental criterion for weak compactness in spaces of compact operators, due to Feder and Saphar [FeS75, Cor. 1.2], yields then that the net

$$A_2T_\gamma B_1 = A_2^{**}K_3T_\gamma B_1 \xrightarrow[\gamma]{w} A_2^{**}SB_1$$

weakly in $K(Z_2, Z_1)$. Thus $L_{A_2}R_{B_1}$ is weakly compact $L(X_2, X_3) \to K(Z_2, Z_1)$, and so is $L_AR_B = L_{A_1}R_{B_2} \circ L_{A_2}R_{B_1}$. □

The proof of Theorem 2.9 is immediate from Lemma 2.10. Recall first that the weakly compact operators $A, B \in W(X)$ factor as $A = A_1 \circ A_2$ and $B = B_1 \circ B_2$ through suitable reflexive spaces Z_1 and Z_2 by the well-known DFJP-construction, see [Wo91, Thm. II.C.5]. If X has the DPP, then $A_2SB_1 \in K(Z_2, Z_1)$ for any $S \in L(X)$, so that (2.9) is satisfied.

It is also possible to prove analogues of Theorem 2.2 for the weak compactness of elementary operators $\mathcal{E}_{A,B}$. The following version of Theorem 2.9 is taken from [S95, Section 2]:

• *Suppose that X has the DPP. Then the elementary operator $\mathcal{E}_{A,B}$ is weakly compact on $L(X)$ if and only if there are m-tuples $U = (U_1, \ldots, U_m), V = (V_1, \ldots, V_m) \in W(X)^m$ so that*

$$\mathcal{E}_{A,B} = \mathcal{E}_{U,V}.$$

It is possible to study multiplications and elementary operators in the general setting of Banach operator ideals in the sense of Pietsch [Pi80]. This extension is motivated by several reasons (one reason is the duality with the nuclear operators used in the proof of Proposition 2.5). Recall that $(I, \|\cdot\|_I)$ is a Banach operator ideal if $I(X,Y) \subset L(X,Y)$ is a linear subspace for any pair X, Y of Banach spaces, $\|\cdot\|_I$ is a complete norm on $I(X,Y)$ and

(i) $x^* \otimes y \in I(X,Y)$ and $\|x^* \otimes y\|_I = \|x^*\| \cdot \|y\|$ for $x^* \in X^*$ and $y \in Y$,
(ii) $ASB \in I(X_1, X_2)$ and $\|ASB\|_I \leq \|A\| \cdot \|B\| \cdot \|S\|_I$ whenever $S \in I(X,Y)$, $A \in L(Y, X_2)$ and $B \in L(X_1, X)$ are bounded operators.

There is a large variety of useful and interesting examples of Banach operator ideals. For instance, K, W, the nuclear operators $(N, \|\cdot\|_I)$ and the class Π_p of the p-summing operators are important examples (we refer to [Pi80] or [DJT95] for further examples). Conditions (i) and (ii) imply that the basic map $L_A R_B$ is bounded $I(X_2, X_3) \to I(X_1, X_4)$, and that in fact

$$\|L_A R_B : I(X_2, X_3) \to I(X_1, X_4)\| = \|A\| \cdot \|B\|. \tag{2.10}$$

for any bounded operators $A \in L(X_3, X_4), B \in L(X_1, X_2)$.

The study of multiplication operators in the framework of Banach operator ideals was initiated by Lindström and Schlüchtermann in [LSch99]. Here one obviously meets the following general problem:

• *Let $(I, \|\cdot\|_I)$ and $(J, \|\cdot\|_J)$ be Banach operator ideals. For which operators A and B does the map $L_A R_B : I(X_2, X_3) \to I(X_1, X_4)$ belong to J?*

Note that Proposition 2.3 admits a more general version. Recall for this that $(I, \|\cdot\|_I)$ is a *closed* Banach operator ideal if $I(X,Y)$ is closed in $(L(X,Y), \|\cdot\|)$ for any pair X, Y. The ideal $(I, \|\cdot\|_I)$ is *injective* if $JS \in I(X,Z)$ for any isometry $J : Y \to Z$ yield that $S \in I(X,Y)$ and $\|JS\|_I = \|S\|_I$. Moreover, $(I, \|\cdot\|_I)$ is *surjective* if $SQ \in I(Z,Y)$ for any metric surjection $Q : Z \to X$ (that is, $\overline{QB_Z} = B_X$) imply that $S \in I(X,Y)$ and $\|SQ\|_I = \|S\|_I$. For instance, K and W are injective and surjective ideals.

Let I and J be Banach operator ideals, and let $A \in L(X_3, X_4)$, $B \in L(X_1, X_2)$, where $X_1, \ldots, X_4$ are any Banach spaces. The following general facts hold, see [LSch99, Section 2].

• *If $A \neq 0 \neq B$ and the map $L_A R_B : I(X_2, X_3) \to I(X_1, X_4)$ belongs to J, then $A \in J(X_3, X_4)$ and $B^* \in J(X_2^*, X_1^*)$.*
• *Assume that I is injective, and that J is closed and injective. If $A \in K(X_3, X_4)$ and $B^* \in J(X_2^*, X_1^*)$, then the map $L_A R_B \colon I(X_2, X_3) \to I(X_1, X_4)$ belongs to J.*
• *Assume that I is surjective, and that J is closed and injective. If $A \in J(X_3, X_4)$ and $B \in K(X_1, X_2)$, then the map $L_A R_B \colon I(X_2, X_3) \to I(X_1, X_4)$ belongs to J.*

Lindström and Schlüchtermann [LSch99] obtained several range inclusion results for the multiplications $L_A R_B$. We state two of their main results. Recall that the operator $U \in L(X,Y)$ is *weakly conditionally compact* if for every bounded sequence $(x_n) \subset X$ there is weakly Cauchy subsequence (Ux_{n_k}). Clearly any weakly compact map is weakly conditionally compact. Note that part (ii) below provides a partial converse of Lemma 2.10. We refer to Section 4, or references such as [LT77] or [Wo91], for more background about unconditional bases.

Theorem 2.11. *Let $X_1, \ldots, X_4$ be Banach spaces and $A \in L(X_3, X_4)$, $B \in L(X_1, X_2)$.*
(i) Suppose that every $S \notin K(X_2, X_3)$ factors through a Banach space Z having an unconditional basis, and that X_3 does not contain any isomorphic copies of c_0. If the map $L_A R_B$ is weakly conditionally compact $L(X_2, X_3) \to L(X_1, X_4)$, then

$$L_A R_B(L(X_2, X_3)) \subset K(X_1, X_4).$$

(ii) Suppose that every $S \notin K(X_2, X_3)$ factors through a Banach space Z having an unconditional basis, and that $L_A R_B$ is weakly compact $L(X_2, X_3) \to L(X_1, X_4)$. Then

$$A^{**} \circ \overline{B_{L(X_2,X_3)}}^{w^*OT} \circ B \subset K(X_1, X_4),$$

where w^*OT denotes the w^*-operator topology in $L(X_2, X_3^{**})$.

The examples included in this section demonstrate that the conditions for $L_A R_B$ to belong to a given operator ideal I usually depend on geometric or structural properties of the Banach spaces involved. However, for suitable classical Banach spaces it is still possible to obtain complete descriptions. We next discuss some non-trivial results from [LST05] related to strict singularity. This class of operators is central for many purposes (such as in perturbation theory and the classification of Banach spaces). The main result of [LST05] completely characterizes the strictly singular multiplications $L_A R_B$ on $L(L^p(0,1))$ for $1 < p < \infty$. The simple form of the characterization is rather unexpected, since the subspace structure of the algebras $L(X)$ is very complicated. The case $p = 2$ is essentially contained in Theorem 2.1, and it is excluded below.

Theorem 2.12. *Let $1 < p < \infty, p \neq 2$, and suppose that $A, B \in L(L^p(0,1))$ are non-zero operators. Then $L_A R_B$ is strictly singular $L(L^p(0,1)) \to L(L^p(0,1))$ if and only if $A \in S(L^p(0,1))$ and $B \in S(L^p(0,1))$.*

In contrast to the simplicity of the statement above the proof of Theorem 2.12 is lengthy and quite delicate, and we are only able to indicate some of the main steps and difficulties here. The implication "$\Leftarrow$" is the non-trivial one (the converse implication follows from the generalities). As the starting point one notes that it suffices to treat the case $2 < p < \infty$, since the map $U \mapsto U^*$ preserves strict singularity on $L^p(0,1)$, see [We77]. Assume to the contrary that there are operators $A, B \in S(L^p(0,1))$ so that $L_A R_B$ is not strictly singular $L(L^p(0,1)) \to L(L^p(0,1))$. Hence there is an infinite-dimensional subspace $N \subset L(L^p(0,1))$ so that $L_A R_B$ is bounded below on N. The first step consists of "modifying" N to obtain a block diagonal sequence $(S_k) \subset \mathcal{F}(L^p(0,1))$, for which the restriction of $L_A R_B$ to $[S_k : k \in \mathbb{N}]$ is still bounded below and the image sequence $(AS_k B)$ is as close as we want to a block diagonal sequence $(U_k) \subset \mathcal{F}(L^p(0,1))$. By a block diagonal sequence (S_k) is here meant that

$$S_k = (P_{m_{k+1}} - P_{m_k}) S_k (P_{m_{k+1}} - P_{m_k}) \quad \text{for } k \in \mathbb{N},$$

where $(m_k) \subset \mathbb{N}$ is some increasing sequence and (P_r) is the sequence of basis projections associated to the Haar basis (h_n) of $L^p(0,1)$. Note that $N \subset L(L^p(0,1))$ so that this reduction cannot be achieved just by a straightforward approximation. In fact, the actual argument proceeds through several auxiliary results.

In the next step one invokes classical estimates on unconditional basic sequences in $L^p(0,1)$ to ensure that

$$\|\sum_k c_k S_k\| \approx \|(c_k)\|_s \quad \text{for } (c_k) \in \ell^s, \tag{2.11}$$

where s satisfies $\frac{1}{2} = \frac{1}{p} + \frac{1}{s}$. The final challenge is to derive a contradiction from (2.11) by a subtle comparison with the Kadec-Pelczynski dichotomy. (This fundamental result [KP62] says that for any normalized basic sequence (f_n) in $L^p(0,1)$, where $2 < p < \infty$, there is a subsequence (f_{n_k}) so that $[f_{n_k} : k \in \mathbb{N}]$ is complemented in $L^p(0,1)$ and (f_{n_k}) is either equivalent to the unit vector basis of ℓ^p or ℓ^2).

Problem 2.13. *Find a simpler approach to Theorem 2.12.*

The delicacy above is further illustrated by the facts that Theorem 2.12 remains true for $X = \ell^p \oplus \ell^q$, see [LST05, Thm. 4.1], but not for $X = \ell^p \oplus \ell^q \oplus \ell^r$, where $1 < p < q < r < \infty$.

Example 2.14. *Let* $1 < p < q < r < \infty$ *and define* $J_1, J_2 \in S(\ell^p \oplus \ell^q \oplus \ell^r)$ *by*

$$J_1(x,y,z) = (0,0,j_1 y), \quad J_2(x,y,z) = (0, j_2 x, 0), \quad (x,y,z) \in \ell^p \oplus \ell^q \oplus \ell^r,$$

where $j_1 : \ell^q \to \ell^r$ *and* $j_2 : \ell^p \to \ell^q$ *are the natural inclusion maps. Then* $L_{J_1} R_{J_2}$ *is not strictly singular* $L(\ell^p \oplus \ell^q \oplus \ell^r) \to L(\ell^p \oplus \ell^q \oplus \ell^r)$.

Indeed, by passing to complemented subspaces it is enough to check that the related composition map $L_{j_1} R_{j_2}$ is not strictly singular $L(\ell^q) \to L(\ell^p, \ell^r)$. This fact follows from the straightforward computation that

$$\| \sum_n a_n j_2^* e_n^* \otimes j_1 e_n \|_{\ell^p \to \ell^r} = \| \sum_n a_n e_n^* \otimes e_n \|_{\ell^q \to \ell^q} = \sup_n |a_n|$$

for $(a_n) \in c_0$, where $(e_n) \subset \ell^q$ is the unit vector basis and $(e_n^*) \subset \ell^{q'}$ is the biorthogonal sequence.

The reference [LST05] also characterizes the strictly singular multiplications $L_A R_B$ on $L(X)$ when X is a $\mathcal{L}^1$-space. This result is based on Theorem 2.9 and the non-trivial fact, essentially due to Bourgain [B81], that here $L(X^{**})$ has the DPP.

Qualitative results for the multiplication operators $L_A R_B$ are often helpful when studying other aspects of multiplication or elementary operators. We also mention an interesting application [BDL01, Section 5], where maps of the form $L_A R_B$ are used to linearize the analytic composition operators $C_\phi : f \mapsto f \circ \phi$ on certain vector-valued spaces of analytic functions. Here ϕ is an analytic self-map of the unit disc $D = \{z \in \mathbf{C} : |z| < 1\}$.

Other developments. There is a quite extensive theory of tensor norms of Banach spaces and tensor products of operators, which parallels the study of the multiplication operators $L_A R_B$. Recall that the norm α, defined on the algebraic tensor products $X \otimes Y$ for all Banach spaces X, Y, is called a *tensor norm* if

(iii) $\alpha(x \otimes y) = \|x\| \cdot \|y\|$ for $x \otimes y \in X \otimes Y$,

(iv) $\|A \otimes B : (X_1 \otimes Y_1, \alpha) \to (X \otimes Y, \alpha)\| \le \|A\| \cdot \|B\|$ for any bounded operators $A \in L(X_1, X)$, $B \in L(Y_1, Y)$ and Banach spaces X_1, X, Y_1, Y.

The α-tensor product $X \widehat{\otimes}_\alpha Y$ is the completion of $(X \otimes Y, \alpha)$. Property (iv) states that any $A \in L(X_1, X)$ and $B \in L(Y_1, Y)$ induce a bounded linear operator $A \widehat{\otimes}_\alpha B : X_1 \widehat{\otimes}_\alpha Y_1 \to X \widehat{\otimes}_\alpha Y$. We refer to [DeF93] for a comprehensive account of tensor norms and tensor products of operators.

Tensor norms and Banach operator ideals are related to each other, but this correspondence is not complete. For instance, recall that $X^* \widehat{\otimes}_\epsilon Y = K(X, Y)$ if X^* or Y has the approximation property (see e.g. [DeF93, 5.3]), while $(X \widehat{\otimes}_\pi Y)^* = L(X^*, Y)$ for any pair X, Y. Given $A \in L(X)$ and $B \in L(Y)$ one may then identify, under appropriate conditions, the tensor product operator $A^* \widehat{\otimes}_\epsilon B$ with the map $L_{A^*} R_B$ and $(A \widehat{\otimes}_\pi B)^*$ with $L_A R_{B^*}$. There are many results which are more natural to state either in terms of multiplication operators or tensor products of operators. An example of this for tensor products is the following celebrated result of J. Holub [Ho70], [Ho74] (see also [DeF93, 34.5]):

- $A \widehat{\otimes}_\epsilon B$ *is a* p*-summing operator whenever* A *and* B *are* p*-summing operators.*

References such as e.g. [DiF76], [Pi87], [CDR89], [DeF93, Chapter 34], [Ra92] and [LSch99] contain qualitative results for tensor products of operators which resemble some of the results of this section for the multiplication operators. Since the elementary operators are the main objects of this survey we have not pursued this aspect.

It is not known whether Vala's result (Theorem 2.1) holds for arbitrary Banach operator ideals (alternatively, for arbitrary tensor norms).

Problem 2.15. *Let $(I, \|\cdot\|_I)$ be an arbitrary Banach operator ideal. Is $L_A R_B$ a compact operator $I(X_2, X_3) \to I(X_1, X_4)$ whenever $A \in K(X_3, X_4)$ and $B \in K(X_1, X_2)$?*

The tensor version of this problem was discussed by Carl, Defant and Ramanujan [CDR89], where one finds a number of partial positive results.

3. Norms and spectra in various settings

This section discusses several results related to the computation of the operator norm and of various spectra of (classes of) elementary operators. It has turned out that computing the norm of reasonably general (classes of) elementary operators is a difficult problem. In fact, only very recently Timoney [Ti05] provided the first general formula for $\|\mathcal{E}_{A,B}\|$ on $L(\ell^2)$, see Theorem 3.10 below.

Recall as our starting point that

$$\|L_A R_B \colon L(X) \to L(X)\| = \|A\| \cdot \|B\|$$

by (2.10) for any Banach space X and any $A, B \in L(X)$. The first non-trivial results concern the norm of the *inner derivations* (or commutator maps)

$$L_A - R_A \colon L(X) \to L(X); \quad S \mapsto AS - SA,$$

determined by $A \in L(X)$. These concrete operators occur in many different contexts. Since $L_{A-\lambda} - R_{A-\lambda} = L_A - R_A$ for any scalar λ, we immediately get the general upper bound

$$\|L_A - R_A\| \leq 2 \cdot \inf_{\lambda \in \mathbb{K}} \|A - \lambda\|, \tag{3.1}$$

which holds for any X (where the scalar field $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$). J.G. Stampfli [St70] showed that the preceding estimate is exact for the norm of the inner derivations on $L(\ell^2)$.

Theorem 3.1. *Let H be a complex Hilbert space and $A \in L(H)$. Then*

$$\|L_A - R_A\| = 2 \cdot \inf\{\|A - \lambda\| : \lambda \in \mathbb{C}\} \tag{3.2}$$

Stampfli's elegant formula also holds in the case of real scalars. Stampfli [St70] extended it to the *generalized derivations* (or intertwining operators) $L_A - R_B$ on $L(\ell^2)$, where $S \mapsto AS - SB$.

Theorem 3.2. *Let H is a complex Hilbert space and $A, B \in L(H)$. Then*

$$\|L_A - R_B\| = \inf\{\|A - \lambda\| + \|B - \lambda\| : \lambda \in \mathbb{C}\} \tag{3.3}$$

Later Fialkow [Fi79, Example 4.14] observed that the operator $A \in L(\ell^2)$, defined by $Ae_{2n} = e_{2n-1}$ and $Ae_{2n-1} = 0$ for $n \in \mathbb{N}$, satisfies

$$\|L_A - R_A \colon C_2 \to C_2\| < 2 \cdot \inf_{\lambda \in \mathbb{C}} \|A - \lambda\|.$$

Here (e_n) is the unit coordinate basis of ℓ^2 and $(C_2, \|\cdot\|_{HS})$ is the Banach ideal of $L(\ell^2)$ consisting of the Hilbert-Schmidt operators. Hence (3.2) fails for arbitrary restrictions $L_A - R_A \colon J \to J$, where J is a Banach ideal of $L(\ell^2)$.

Let H be a complex Hilbert space. Fialkow [Fi79] called the operator $A \in L(H)$ *S-universal* if $\|L_A - R_A \colon J \to J\| = 2 \cdot \inf\{\|A - \lambda\| : \lambda \in \mathbb{C}\}$ for all Banach ideals $J \subset L(H)$. Barraa and Boumazgour [BB01] obtained, in combination with earlier results of Fialkow, the following neat characterization of S-universality. Let $W(A) = \{(Ax, x) : x \in S_H\}$ be the numerical range of $A \in L(H)$.

Theorem 3.3. *Let H be a complex Hilbert space and $A \in L(H)$. Then the following conditions are equivalent.*
(i) A is S-universal,
(ii) $\|L_A - R_A \colon C_2 \to C_2\| = 2 \cdot \inf_{\lambda \in \mathbb{C}} \|A - \lambda\|$,
(iii) $diam(W(A)) = 2 \cdot \inf_{\lambda \in \mathbb{C}} \|A - \lambda\|$,
(iv) $diam(\sigma(A)) = 2 \cdot \inf_{\lambda \in \mathbb{C}} \|A - \lambda\|$. (Here $\sigma(A)$ is the spectrum of A.)

There are several proofs of Stampfli's formula (3.2), see e.g. [AM03, Section 4.1]. We briefly discuss one of the approaches from [St70] of this fundamental result in order to convey an impression of the tools involved here. The *maximal numerical range* of $A \in L(H)$ is

$$W_0(A) = \{\lambda \in \mathbb{C}\colon \lambda = \lim_n (Ax_n, x_n), \text{where } (x_n) \subset S_H \text{ and } \|Ax_n\| \to \|A\|\}.$$

Here $(\cdot, \cdot)$ is the inner product on H. The set $W_0(A)$ is known to be non-empty, closed and convex.

Sketch of the proof of Theorem 3.1. We first claim that if $\mu \in W_0(A)$ then

$$\|L_A - R_A\| \geq 2(\|A\|^2 - |\mu|^2)^{1/2}. \tag{3.4}$$

Indeed, by assumption there is a sequence $(x_n) \subset S_H$ so that $\mu = \lim_n (Ax_n, x_n)$ and $\|A\| = \lim_n \|Ax_n\|$. Write $Ax_n = \alpha_n x_n + \beta_n y_n$ for $n \in \mathbb{N}$, where $y_n \in \{x_n\}^\perp$ and $\|y_n\| = 1$. Note that $\alpha_n = (Ax_n, x_n) \to \mu$ and $|\alpha_n|^2 + |\beta_n|^2 = \|Ax_n\|^2 \to \|A\|^2$ as $n \to \infty$. Define the rank-2 operators $V_n \in L(H)$ by

$$V_n = (x_n \otimes x_n - y_n \otimes y_n) \circ P_n,$$

where P_n is the orthogonal projection onto $[x_n, y_n]$. Here $(u \otimes v)x = (x, u)v$ for $u, v, x \in H$. Thus $\|V_n\| = 1$ for $n \in \mathbb{N}$. We obtain that

$$\begin{aligned} \lim_n \|AV_n x_n - V_n A x_n\| &= \lim_n \|\alpha_n x_n + \beta_n y_n - (\alpha_n x_n - \beta_n y_n)\| \\ &= \lim_n 2|\beta_n| = 2(\|A\|^2 - |\mu|^2)^{1/2} \end{aligned}$$

Since $\|L_A - R_A\| \geq \limsup_n \|AV_n x_n - V_n A x_n\|$ it follows that (3.4) holds.

Observe next that if $0 \in W_0(A - \lambda_0)$ for some scalar $\lambda_0 \in \mathbb{C}$, then (3.4) yields the lower estimate

$$\|L_A - R_A\| = \|L_{A-\lambda_0} - R_{A-\lambda_0}\| \geq 2\|A - \lambda_0\| \geq 2 \inf_{\lambda \in \mathbb{C}} \|A - \lambda\|.$$

Hence it follows from (3.1) that (3.2) holds.

The non-trivial part of the argument is to find $\lambda_0 \in \mathbb{C}$ so that $0 \in W_0(A - \lambda_0)$. This part is quite well-documented in the literature so we just refer to Stampfli [St70](who included two different approaches), [Fi92, Section 2] or [AM03, Thm. 4.1.17]). □

Stampfli asked whether (3.2) also holds for the inner derivations on $L(X)$, where X is an arbitrary Banach space. This was disproved by the following example of Johnson [J71].

Example 3.4. Let $1 < p < \infty$ and $p \neq 2$. Then there is a rank-1 operator $A \in L(\ell^p)$ for which

$$\|L_A - R_A\| < 2 \cdot \inf\{\|A - \lambda\| : \lambda \in \mathbb{C}\}.$$

Johnson [J71] also provided examples of spaces X where (3.2) does hold.

Example 3.5. Let $\ell^1_n(\mathbb{R}) = (\mathbb{R}^n, \|\cdot\|_1)$. Then

$$\|L_A - R_A\| = 2 \cdot \inf\{\|A - \lambda\| : \lambda \in \mathbb{R}\}$$

for any $A \in L(\ell^1_n(\mathbb{R}))$.

Above $\ell^1_n(\mathbb{R})$ is not uniformly convex. Subsequently Kyle [Ky77] obtained an elegant connection between Stampfli's formula and isometric characterizations of Hilbert spaces within the class of uniformly convex spaces.

Theorem 3.6. *Let X be a uniformly convex Banach space over the scalars $\mathbb{K}$. Then the following conditions are equivalent.*
(i) X is isometric to a Hilbert space,
(ii) $\|L_A - R_A\| = 2 \cdot \inf_{\lambda\in\mathbb{K}} \|A - \lambda\|$ holds for any $A \in L(X)$,
(iii) $\|L_A - R_A\| = 2 \cdot \inf_{\lambda\in\mathbb{K}} \|A - \lambda\|$ holds for any rank-1 operator $A \in L(X)$.

There has been much recent work concerning the computation of norms (of classes) of elementary operators. An optimal outcome would be a formula for

$$\|\sum_{j=1}^{n} L_{A_j} R_{B_j} : L(X) \to L(X)\|$$

which in some sense involves the coefficients $A = (A_1, \ldots, A_n), B = (B_1, \ldots, B_n) \in L(X)^n$ of a given elementary operator $\mathcal{E}_{A,B}$ so that their contribution to the norm is clarified in some non-trivial sense (at least in the case where X is a Hilbert space). One obvious obstruction is the non-uniqueness of the representation of such operators. Runde [Run00] observed that $\|\mathcal{E}_{A,B}\|$ is not symmetric in the sense that the norms do not remain uniformly bounded in the flip correspondence $\mathcal{E}_{A,B} \to \mathcal{E}_{B,A}$ (note that this is not well-defined as a map). Timoney [Ti01] gave the following simplified version of Runde's instructive example.

Example 3.7. Suppose that X is a Banach space with a normalized basis (e_n) and biorthogonal sequence $(e_n^*) \subset X^*$. Put $A_j = e_j^* \otimes e_1$ and $B_j = e_1^* \otimes e_j$ for $j \in \mathbb{N}$. Then

$$\|\sum_{j=1}^{n} L_{A_j} R_{B_j}\| \geq n, \quad \|\sum_{j=1}^{n} L_{B_j} R_{A_j}\| \leq C, \quad n \in \mathbb{N},$$

where C is the basis constant of (e_n).

Proof. For $S \in L(X)$ one gets that

$$\sum_{j=1}^{n} L_{A_j} R_{B_j}(S) = \sum_{j=1}^{n} (e_j^* \otimes e_1) \circ S \circ (e_1^* \otimes e_j) = (\sum_{j=1}^{n} \langle e_j^*, Se_j\rangle) e_1^* \otimes e_1.$$

Thus $\|\sum_{j=1}^{n} L_{A_j} R_{B_j}\| \geq n\|e_1^* \otimes e_1\| = n$ by choosing $S = I_X$. Moreover, for $S \in L(X)$ and $x \in X$ one obtains that

$$\begin{aligned}
\|\sum_{j=1}^{n} L_{B_j} R_{A_j}(S)x\| &= \|\sum_{j=1}^{n} (e_1^* \otimes e_j) \circ S \circ (e_j^* \otimes e_1)x = \|\langle e_1^*, Se_1\rangle \sum_{j=1}^{n} e_j^*(x) e_j\| \\
&\leq \|S\| \cdot \|P_n\| \cdot \|\sum_{j=1}^{\infty} e_j^*(x) e_j\| \leq C \cdot \|S\| \cdot \|x\|.
\end{aligned}$$

Above $x = \sum_{j=1}^{\infty} e_j^*(x) e_j$ for $x \in X$, P_n denotes the natural basis projection $X \to [e_1, \ldots, e_n]$ and $C = \sup_m \|P_m\|$ is the basis constant. □

By exercising somewhat more care in the argument (see [Ti01, Thm. 1]) it is enough above to assume just that the basic sequence (e_n) spans a complemented subspace of X.

Let X be a Banach space and $A, B \in L(X)$. Clearly the symmetrized elementary operator $L_A R_B + L_B R_A$, for which $S \mapsto ASB + BSA$ for $S \in L(X)$, satisfies

$$\|L_A R_B + L_B R_A : L(X) \to L(X)\| \leq 2\|A\| \cdot \|B\|.$$

For these maps it is natural to look for lower bounds having the form

$$\|L_A R_B + L_B R_A : L(X) \to L(X)\| \geq c_X \|A\| \cdot \|B\|, \tag{3.5}$$

for some constant $c_X > 0$, possibly depending on X. In particular, Mathieu [Ma89] conjectured that $c_H = 1$. Recently Mathieu's conjecture was independently solved, using different methods, by Timoney [Ti03a] and by Blanco, Boumazgour and Ransford [BBR04].

Theorem 3.8. *Let H be a Hilbert space. Then*

$$\|L_A R_B + L_B R_A : L(H) \to L(H)\| \geq \|A\| \cdot \|B\|$$

for any $A, B \in L(H)$.

Earlier Stacho and Zalar [SZ96] showed that Mathieu's conjecture holds for self-adjoint $A, B \in L(H)$, and that

$$\|L_A R_B + L_B R_A : L(H) \to L(H)\| \geq 2(\sqrt{2} - 1)\|A\| \cdot \|B\|$$

for any $A, B \in L(H)$. The bound $2(\sqrt{2} - 1)$ occurs naturally in the following result due Blanco, Boumazgour and Ransford [BBR04, Thm. 5.1 and Prop. 5.3].

Theorem 3.9. *In* (3.5) *one has that $c_X \geq 2(\sqrt{2} - 1)$ for any Banach space X. The bound $2(\sqrt{2} - 1)$ cannot be improved e.g. on $X = (\mathbb{R}^2, \|\cdot\|_\infty)$, ℓ^∞ or ℓ^1.*

The general estimate in Theorem 3.9 also holds for the norm of the restrictions $L_A R_B + L_B R_A : I(X) \to I(X)$, where I is a Banach operator ideal.

Very recently Timoney [Ti05], building on his earlier work [Ti03b], obtained a couple of general formulas for the norm $\|\mathcal{E}_{A,B}\|$ in the Hilbert space case. His work provides a solution of the norm problem which involves matrix numerical ranges and a notion of tracial geometric mean. We briefly describe his solution, though we are not able to include any details here. The *tracial geometric mean* of the positive (semi-definite) $n \times n$-matrices U, V is

$$tgm(U, V) = trace\sqrt{\sqrt{U}V\sqrt{U}} = \sum_{j=1}^{n} \sqrt{\lambda_j(UV)}.$$

Here $\sqrt{\cdot}$ denotes the positive square root, and $(\lambda_j(UV))$ are the eigenvalues of UV ordered in non-increasing order and counting multiplicities. For the n-tuple $A = (A_1, \ldots, A_n) \in L(H)^n$ and $x \in H$ one introduces the scalar $n \times n$-matrix

$$Q(A, x) = ((A_i^* A_j x, x))_{i,j=1}^n = ((A_j x, A_i x))_{i,j=1}^n.$$

The first version [Ti05, Thm. 1.4] of Timoney's formula reads as follows.

Theorem 3.10. *For any $A = (A_1, \ldots, A_n), B = (B_1, \ldots, B_n) \in L(H)^n$ one has*

$$\|\mathcal{E}_{A,B} \colon L(H) \to L(H)\| = \sup\{tgm(Q(A^*, x), Q(B, y)) \colon x, y \in S_H\},$$

where $A^* = (A_1^*, \ldots, A_n^*)$.

Next put

$$\|(x_1, \ldots, x_n)\|_{S1} = trace\sqrt{((x_i, x_j))_{i,j=1}^n}$$

for $(x_1, \ldots, x_n) \in H^n$. This defines a norm on H^n, see [Ti05, Lemma 1.7]. For $A = (A_1, \ldots, A_n) \in L(H)^n$ let $\|A\|_{S1}$ denote the norm of A considered as an operator $H \to (H^n, \|\cdot\|_{S1})$. One gets the following alternative formula [Ti05, Thm. 1.10] for the norm.

Theorem 3.11. *For any $A = (A_1, \ldots, A_n), B = (B_1, \ldots, B_n) \in L(H)^n$ one has*

$$\|\mathcal{E}_{A,B} \colon L(H) \to L(H)\| = \sup\{\|\sqrt{Q(B, y)^t} A^*\|_{S1} \colon y \in S_H\},$$

where $A^* = (A_1^*, \ldots, A_n^*)$ and $Q(B, y)^t$ is the transpose of the matrix $Q(B, y)$.

Timoney [Ti05] also established versions of Theorems 3.10 and 3.11 for the norm of elementary operators on arbitrary C*-algebras $\mathcal{A}$. Moreover, he characterized the compact elementary operators $\mathcal{A} \to \mathcal{A}$, thus providing a complete generalization of Theorem 2.2 to the setting of C*-algebras (see [AM03, 5.3.26] for earlier results of Mathieu in the case of prime algebras).

For further results about the norms of elementary operators on $L(H)$ or on classes of C*-algebras we refer to e.g. [AM03, Sections 4.1,4.2 and 5.4], [AST05], [Ti05] (and the references therein), as well as to Theorems 4.13 and 4.14 concerning elementary operators on the Calkin algebra.

Let X be a complex Banach space and $A, B \in L(X)$. It is an easy exercise to check that the spectrum $\sigma(L_A) = \sigma(A) = \sigma(R_A)$ for any operator A. Since $L_A R_B = R_B L_A$ it follows immediately from elementary Gelfand theory that the spectra of $L_A R_B$ and $L_A - R_B$ satisfy

$$\sigma(L_A R_B) \subset \sigma(A)\sigma(B), \quad \sigma(L_A - R_B) \subset \sigma(A) - \sigma(B),$$

where $\sigma(A) - \sigma(B) \equiv \{\alpha - \beta \colon \alpha \in \sigma(A), \beta \in \sigma(B)\}$. Lumer and Rosenblum [LR59] showed the exact formula

$$\sigma(\sum_{j=1}^{n} L_{f_j(A)} R_{g_j(B)}) = \{\sum_{j=1}^{n} f_j(\alpha) g_j(\beta) \colon \alpha \in \sigma(A), \beta \in \sigma(B)\}, \tag{3.6}$$

which holds whenever f_j is holomorphic in a neighborhood of $\sigma(A)$ and g_j is holomorphic in a neighborhood of $\sigma(B)$ for $j = 1, \dots, n$. (The result itself is attributed to Kleinecke in [LR59].) Hence simple choices in (3.6) of the holomorphic functions imply that in fact

$$\sigma(L_A R_B) = \sigma(A)\sigma(B), \quad \sigma(L_A - R_B) = \sigma(A) - \sigma(B).$$

For a long time it remained a considerable challenge to compute the spectrum $\sigma(\mathcal{E}_{A,B})$ of general elementary operators. A satisfactory formula was eventually obtained by Curto [Cu83] for Hilbert spaces X, and this result was later substantially improved by Curto and Fialkow [CuF87] (again for Hilbert spaces) and Eschmeier [E88] (for arbitrary Banach spaces). (Some of these facts were announced by Fainshtein [F84].) Here one expresses the spectrum $\sigma(\mathcal{E}_{A,B})$ and the essential spectrum $\sigma_e(\mathcal{E}_{A,B})$ in terms of the Taylor joint spectrum and the Taylor joint essential spectrum of the n-tuples $(A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$. We refer to the survey [Cu92] for further references to the numerous intermediary results (including those for tensor products of operators) that culminated in Theorems 3.12 and 3.13 below.

Let $A = (A_1, \dots, A_n) \in L(X)^n$ be an n-tuple such that the set $\{A_1, \dots, A_n\}$ commutes. The Taylor joint spectrum $\sigma_T(A)$ consists of $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ so that the Koszul complex corresponding to $A - \lambda$ is not exact. Actually we will not require the precise homological definition here, and we refer e.g. to [Ta70], [Cu88] or [Mu02] for the details. The set $\sigma_T(A) \subset \mathbb{C}^n$ is compact and non-empty. We will use below the convenient notation

$$U \circ V \equiv \{\sum_{j=1}^{n} \alpha_j \beta_j : (\alpha_1, \dots, \alpha_n) \in U, (\beta_1, \dots, \beta_n) \in V\}$$

for subsets $U, V \subset \mathbb{C}^n$. It will also be convenient to discuss simultaneously the spectrum of the restrictions $\mathcal{E}_{A,B} : I \to I$, where $I \subset L(X)$ is any Banach ideal, since the results does not depend on I and the arguments are similar.

Theorem 3.12. *Let X be a complex Banach space and $I \subset L(X)$ be any Banach ideal. Suppose that $A = (A_1, \dots, A_n), B = (B_1, \dots, B_n) \in L(X)^n$ are n-tuples such* *that* $\{A_1,\dots,A_n\}$ *and* $\{B_1,\dots,B_n\}$ *are commuting sets. Then*

$$\sigma(\mathcal{E}_{A,B}\colon I\to I)=\sigma_T(A)\circ\sigma_T(B). \tag{3.7}$$

The essential spectrum of $S\in L(X)$ is

$$\sigma_e(S)=\{\lambda\in\mathbb{C}:\lambda-S \text{ is not a Fredholm operator}\}.$$

Recall that $S\in L(X)$ is a Fredholm operator if there are operators $T\in L(X)$ and $K_1,K_2\in K(X)$ so that $ST=I_X+K_1$ and $TS=I_X+K_2$. Thus $\sigma_e(S)$ is the spectrum of the quotient element $S+K(X)$ in the corresponding Calkin algebra $L(X)/K(X)$. The references [E88] and [CuF87] also compute the essential spectrum $\sigma_e(\mathcal{E}_{A,B})$ in the preceding setting. Let $A=(A_1,\dots,A_n)\in L(X)^n$ be a commuting n-tuple. We recall that the Taylor joint essential spectrum $\sigma_{Te}(A)$ consists of the $\lambda\in\mathbb{C}^n$ for which the Koszul complex of $A-\lambda$ is not Fredholm (see again e.g. [E88] or [Cu92] for the precise definition).

Theorem 3.13. *Let X be a complex Banach space and let $I\subset L(X)$ be any Banach ideal. Suppose that $A=(A_1,\dots,A_n), B=(B_1,\dots,B_n)\in L(X)^n$ are n-tuples such that $\{A_1,\dots,A_n\}$ and $\{B_1,\dots,B_n\}$ are commuting sets. Then*

$$\sigma_e(\mathcal{E}_{A,B}\colon I\to I)=\sigma_{Te}(A)\circ\sigma_T(B)\cup\sigma_T(A)\circ\sigma_{Te}(B). \tag{3.8}$$

Theorems 3.12 and 3.13 remain valid for the spectrum and the essential spectrum of the analogous tensor product operators $\sum_{j=1}^n A_j\otimes B_j$ with respect to any tensor norm (this is the explicit point of view in [E88]). The arguments in [E88] and [CuF87] apply multivariable spectral theory and some homological algebra (the requisite background is discussed e.g. in the surveys [Cu88] and [Cu92]). A central idea is to determine the complete spectral picture and compute, or in the Banach space case to suitably estimate, the Taylor joint (essential) spectra of the commuting $2n$-tuples $(A\otimes I_X, I_X\otimes B)=(A_1\otimes I,\dots,A_n\otimes I, I\otimes B_1,\dots,I\otimes B_n)$ or $(L_A,R_B)=(L_{A_1},\dots,L_{A_n},R_{B_1},\dots,R_{B_n})$. Theorems 3.12 and 3.13 are then obtained by applying the polynomial spectral mapping property to $P:\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}$, where $P(z,w)=\sum_{j=1}^n z_jw_j$ for $z,w\in\mathbb{C}^n$. The references [E88] and [CuF87] contain plenty of additional information related to other classical subsets of the spectrum as well as index formulas. By contrast we will provide below minimalist approaches to (3.7) and (3.8), which are based on ideas from [S95].

Before proving (3.7) we mention that it is even possible to identify the *weak essential spectrum* $\sigma_w(\mathcal{E}_{A,B})$ in several situations. Here

$$\sigma_w(S)=\{\lambda\in\mathbb{C}:\lambda-S \text{ is not invertible modulo } W(Y)\}$$

for $S\in L(Y)$. Recall that $S\in L(Y)$ is invertible modulo $W(Y)$ if there are $T\in L(Y)$ and $V_1,V_2\in W(Y)$ so that $ST=I_Y+V_1$ and $TS=I_Y+V_2$. We refer e.g. to Section 2 and Corollary 4.2 for results about weakly compact elementary operators. The following results were obtained in [ST94].

Theorem 3.14. *(i) Let $1<p<\infty$ and $A,B\in L(\ell^p)$. Then*

$$\sigma_w(L_AR_B)=\sigma_e(A)\sigma_e(B).$$

(ii) Let X be an arbitrary complex Banach space and $A,B\in L(X)$. Then

$$\sigma_w(A^*)\sigma(B)\cup\sigma(A)\sigma_w(B)\subset\sigma_w(L_AR_B)\subset\sigma_e(A)\sigma(B)\cup\sigma(A)\sigma_e(B). \tag{3.9}$$

For instance, if X^* has the Dunford-Pettis property, then one obtains from (3.9) that

$$\sigma_w(L_AR_B)=\sigma_e(L_AR_B)=\sigma_e(A)\sigma(B)\cup\sigma(A)\sigma_e(B).$$

For this equality one has to recall a few well-known facts. Firstly, if Y has the DPP and $V\in W(Y)$, then I_Y+V is a Fredholm operator. In fact, by assumption

$V^2 \in K(Y)$ so that $I_Y - V^2 = (I_Y + V)(I_Y - V) = (I_Y - V)(I_Y + V)$ is a Fredholm operator, whence $I_Y + V$ is also Fredholm. This yields that $\sigma_w(S) = \sigma_e(S)$ for any $S \in L(Y)$. Secondly, X has the DPP if X^* has this property. Finally, $\sigma_e(S^*) = \sigma_e(S)$ for any operator S.

Subsequently Saksman [S95] extended Theorem 3.14 to the weak essential spectra of elementary operators in certain cases. We state a couple of results from [S95].

Theorem 3.15. *Let X be a complex Banach space and let $A = (A_1, \ldots, A_n), B = (B_1, \ldots, B_n) \in L(X)^n$ be commuting n-tuples.*
(i) If $X = \ell^p$ and $1 < p < \infty$, then

$$\sigma_w(\mathcal{E}_{A,B}) = \sigma_{Te}(A) \circ \sigma_{Te}(B).$$

(ii) If X^* has the DPP, then

$$\sigma_w(\mathcal{E}_{A,B}) = \sigma_{Te}(A) \circ \sigma_T(B) \cup \sigma_T(A) \circ \sigma_{Te}(B).$$

The ideas underlying Theorem 3.15, and the earlier Theorem 3.14 for the case $L_A R_B$, yield "elementary" approaches to to Theorems 3.12 and 3.13 that do not use any homological algebra (as was pointed out on [S95, p. 182]). In order to give the reader some impressions of the techniques involved in computing the spectra of elementary operators we present here a fairly detailed argument for Theorem 3.12 along these lines. We also point out the main additional ideas needed for a proof of Theorem 3.13 using elementary tools, see Remark 3.17 below.

We begin by recalling some classical concepts of joint spectra. Let X be a complex Banach space and let $A = (A_1, \ldots, A_n) \in L(X)^n$ be a n-tuple of commuting operators. The (joint) approximative point spectrum $\sigma_\pi(A)$ of $A = (A_1, \ldots, A_n)$ consists of the points $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ for which

$$\inf_{x \in S_X} \sum_{j=1}^{n} \|(A_j - \lambda_j)x\| = 0.$$

The joint approximative spectrum of $A = (A_1, \ldots, A_n)$ is then

$$\sigma_a(A) = \sigma_\pi(A) \cup \sigma_\pi(A^*),$$

where $A^* = (A_1^*, \ldots, A_n^*)$. The left spectrum $\sigma_l(A)$ of $A = (A_1, \ldots, A_n)$ consists of $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that $\sum_{j=1}^n S_j(A_j - \lambda_j) \neq I_X$ for all n-tuples $(S_1, \ldots, S_n) \in L(X)^n$. Similarly, $(\lambda_1, \ldots, \lambda_n) \in \sigma_r(A)$ if $\sum_{j=1}^n (A_j - \lambda_j)S_j \neq I_X$ for $(S_1, \ldots, S_n) \in L(X)^n$. The Harte joint spectrum of $A = (A_1, \ldots, A_n)$ is

$$\sigma_H(A) = \sigma_l(A) \cup \sigma_r(A).$$

For a single operator $S \in L(X)$ (that is, the case $n = 1$) one has $\sigma_T(S) = \sigma_H(S) = \sigma_a(S) = \sigma(S)$. According to the polynomial spectral mapping property for the Taylor spectrum one has

$$\sigma(P(A_1, \ldots, A_n)) = P(\sigma_T(A_1, \ldots, A_n)).$$

for any commuting n-tuple $A = (A_1, \ldots, A_n) \in L(X)^n$ and for any scalar polynomial $P\colon \mathbb{C}^n \to \mathbb{C}$. This property also holds for the joint spectra $\sigma_H(\cdot)$ and $\sigma_a(\cdot)$. We refer e.g to [Cu88] for a further discussion of multivariable spectral theory.

The following technical observation will be crucial. This fact goes back to Curto [Cu86, Thm. 3.15] (see also [S95, Prop. 1]).

Lemma 3.16. *Let $A = (A_1, \ldots, A_n), B = (B_1, \ldots, B_n) \in L(X)^n$ be commuting n-tuples, and $P\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be a polynomial. Then*

$$P(\sigma_T(A) \times \sigma_T(B)) = P(\sigma_H(A) \times \sigma_H(B)) = P(\sigma_a(A) \times \sigma_a(B)).$$

Proof. We verify the inclusion $P(\sigma_T(A) \times \sigma_T(B)) \subset P(\sigma_a(A) \times \sigma_a(B))$. The other inclusions are similar (actually, one could also apply the fact that $\sigma_a(A) \subset \sigma_H(A) \subset \sigma_T(A)$ for any commuting n-tuple A, see e.g. [Cu88]).

Let $P(\alpha_0, \beta_0) \in P(\sigma_T(A) \times \sigma_T(B))$, and consider the polynomial $Q_1 : \mathbb{C}^n \to \mathbb{C}$, where $Q_1(z) = P(z, \beta_0)$. It follows from the polynomial spectral mapping theorems, and the fact that these spectra coincide for a single operator, that

$$P(\alpha_0, \beta_0) = Q_1(\alpha_0) \in Q_1(\sigma_T(A)) = \sigma(Q_1(A)) = Q_1(\sigma_a(A)).$$

Hence there is $\alpha_1 \in \sigma_a(A)$ so that $P(\alpha_0, \beta_0) = Q_1(\alpha_1) = P(\alpha_1, \beta_0)$. By applying the same argument to $Q_2 : \mathbb{C}^n \to \mathbb{C}$, where $Q_2(w) = P(\alpha_1, w)$, we get $\beta_1 \in \sigma_a(B)$ so that

$$P(\alpha_0, \beta_0) = P(\alpha_1, \beta_0) = Q_2(\beta_0) = Q_2(\beta_1) = P(\alpha_1, \beta_1) \in P(\sigma_a(A) \times \sigma_a(B)).$$

□

Proof of Theorem 3.12. The strategy is to prove the inclusions

$$\sigma_H((L_A, R_B); L(I)) \subset \sigma_H(A) \times \sigma_H(B), \tag{3.10}$$

$$P(\sigma_a(A) \times \sigma_a(B)) \subset \sigma(P(L_A, R_B) : I \to I), \tag{3.11}$$

for any polynomial $P : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$. Above $\sigma_H((L_A, R_B); L(I))$ denotes the Harte spectrum of the commuting $2n$-tuple $(L_{A_1}, \ldots, L_{A_n}, R_{B_1}, \ldots, R_{B_n})$ in the algebra $L(I)$. By applying Lemma 3.16 with the choice $P(z, w) = \sum_{j=1}^n z_j w_j$ to (3.10) and (3.11) we get the desired identities

$$\sigma(\mathcal{E}_{A,B} : I \to I) = \sigma_T(A) \circ \sigma_T(B) = \sigma_H(A) \circ \sigma_H(B) = \sigma_a(A) \circ \sigma_a(B).$$

It is enough towards (3.10) to verify that

$$\sigma_l((L_A, R_B); L(I)) \subset \sigma_l(A) \times \sigma_r(B), \tag{3.12}$$

$$\sigma_r((L_A, R_B); L(I)) \subset \sigma_r(A) \times \sigma_l(B). \tag{3.13}$$

These inclusions were probably first noted by Harte. If $\lambda \notin \sigma_l(A)$ then there is an n-tuple $S = (S_1, \ldots, S_n) \in L(X)^n$ so that $\sum_{j=1}^n S_j(A_j - \lambda_j) = I_X$. It follows that

$$\sum_{j=1}^n L_{S_j} L_{A_j - \lambda_j} + \sum_{j=1}^n 0 \circ R_{B_j - \mu_j} = Id_I,$$

so that $(\lambda, \mu) \notin \sigma_l((L_A, R_B); L(I))$. If $\mu \notin \sigma_r(B)$ then there is $T = (T_1, \ldots, T_n) \in L(X)^n$ so that $\sum_{j=1}^n (B_j - \mu_j) T_j = I_X$. We get that

$$\sum_{j=1}^n 0 \circ L_{A_j - \lambda_j} + \sum_{j=1}^n R_{T_j} R_{B_j - \mu_j} = Id_I,$$

that is, $(\lambda, \mu) \notin \sigma_l((L_A, R_B); L(I))$. The verification of (3.13) is similar.

The proof of the lower inclusion (3.11) is the crucial step of the argument. Suppose that $(\lambda, \mu) \in \sigma_a(A) \times \sigma_a(B)$. We first factorize

$$P(z, w) - P(\lambda, \mu) = \sum_{j=1}^n G_j(z, w)(z_j - \lambda_j) + \sum_{j=1}^n H_j(z, w)(w_j - \mu_j),$$

where G_j and H_j are suitable polynomials for $j = 1, \ldots, n$, so that

$$\Phi = P(L_A, R_B) - P(\lambda, \mu) = \sum_{j=1}^n G_j(L_A, R_B) \circ L_{A_j - \lambda_j} + \sum_{j=1}^n H_j(L_A, R_B) \circ R_{B_j - \mu_j}$$

defines a bounded operator on I. Assume next to the contrary that $P(\lambda,\mu) \notin \sigma(P(L_A,R_B); I \to I)$, so that Φ is invertible $I \to I$. Since $\sigma_a(A) = \sigma_\pi(A) \cup \sigma_\pi(A^*)$, and similarly for $\sigma_a(B)$, we get four cases which are all handled somewhat differently.

Case 1. $\lambda \in \sigma_\pi(A)$, $\mu \in \sigma_\pi(B^*)$. There are sequences $(x_k) \subset S_X$ and $(x_k^*) \subset S_{X^*}$ so that $\|(A_j - \lambda_j)x_k\| \to 0$ and $\|(B_j^* - \mu_j)x_k^*\| \to 0$ as $k \to \infty$ for each $j = 1,\ldots,n$. Consider the rank-1 operator $x_k^* \otimes x_k \in I$, for which $\|x_k^* \otimes x_k\|_I = 1$ for $k \in \mathbb{N}$. Hence

$$\|(L_{A_j-\lambda_j})(x_k^* \otimes x_k)\|_I = \|x_k^*\| \cdot \|(A_j - \lambda_j)x_k\| \to 0,$$
$$\|(R_{B_j-\mu_j})(x_k^* \otimes x_k)\|_I = \|(B_j^* - \mu_j)x_k^*\| \cdot \|x_k\| \to 0$$

as $k \to \infty$ for $j = 1,\ldots,n$. Here $\|\cdot\|_I$ is the norm on the Banach ideal I. Deduce that

$$\Phi(x_k^* \otimes x_k) = \sum_{j=1}^n G_j(L_A,R_B)(L_{A_j-\lambda_j})(x_k^* \otimes x_k) + \sum_{j=1}^n H_j(L_A,R_B)(R_{B_j-\mu_j})(x_k^* \otimes x_k)$$

converges to 0 in I as $k \to \infty$, which contradicts the fact that Φ is invertible.

Case 2. $\lambda \in \sigma_\pi(A^*)$, $\mu \in \sigma_\pi(B)$. There are $(x_k) \subset S_X$ and $(x_k^*) \subset S_{X^*}$ so that $\|(A_j^* - \lambda_j)x_k^*\| \to 0$ and $\|(B_j - \mu_j)x_k\| \to 0$ as $k \to \infty$ for $j = 1,\ldots,n$. Define a linear functional ψ_k on I by

$$\psi_k(S) = \langle x_k^*, Sx_k\rangle, \quad S \in I.$$

Thus $\psi_k \in I^*$ and $\|\psi_k\| = 1$ for $k \in \mathbb{N}$ since I is a Banach ideal. Since $G_j(L_A,R_B) \circ L_{A_j-\lambda_j} = L_{A_j-\lambda_j} \circ G_j(L_A,R_B)$ and $H_j(L_A,R_B) \circ R_{B_j-\mu_j} = R_{B_j-\mu_j} \circ H_j(L_A,R_B)$ for $j = 1,\ldots,n$ we get for any $S \in I$ that

$$\begin{aligned}
|\Phi^*\psi_k(S)| &\le \sum_{j=1}^n |\langle (A_j^* - \lambda_j)x_k^*, (G_j(L_A,R_B)(S))x_k\rangle| \\
&+ \sum_{j=1}^n |\langle x_k^*, (H_j(L_A,R_B)(S))((B_j - \mu_j)x_k)\rangle| \\
&\le c\sum_{j=1}^n \|(A_j^* - \lambda_j)x_k^*\| \cdot \|S\|_I + d\sum_{j=1}^n \|(B_j - \mu_j)x_k\| \cdot \|S\|_I.
\end{aligned}$$

Here $c = \max_{j\le n}\|G_j(L_A,R_B)\|$ and $d = \max_{j\le n}\|H_j(L_A,R_B)\|$ considered as operators $I \to I$. This implies that $\|\Phi^*(\psi_k)\| \to 0$ as $k \to \infty$, which again contradicts the invertibility of Φ.

Case 3. $\lambda \in \sigma_\pi(A)$, $\mu \in \sigma_\pi(B)$. There are $(x_k) \subset S_X$ and $(y_k) \subset S_X$ so that $\|(A_j - \lambda_j)x_k\| \to 0$ and $\|(B_j - \mu_j)y_k\| \to 0$ as $k \to \infty$ for $j = 1,\ldots,n$. Fix $x^*, y^* \in S_{X^*}$ and consider the normalized rank-1 operators $U_k = x^* \otimes x_k, V_k = y^* \otimes y_k \in I$ for $k \in \mathbb{N}$. Here $\|(A_j - \lambda_j)U_k\|_I = \|(A_j - \lambda_j)x_k\| \to 0$ and $\|(B_j - \mu_j)V_k\|_I \to 0$ as $k \to \infty$ for $j = 1,\ldots,n$. Note next that $\Phi \circ L_{A_j-\lambda_j} = L_{A_j-\lambda_j} \circ \Phi$ for each j, since $\{L_{A_1},\ldots,L_{A_n},R_{B_1},\ldots,R_{B_n}\}$ commutes by assumption. Hence

$$\Phi \circ L_{A_j-\lambda_j} \circ \Phi^{-1} \circ L_{U_k} = L_{(A_j-\lambda_j)U_k} \to 0$$

as $k \to \infty$, considered as operators on I, for each j. This means that $L_{A_j-\lambda_j} \circ \Phi^{-1} \circ L_{U_k} \to 0$ as $k \to \infty$, since Φ is invertible on I by assumption. We get that

$$\begin{aligned}
R_{V_k}L_{U_k} &= R_{V_k} \circ \Phi \circ \Phi^{-1} \circ L_{U_k} = \sum_{j=1}^n R_{V_k} \circ G_j(L_A,R_B) \circ L_{A_j-\lambda_j} \circ \Phi^{-1} \circ L_{U_k} \\
&+ \sum_{j=1}^n R_{V_k} \circ R_{B_j-\mu_j} \circ H_j(L_A,R_B) \circ \Phi^{-1} \circ L_{U_k} \to 0
\end{aligned}$$

as $k \to \infty$. Above we also used the fact that $H_j(L_A, R_B)$ and $R_{B_j-\mu_j}$ commute for each j. This contradicts the fact that $\|R_{V_k}L_{U_k} : I \to I\| = \|U_k\| \cdot \|U_k\| = 1$ for $k \in \mathbb{N}$ by (2.10).

Case 4. $\lambda \in \sigma_\pi(A^*)$, $\mu \in \sigma_\pi(B^*)$. There are sequences $(x_k^*), (y_k^*) \subset S_{X^*}$ for which $\|(A_j^* - \lambda_j)x_k^*\| \to 0$ and $\|(B_j^* - \mu_j)y_k^*\| \to 0$ as $k \to \infty$ for $j = 1, \ldots, n$. Fix $x, y \in S_X$ and consider $U_k = x_k^* \otimes x, V_k = y_k^* \otimes y \in I$ for $k \in \mathbb{N}$. Thus $\|U_k(A_j - \lambda_j)\|_I \to 0$ and $\|V_k(B_j - \mu_j)\|_I \to 0$ as $k \to \infty$ for $j = 1, \ldots, n$. By arguing as in Case 3 we get that $L_{U_k} \circ \Phi^{-1} \circ L_{A_j - \lambda_j} \to 0$ as $k \to \infty$ for $= 1, \ldots, n$, from which we again deduce the contradiction that $\|L_{U_k}R_{V_k} : I \to I\| \to 0$ as $k \to \infty$. $\square$

Remark 3.17. Perhaps more interestingly, one may also recover an alternative proof of Theorem 3.13 by suitably modifying the arguments of [S95, Prop. 10 and Thm. 11], see the Remark on [S95, p. 182]. However, the argument is more delicate than the previous one, and here we just indicate the main additional ideas. In this case the strategy is to show the inclusions

$$\sigma_{He}((L_A, R_B); L(I)) \subset \sigma_{He}(A) \times \sigma_H(B) \cup \sigma_H(A) \times \sigma_{He}(B), \tag{3.14}$$

$$\sigma_{He}(A) \times \sigma_a(B) \cup \sigma_a(A) \times \sigma_{He}(B^*) \subset \sigma''((L_A, R_B); \mathcal{C}(I)). \tag{3.15}$$

In (3.15) one meets a crucial observation, that is, the advantage of using the bicommutant spectrum $\sigma''((L_A, R_B); \mathcal{C}(I))$. Above $\sigma''((L_A, R_B); \mathcal{C}(I))$ is the algebraic joint spectrum of $(L_{A_1} + K(I), \ldots, L_{A_n} + K(I), R_{B_1} + K(I), \ldots, R_{B_n} + K(I))$ in the (commutative) bicommutant subalgebra

$$\{L_{A_1} + K(I), \ldots, L_{A_n} + K(I), R_{B_1} + K(I), \ldots, R_{B_n} + K(I)\}'', \subset \mathcal{C}(I) = L(I)/K(I).$$

and $\sigma_{He}(A) = \sigma_{le}(A) \cup \sigma_{re}(A)$ is the Harte spectrum of $(L_{A_1 + K(X)}, \ldots, L_{A_n + K(X)})$ computed on the Calkin algebra $\mathcal{C}(X) = L(X)/K(X)$. Another important tool in the proof is the construction of Fredholm inverses for operator n-tuples on X assuming their existence on the level of elementary operators on I.

The following approximation problem for the inverses of elementary operators has some practical interest. Note that the class $\mathcal{E}(L(X))$ of all elementary operators is a subalgebra of $L(L(X))$ for any Banach space X.

Problem 3.18. *Does the inverse $\mathcal{E}_{A,B}^{-1} \in \overline{\mathcal{E}(L(X))}$ whenever the elementary operator $\mathcal{E}_{A,B}$ is invertible on $L(X)$? Here $\overline{\mathcal{E}(L(X))}$ is the uniform closure of the subalgebra $\mathcal{E}(L(X))$ in $L(L(X))$.*

For instance, the above holds for invertible generalized derivations $L_A - R_B$, where $A, B \in L(X)$, or for invertible $\mathcal{E}_{A,B}$ if $A, B \in L(H)^n$ are commuting n-tuples of normal operators on a Hilbert space H. This is seen for $L_A - R_B$ by applying an integral representation of $(L_A - R_B)^{-1}$ due to Rosenblum [Ro56, Thm. 3.1]. The case of normal operators is an unpublished observation [ST]. One should mention here a striking approximation result due to Magajna [M93, Cor. 2.3]:

• *Let $\mathcal{A}$ be a C^*-algebra, and suppose that $\phi : \mathcal{A} \to \mathcal{A}$ is a bounded linear operator so that $\phi(I) \subset I$ for any closed 2-sided ideals $I \subset \mathcal{A}$. Then $\phi \in \overline{\mathcal{E}(\mathcal{A})}^{SOT}$, the closure in the strong operator topology of $L(\mathcal{A})$.*

Above $\mathcal{E}(\mathcal{A})$ is the class of elementary operators on $\mathcal{A}$. Magajna's result yields some information related to Problem 3.18. Suppose that $\mathcal{E}_{A,B}$ is invertible on $L(\ell^2)$ for the n-tuples $A, B \in L(\ell^2)^n$. By applying Magajna's result to $\phi = \mathcal{E}_{A,B}^{-1}$ we get that

$$\mathcal{E}_{A,B}^{-1} \in \overline{\mathcal{E}(L(X))}^{SOT}.$$

In fact, $K(\ell^2)$ is the only non-trivial ideal of $L(\ell^2)$ and $\mathcal{E}_{A,B}(K(\ell^2)) = K(\ell^2)$ since $(\mathcal{E}_{A,B|K(\ell^2)})^{**} = \mathcal{E}_{A,B}$ by (4.17) and (4.18).

Other developments. McIntosh, Pryde and Ricker [MPR88] estimate the growth of the norm $\|S\|$ of solutions to the elementary operator equation

$$\mathcal{E}_{A,B}(S) = Y$$

for commuting n-tuples $A, B \in L(X)^n$ consisting of generalized scalar operators. (Recall that $S \in L(X)$ is a generalized scalar operator if there is $s \geq 0$ and $C < \infty$ so that $\|exp(itS)\| \leq C(1+|t|^s)$ for $t \in \mathbb{R}$.) Recently Shulman and Turowska [ShT05] have obtained an interesting approach to some operator equations that include those arising from elementary operators.

Moreover, we note that Arendt, Räbiger and Sourour [ARS94] discuss the spectrum of the map $S \mapsto AS + SB$ in the setting of unbounded operators A, B.

4. Elementary operators on Calkin algebras

Let X be an arbitrary Banach space and let $\mathcal{E}_{A,B} = \sum_{j=1}^n L_{A_j} R_{B_j}$ be the elementary operator on $L(X)$ associated to $A = (A_1, \ldots, A_n), B = (B_1, \ldots, B_n) \in L(X)^n$. Since $K(X)$ is a closed 2-sided ideal of $L(X)$ the operator $\mathcal{E}_{A,B}$ induces the related elementary operator

$$\mathcal{E}_{a,b};\ s \mapsto \sum_{j=1}^n a_j s b_j, \quad \mathcal{C}(X) \to \mathcal{C}(X),$$

on the Calkin algebra $\mathcal{C}(X) = L(X)/K(X)$, where we denote quotient elements by $s = S + K(X) \in \mathcal{C}(X)$ for $S \in L(X)$. The quotient norm $\|S\|_e \equiv dist(S, K(X))$ for $S \in L(X)$ is called the essential norm. (We will change here freely between these notations.)

In this section we will see that the operators $\mathcal{E}_{a,b}$ on the quotient algebra $\mathcal{C}(X)$ has several remarkable properties which are not shared by $\mathcal{E}_{A,B}$ on $L(X)$. Roughly speaking, $\mathcal{E}_{a,b}$ are quite "rigid" operators and this also tells something about $\mathcal{E}_{A,B}$. Let H be a Hilbert space. Fong and Sourour [FS79, p. 856] asked *whether the compactness of $\mathcal{E}_{a,b} : \mathcal{C}(H) \to \mathcal{C}(H)$ actually implies that $\mathcal{E}_{a,b} = 0$, that is, whether the Calkin algebra $\mathcal{C}(H)$ admits any non-trivial compact elementary operators $\mathcal{E}_{a,b}$.*

Clearly such a property does not hold for the elementary operators on $L(X)$, since already $L_A R_B$ is a non-zero finite rank operator on $L(X)$ whenever $A \neq 0 \neq B$ are finite rank operators on X. We next recall why the Fong-Sourour conjecture holds for the simplest operators $L_a R_b$ to get a feeling for the matter. In fact, if $A, B \notin K(H)$ then by a simple modification of the argument of Example 2.6 (for $p = 2$) one finds $A_1, A_2, B_1, B_2 \in L(H)$ so that $A_1 A A_2 = I_H = B_1 B B_2$. It follows that

$$I_{\mathcal{C}(H)} = L_{a_1} R_{b_2} \circ L_a R_b \circ L_{a_2} R_{b_1},$$

whence $L_a R_b$ is not weakly compact on $\mathcal{C}(H)$.

The Fong-Sourour conjecture was independently solved in the positive by Apostol and Fialkow [AF86], Magajna [M87] and Mathieu [Ma88]. Apostol and Fialkow [AF86, Thm. 4.1] established the stronger result that

$$\|\mathcal{E}_{a,b} : \mathcal{C}(\ell^2) \to \mathcal{C}(\ell^2)\| = dist(\mathcal{E}_{a,b}, K(\mathcal{C}(\ell^2))) \tag{4.1}$$

for arbitrary n-tuples A, B of $L(\ell^2)$. Magajna's solution is algebraic in nature, while Mathieu used tools from C*-algebras. Actually, Mathieu established the somewhat stronger result that $\mathcal{C}(\ell^2)$ does not admit any non-zero weakly compact elementary operators $\mathcal{E}_{a,b}$.

It turns out that (4.1) is a particular case of a rigidity phenomenon for elementary operators $\mathcal{E}_{a,b}$ on the Calkin algebra $\mathcal{C}(X)$, where X is any Banach space having an unconditional Schauder basis. Recall that the normalized Schauder basis (e_j) of X is *unconditional* if $\sum_{j=1}^\infty \theta_j a_j e_j$ converges in X for any sequence $(\theta_j) \in \{-1, 1\}^{\mathbf{N}}$ of signs whenever $\sum_{j=1}^{\infty} a_j e_j$ converges in X. The unconditional basis constant of (e_j) is

$$C = \sup\{\|M_\theta\| : \theta = (\theta_j) \in \{-1,1\}^{\mathbf{N}}\},$$

where the diagonal operators $M_\theta \in L(X)$ are given by $M_\theta(\sum_{j=1}^{\infty} a_j e_j) = \sum_{j=1}^{\infty} \theta_j a_j e_j$ for $\sum_{j=1}^{\infty} a_j e_j \in X$. The basis (e_j) is 1-*unconditional* if $C = 1$. An important consequence of unconditionality is that X admits plenty of nice projections: for any non-empty subset $A \subset \mathbb{N}$

$$P_A(\sum_{j=1}^{\infty} a_j e_j) = \sum_{j\in A} a_j e_j \quad \text{for } \sum_{j=1}^{\infty} a_j e_j \in X,$$

defines a projection $P_A \in L(X)$ such that $\|P_A\| \le C$. Recall that the class of Banach spaces having an unconditional basis is substantial: it contains e.g. the sequence spaces ℓ^p $(1 \le p < \infty)$ and c_0, as well as the function spaces $L^p(0,1)$ $(1 < p < \infty)$ and H^1. By contrast, $L^1(0,1)$ and $C(0,1)$ do not have any unconditional bases, cf. [LT77, 1.d.1] or [Wo91, II.D.10 and II.D.12].

The following result from [ST99, Thm. 3] extends the Apostol-Fialkow formula (4.1) to spaces X having an unconditional basis. If the unconditional constant $C > 1$ for X, then the identity (4.1) is here replaced by inequalities between $\|\mathcal{E}_{a,b}\|$ and $dist(\mathcal{E}_{a,b}, K(\mathcal{C}(X)))$ that involve C.

Theorem 4.1. *Suppose that X is a Banach space having an unconditional basis (e_j) with unconditional basis constant C. Let $A, B \in L(X)^n$ be arbitrary n-tuples. Then the elementary operators $\mathcal{E}_{A,B} : L(X) \to L(X)$ and $\mathcal{E}_{a,b} : \mathcal{C}(X) \to \mathcal{C}(X)$ satisfy*

$$dist(\mathcal{E}_{a,b}, W(\mathcal{C}(X))) \ge C^{-4}\|\mathcal{E}_{a,b} : \mathcal{C}(X) \to \mathcal{C}(X)\|, \tag{4.2}$$

$$dist(\mathcal{E}_{A,B}, W(L(X))) \ge C^{-4}\|\mathcal{E}_{a,b} : \mathcal{C}(X) \to \mathcal{C}(X)\|. \tag{4.3}$$

In particular, if (e_j) is a 1-unconditional basis, then

$$\|\mathcal{E}_{a,b} : \mathcal{C}(X) \to \mathcal{C}(X)\| = dist(\mathcal{E}_{a,b}, K(\mathcal{C}(X))) = dist(\mathcal{E}_{a,b}, W(\mathcal{C}(X))),$$

and

$$\|\mathcal{E}_{a,b}\| \le dist(\mathcal{E}_{A,B}, W(L(X))) \le \|\mathcal{E}_{A,B} : L(X) \to L(X)\|. \tag{4.4}$$

Recall that e.g. ℓ^p $(1 \le p < \infty)$, c_0 and direct sums such as $\ell^p \oplus \ell^q$ $(1 \le p < q < \infty)$ have 1-unconditional bases, but the unconditional constant of any unconditional basis of $L^p(0,1)$ for $1 < p < \infty$, $p \ne 2$, is strictly greater that 1 (see e.g. [Wo91, II.D.13 and p. 68]).

In particular, Theorem 4.1 solves a generalized version of the Fong-Sourour conjecture for elementary operators on $\mathcal{C}(X)$ for this class of Banach spaces. Part (ii) should be compared with Proposition 2.5.

Corollary 4.2. *Suppose that X is a Banach space having an unconditional basis (e_j), and let $A, B \in L(X)^n$ be arbitrary n-tuples.*
(i) If $\mathcal{E}_{a,b}$ is weakly compact $\mathcal{C}(X) \to \mathcal{C}(X)$, then $\mathcal{E}_{a,b} = 0$.
(ii) If $\mathcal{E}_{A,B}$ is weakly compact $L(X) \to L(X)$, then $\mathcal{E}_{a,b} = 0$ (so that $\mathcal{E}_{A,B}(L(X)) \subset K(X)$).

Surprisingly enough, it is actually possible to improve the estimate (4.4) from Theorem 4.1 in the case $X = \ell^p$.

Theorem 4.3. *Let $1 < p < \infty$ and $A, B \in L(\ell^p)^n$ be arbitrary n-tuples. Then*

$$\|\mathcal{E}_{a,b} : \mathcal{C}(\ell^p) \to \mathcal{C}(\ell^p)\| = dist(\mathcal{E}_{A,B}, W(L(\ell^p))) = dist(\mathcal{E}_{A,B}, K(L(\ell^p))).$$

The proof of (4.1) in [AF86] is based on Voiculescu's non-commutative Weyl - von Neumann theorem (see e.g. [Da96, Section II.5] for a description of this result). By contrast, the proof of Theorem 4.1 is quite different and it draws on fundamental properties of unconditional bases. The following simple facts will also be used here.

• *if* $(x_n) \subset X$ *is a normalized weak-null sequence, then*

$$\|S\|_e \geq \limsup_{n\to\infty} \|Sx_n\| \quad \text{for } S \in L(X) \tag{4.5}$$

(This holds since $\|Ux_n\| \to 0$ as $n \to \infty$ for any compact operator $U \in K(X)$.)

• *Let* (e_j) *be a Schauder basis for* X *so that* $\|Q_n\| = 1$ *for all* $n \in \mathbb{N}$*, where* Q_n *is the natural basis projection* $X \to [e_r : r \geq n+1]$ *(this property holds e.g. if* (e_j) *is a* 1*-unconditional basis). Then*

$$\|S\|_e = \lim_{n\to\infty} \|Q_nS\| \quad \text{for } S \in L(X). \tag{4.6}$$

Proof of Theorem 4.1 (sketch). For notational simplicity we may assume that the basis (e_j) is 1-unconditional on X. Otherwise we just pass to the equivalent norm $|\cdot|$ on X, where

$$|x| = \sup\{\|M_\theta x\| : \theta = (\theta_j) \in \{-1,1\}^{\mathbf{N}}\}, \quad x \in X,$$

and (e_j) is an 1-unconditional basis in $(X, |\cdot|)$. Above $\|x\| \leq |x| \leq C\|x\|$ for $x \in X$, where C is the unconditional constant of the basis (e_j) in $(X, \|\cdot\|)$.

Suppose that after normalization one has $\|\mathcal{E}_{a,b} : \mathcal{C}(X) \to \mathcal{C}(X)\| = 1$ for the n-tuples $A = (A_1, \ldots, A_n), B = (B_1, \ldots, B_n) \in L(X)^n$. In order to show (4.2) we are required to verify that

$$dist(\mathcal{E}_{a,b}, W(\mathcal{C}(X))) \geq 1. \tag{4.7}$$

The strategy of the argument is to construct, for any given $\varepsilon > 0$, an operator $T \in L(X)$ and a sequence $(S_j) \subset L(X)$ so that

$$1 = \|T\|_e \leq \|T\| < 1 + \varepsilon, \tag{4.8}$$

$$1 = \|S_j\|_e = \|S_j\|, \quad j \in \mathbb{N}, \tag{4.9}$$

$$\sup_j |\lambda_j| \leq \|\sum_{j=1}^\infty \lambda_j S_j\|_e \leq \|\sum_{j=1}^\infty \lambda_j S_j\| \leq 2 \cdot \sup_j |\lambda_j| \quad \text{for all } (\lambda_j) \in c_0, \tag{4.10}$$

$$dist(\mathcal{E}_{A,B}(TS_j), K(X)) \geq 1 - \varepsilon, \quad j \in \mathbb{N}. \tag{4.11}$$

To verify (4.7) from these conditions suppose that $\mathcal{V} \in W(\mathcal{C}(X))$ is arbitrary. By (4.10) the closed linear span $[s_j : j \in \mathbb{N}] \subset \mathcal{C}(X)$ is isomorphic to c_0, so that $s_j \xrightarrow{w} 0$ in $\mathcal{C}(X)$ as $n \to \infty$. Since c_0 has the DPP and $\mathcal{V} \circ L_t$ is weakly compact on $\mathcal{C}(X)$, it follows that $\|\mathcal{V}(ts_j)\| \to 0$ as $j \to \infty$. As $\|ts_j\| \leq \|T\|_e \cdot \|S_j\|_e \leq 1$ for $j \in \mathbb{N}$ we get that

$$\|\mathcal{E}_{a,b} - \mathcal{V}\| \geq \limsup_{j\to\infty} \|\mathcal{E}_{a,b}(ts_j) - \mathcal{V}(ts_j)\| \geq \limsup_{j\to\infty} \|\mathcal{E}_{a,b}(ts_j)\| \geq 1 - \varepsilon$$

from (4.11). The verification of (4.3) follows a similar outline, since $[S_j : j \in \mathbb{N}] \subset L(X)$ is also linearly isomorphic to c_0 by (4.10).

The heart of the argument lies in the construction of $T \in L(X)$ and $(S_j) \subset L(X)$ which satisfy (4.8) - (4.11). We indicate some of the ideas for completeness. Let $P_m \in L(X)$ be the natural basis projection of X onto $[e_1, \ldots, e_m]$, and $Q_m = I - P_m$ for $m \in \mathbb{N}$. Moreover, put $P_{(m,n]} = P_n - P_m$ for $m < n$, where $(m,n] = \{m+1, \ldots, n\}$.

Since $\|\mathcal{E}_{a,b}\| = 1$ there is $T \in L(X)$ such that $1 = \|T\|_e \leq \|T\| < 1 + \varepsilon$ and $\|\mathcal{E}_{A,B}(T)\|_e \geq 1 - \frac{\varepsilon}{8}$. By an inductive process it is possible to choose increasing sequences $(m_k), (n_k) \subset \mathbb{N}$ and normalized sequences $(y_j) \subset S_Y$ and $(y_j^*) \subset S_{Y^*}$ so that the following properties are satisfied:

$$0 = m_1 < n_1 < m_2 < n_2 < \dots, \tag{4.12}$$

$$Q_k^* y_k^* = y_k^*, \quad k \in \mathbb{N}, \tag{4.13}$$

$$\langle y_k^*, \mathcal{E}_{A,B}(TU_k) y_k \rangle > 1 - \frac{\varepsilon}{2}, \quad k \in \mathbb{N}, \tag{4.14}$$

$$|\langle y_k^*, \mathcal{E}_{A,B}(TU_l) y_k \rangle| \le \varepsilon \cdot 2^{-l-k}, \quad k, l \in \mathbb{N},\ k \neq l. \tag{4.15}$$

Above we have denoted $U_k = P_{(m_k, n_k]}$ for $k \in \mathbb{N}$. The induction is fairly lengthy and delicate, and we are forced to refer to [ST99, pp. 8-9] for the actual details.

The operators (S_j) will be chosen as suitable disjointly supported basis projections on X that are obtained by "cut-and-paste" as follows. First fix a disjoint partition $\mathbb{N} = \cup_{i=1}^{\infty} N_i$ into infinite sets, and let $M_i = \cup_{k \in N_i} (m_k, n_k]$ for $i \in \mathbb{N}$. Put

$$S_j = P_{M_j} \quad \text{for } j \in \mathbb{N},$$

the basis projection on X related to $M_j \subset \mathbb{N}$. It follows from the 1-unconditionality of (e_j) that $\|M_\lambda\| = \sup_{j=1} |\lambda_j|$ for any bounded real-valued $\lambda = (\lambda_j) \in \ell^\infty$, so that (by splitting into real and imaginary parts)

$$\sup_{j \in \mathbb{N}} |\lambda_j| \le \|M_\lambda\| \le 2 \cdot \sup_{j \in \mathbb{N}} |\lambda_j| \quad \text{for all } \lambda = (\lambda_j) \in \ell^\infty$$

Since the sets M_j are pairwise disjoint this fact yields that

$$\|\sum_{j=1}^{\infty} \lambda_j S_j\|_e \le \|\sum_{j=1}^{\infty} \lambda_j S_j\| \le 2 \cdot \sup_j |\lambda_j|, \quad (\lambda_j) \in c_0.$$

To obtain the lower bound in (4.10) fix $k \in \mathbb{N}$ and enumerate $N_k = \{k(m) : m \in \mathbb{N}\}$. From (4.5) we get that $\| \sum_{j=1}^{\infty} \lambda_j S_j \|_e \ge \limsup_{m \to \infty} |\lambda_k| \cdot \|S_k e_{k(m)}\| = |\lambda_k|$. Thus (4.10) holds.

Finally, (4.13) - (4.15) are used to enforce (4.11). Put again $N_k = \{k(m) : m \in \mathbb{N}\}$ for $k \in \mathbb{N}$. Since $S_k = \sum_{m=1}^{\infty} U_{k(m)}$ in the strong operator topology in $L(X)$, we get from (4.13) - (4.15) that

$$\begin{aligned} \|Q_{k(m)} \mathcal{E}_{A,B}(TS_k)\| &\ge |\langle y_{k(m)}^*, Q_{k(m)} \mathcal{E}_{A,B}(TS_k) y_{k(m)} \rangle| = |\langle y_{k(m)}^*, \mathcal{E}_{A,B}(TS_k) y_{k(m)} \rangle| \\ &\ge |\langle y_{k(m)}^*, \mathcal{E}_{A,B}(TU_{k(m)}) y_{k(m)} \rangle| - \sum_{l; l \neq m} |\langle y_{k(m)}^*, \mathcal{E}_{A,B}(TU_{k(l)}) y_{k(m)} \rangle| \\ &\ge 1 - \frac{\varepsilon}{2} - \varepsilon \sum_{l; l \neq m} 2^{-k(m)-k(l)} > 1 - \varepsilon. \end{aligned}$$

Finally, since $\|Q_{k(m)}\| = 1$ for each m by the 1-unconditionality we get

$$dist(\mathcal{E}_{A,B}(TS_k), K(X)) \ge \limsup_{m \to \infty} \|Q_{k(m)} \mathcal{E}_{A,B}(TS_k)\| \ge 1 - \varepsilon.$$

from (4.6) and the preceding estimate. □

Theorem 4.1 remains valid if X has an unconditional finite-dimensional Schauder decomposition (the Schatten classes C_p for $1 \le p < \infty$ are concrete examples of spaces having this property, but failing to have an unconditional basis, cf. [DJT95, pp. 364-368]). The exact class of Banach spaces for which Theorem 4.1 holds remains unknown.

Problem 4.4. *Does Theorem 4.1 hold for $\mathcal{C}(X)$ if X is a (classical) Banach space that fails to have any unconditional structure, e.g. if $X = L^1(0,1)$ or $X = C(0,1)$? An obvious starting point is to check whether $dist(L_aR_b, K(\mathcal{C}(X)))$ is comparable to $\|L_aR_b\|$.*

In [ST98, Thm. 7] one obtains a somewhat complementary rigidity result for non-zero elementary operators $\mathcal{E}_{a,b}$ on $\mathcal{C}(X)$, where X is a reflexive space having an unconditional basis. For such spaces one gets an alternative approach to the Fong-Sourour type results in Corollary 4.2 by showing that the non-zero operators $\mathcal{E}_{a,b}$ are automatically non strictly singular (note that this yields yet another proof of the original conjecture for $\mathcal{C}(\ell^2)$). The argument does not give any information about $\|\mathcal{E}_{a,b}\|$, but it is a simpler variant of the ideas underlying Theorem 4.1.

For $S \in L(Y)$ let $R(S) \in L(Y^{**}/Y)$ be the operator defined by

$$R(S)(y^{**} + Y) = S^{**}y^{**} + Y, \quad y^{**} \in Y^{**}.$$

It is a general fact that if S is bounded below on the subspace $M \subset Y$ then $R(S)$ is bounded below on the subspace $M^{**}/M \subset Y^{**}/Y$, see e.g. [GST95, Prop. 1.4]. We will apply this observation to the restriction $\widetilde{\mathcal{E}}_{A,B} = \mathcal{E}_{A,B|K(X)}$, for which

$$R(\widetilde{\mathcal{E}}_{A,B}) = \mathcal{E}_{a,b}. \tag{4.16}$$

This holds in trace duality for any n-tuples $A = (A_1, \ldots, A_n), B = (B_1, \ldots, B_n) \in L(X)^n$, provided e.g. X is a reflexive Banach space having an unconditional basis. In fact, in this case $\mathcal{C}(X)^* = K(X)^{\perp}$, and from (2.6) - (2.7) we get that

$$(\mathcal{E}_{A,B} : K(X) \to K(X))^* = \mathcal{E}_{A^*,B^*} : N(X^*) \to N(X^*), \tag{4.17}$$

$$(\mathcal{E}_{A^*,B^*} : N(X^*) \to N(X^*))^* = \mathcal{E}_{A,B} : L(X) \to L(X)), \tag{4.18}$$

where $A^* = (A_1^*, \ldots, A_n^*), B^* = (B_1^*, \ldots, B_n^*)$ (see the discussion following Proposition 2.5).

Theorem 4.5. *Suppose that X is a reflexive Banach space having an unconditional basis (e_j), and let $A, B \in L(X)^n$ be arbitrary n-tuples. If $\mathcal{E}_{a,b} \neq 0$, then $\mathcal{E}_{a,b}$ fixes a copy of the non-separable quotient space ℓ^∞/c_0: there is a subspace $M \subset \mathcal{C}(X)$ so that M is isomorphic to ℓ^∞/c_0 and $\mathcal{E}_{a,b}$ is bounded below on M.*

Sketch. Assume without loss of generality that $\|\mathcal{E}_{a,b}\| = 1$. Let $\varepsilon > 0$ and pick $T \in L(X)$ so that $1 = \|T\|_e \leq \|T\| < 1 + \varepsilon$ and $\|\mathcal{E}_{A,B}(T)\| > 1 - \varepsilon$.

By a gliding hump argument (see [ST98, p. p. 233] for the details) one obtains an increasing sequence $(m_k) \subset \mathbb{N}$ and a sequence $(y_k) \subset S_X$ so that

$$\|\mathcal{E}_{A,B}(TU_j)y_j\| > 1 - \varepsilon, \quad j \in \mathbb{N}, \tag{4.19}$$

where $U_j = P_{(m_j, m_{j+1}]}$ for all j. (Here we retain the notation from Theorem 4.1.) Unconditionality implies that

$$\|\sum_{j=1}^{\infty} \lambda_j U_j\| \approx \sup_j |\lambda_j| \quad \text{for all } (\lambda_j) \in c_0.$$

Thus $U_j \xrightarrow{w} 0$ in $K(X)$ as $j \to \infty$, so that (TU_j) and $(\mathcal{E}_{A,B}(TU_j))$ are weak-null sequences of $K(X)$, but $(\mathcal{E}_{A,B}(TU_j))$ is not norm-null by (4.19). Hence the Bessaga-Pelczynski basic sequence selection principle [LT77, 1.a.12] produces a subsequence (U_{j_k}) so that both (TU_{j_k}) and $(\mathcal{E}_{A,B}(TU_{j_k}))$ are basic sequences in $K(X)$. This means that

$$\|\sum_{k=1}^{\infty} \lambda_j TU_{j_k}\| \approx \|\sum_{k=1}^{\infty} \lambda_j \mathcal{E}_{A,B}(TU_{j_k})\| \approx \sup_k |\lambda_k| \quad \text{for all } (\lambda_k) \in c_0,$$

so that the restriction $\widetilde{\mathcal{E}}_{A,B} = \mathcal{E}_{A,B|K(X)}$ is bounded below on the subspace $N = [TU_{j_k} : k \in \mathbb{N}] \approx c_0$ in $K(X)$. It follows from the facts cited prior to this theorem that $\mathcal{E}_{a,b}$ is bounded below on the subspace $N^{**}/N \approx \ell^\infty/c_0$. □

The circle of results (Theorems 4.1 and 4.5) related to the Fong-Sourour conjecture concern the non-existence of small non-zero elementary operators $\mathcal{E}_{a,b}$ on a natural class of Calkin algebras. We next discuss a different rigidity property from [ST98] for $\mathcal{E}_{a,b}$ on $\mathcal{C}(X)$ for more special spaces X. These properties are related to the size of the kernel $Ker(\mathcal{E}_{a,b})$ and the cokernel $\mathcal{C}(X)/\overline{Im(\mathcal{E}_{a,b})}$. We first recall some earlier results which served as motivation. Gravner [G86] noted the following surprising facts, which contain earlier results on $\mathcal{C}(\ell^2)$ by Fialkow for the generalized derivation $L_a - R_b$ and by Weber for L_aR_b. The operators $\mathcal{E}_{A,B}$ do not enjoy such rigidity properties on $L(X)$ for any space X (just consider a left multiplication L_A).

Fact 4.6. *Let $A = (A_1, \ldots, A_n), B = (B_1, \ldots, B_n) \in L(\ell^2)^n$ be n-tuples so that $\{A_1, \ldots, A_n\}$ or $\{B_1, \ldots, B_n\}$ are commuting sets in $L(\ell^2)$.*

(i) If $\mathcal{E}_{a,b}$ is injective on $\mathcal{C}(\ell^2)$, then $\mathcal{E}_{a,b}$ is actually bounded below on $\mathcal{C}(\ell^2)$.

(ii) If the range $\mathcal{E}_{a,b}(\mathcal{C}(\ell^2))$ is dense in $\mathcal{C}(\ell^2)$, then $\mathcal{E}_{a,b}$ is a surjection $\mathcal{C}(\ell^2) \to \mathcal{C}(\ell^2)$.

Results from [LeS71] and [AT86] yield some similar mapping results for the basic maps L_a, R_b and L_aR_b on $\mathcal{C}(X)$ for a number of classical Banach spaces X.

Fact 4.7. *Suppose that $A, B \in L(X)$.*

(i) Let $X = \ell^p$ $(1 \leq p < \infty)$, c_0 or $L^p(0,1)$ $(1 \leq p < \infty)$. Then L_a is injective $\mathcal{C}(X) \to \mathcal{C}(X)$ if and only if L_a is bounded below on $\mathcal{C}(X)$.

(ii) Let $X = \ell^p$ $(1 < p \leq \infty)$, c_0, or $L^p(0,1)$ $(1 < p \leq \infty)$. Then R_b is injective $\mathcal{C}(X) \to \mathcal{C}(X)$ if and only if R_b is bounded below on $\mathcal{C}(X)$.

(iii) Let $X = \ell^p$ $(1 < p < \infty)$, c_0, or $L^p(0,1)$ $(1 < p < \infty)$. Then L_aR_b is injective $\mathcal{C}(X) \to \mathcal{C}(X)$ if and only if L_aR_b is bounded below on $\mathcal{C}(X)$.

Note that part (iii) follows easily from (i) and (ii), since $L_aR_b = R_bL_a$. In fact, if L_aR_b is injective on $\mathcal{C}(X)$, then L_a and R_b are both injective and consequently also bounded below on $\mathcal{C}(X)$.

Fact 4.7 originated in studies of properties of the semi-Fredholm classes $\Phi_+(X)$ and $\Phi_-(X)$ in $\mathcal{C}(X)$. Recall that the operator $S \in \Phi_+(X)$ if its range $Im(S)$ is closed in X and its kernel $Ker(S)$ is finite-dimensional, while $S \in \Phi_-(X)$ if $Im(S)$ has finite codimension in X (thus $\Phi(X) = \Phi_+(X) \cap \Phi_-(X)$ are the Fredholm operators). We refer e.g. to [LeS71], [AT86], [AT87] and [T94] for a more careful discussion and for further results about the classes $\Phi_\pm(X)$ modulo the compact operators.

Facts 4.6 and 4.7 suggest the question whether similar rigidity facts would hold for general elementary operators $\mathcal{E}_{a,b}$ on $\mathcal{C}(X)$. The following unexpected example from [AT87] demonstrates that some limitations apply.

Example 4.8. *There is a reflexive Banach space X, which fails to have the compact approximation property, and an isometric embedding $J \in L(X)$ so that*

(i) *L_j is one-to-one but not bounded below on $\mathcal{C}(X)$,*

(ii) *R_{j^*} is one-to-one but not bounded below on $\mathcal{C}(X^*)$.*

In contrast with Example 4.8 it was shown in [ST98, Thm. 3 and 4] that one has the following striking dichotomies for *arbitrary* elementary operators $\mathcal{E}_{a,b}$ on $\mathcal{C}(\ell^p)$. We stress that the generality is much greater compared to Fact 4.6 also in the classical case $p = 2$, since there are no commutativity assumptions on the n-tuples.

Theorem 4.9. *Let $1 < p < \infty$ and $A = (A_1, \dots, A_n), B = (B_1, \dots, B_n) \in L(\ell^p)^n$ be arbitrary n-tuples. If $\mathcal{E}_{a,b}$ is not bounded below on $\mathcal{C}(\ell^p)$, then its kernel $Ker(\mathcal{E}_{a,b})$ is non-separable. In particular, if $\mathcal{E}_{a,b}$ is injective on $\mathcal{C}(\ell^p)$ then there is $c > 0$ so that*

$$\|\mathcal{E}_{A,B}(S)\|_e \geq c\|S\|_e, \quad S \in L(\ell^p).$$

Theorem 4.10. *Let $1 < p < \infty$ and $A = (A_1, \dots, A_n), B = (B_1, \dots, B_n) \in L(\ell^p)^n$ be arbitrary n-tuples. If the range $\mathcal{E}_{a,b}(\mathcal{C}(\ell^p)) \neq \mathcal{C}(\ell^p)$, then the quotient*

$$\mathcal{C}(\ell^p)/\overline{Im(\mathcal{E}_{a,b})}$$

is non-separable. In particular, if $\mathcal{E}_{a,b}(\mathcal{C}(\ell^p))$ is dense in $\mathcal{C}(\ell^p)$, then $\mathcal{E}_{a,b}$ is a surjection.

Proof of Theorem 4.9 (sketch). Let $A = (A_1, \dots, A_n), B = (B_1, \dots, B_n) \in L(\ell^p)^n$ be n-tuples so that $\mathcal{E}_{a,b}$ is not bounded below on $\mathcal{C}(\ell^p)$. Recall that by (4.16) we have $R(\widetilde{\mathcal{E}}_{A,B}) = \mathcal{E}_{a,b}$, where $\widetilde{\mathcal{E}}_{A,B} = \mathcal{E}_{A,B|K(\ell^p)}$ is the restriction to $K(\ell^p)$ and

$$R(\widetilde{\mathcal{E}}_{A,B})(S + K(\ell^p)) = \mathcal{E}_{A,B}(S) + K(\ell^p), \quad S + K(\ell^p) \in \mathcal{C}(\ell^p).$$

By the crucial fact cited prior to Theorem 4.5 our assumption yields that $\widetilde{\mathcal{E}}_{A,B}$ is not bounded below on $K(\ell^p)$.

Let P_n and $Q_n = I - P_n$ again denote the natural basis projections with respect to the unit vector basis (e_k) in ℓ^p for $n \in \mathbb{N}$.

Step 1. We claim that for any $\varepsilon > 0$ and $m \in \mathbb{N}$ there is $S \in K(\ell^p)$ so that

$$\|S\| = 1, \quad Q_m S Q_m = S, \quad \text{and } \|\widetilde{\mathcal{E}}_{A,B}(S)\| < \varepsilon.$$

Fix $m \in \mathbf{N}$ and put $E_m = Q_m(\ell^p) = [e_s : s \geq m+1]$. The key observation here is that the related elementary operator

$$\mathcal{E}_{aj_m,q_m b} : \mathcal{C}(E_m) \to \mathcal{C}(\ell^p)$$

fails also to be bounded below. Here J_m denotes the inclusion $E_m \subset \ell^p$ and $AJ_m = (A_1 J_m, \dots, A_n J_m), Q_m B = (Q_m B_1, \dots, Q_m B_n)$. One should check that $\mathcal{E}_{aj_m,q_m b}$ is obtained from the restriction $\widetilde{\mathcal{E}}_{AJ_m,Q_m B} : K(E_m) \to K(\ell^p)$ in trace duality, that is

$$R(\widetilde{\mathcal{E}}_{AJ_m,Q_m B}) = \mathcal{E}_{aj_m,q_m b}.$$

Note that with only a minor loss of precision one may view the operators $\mathcal{E}_{aj_m,q_m b}$ and $\mathcal{E}_{a,b}$ as practically the same. This is so because $Q_m = I - P_m$, where P_m has finite rank and the codimension of E_m is finite – hence one expects that these differences are wiped away at the Calkin level. In particular, we thus know that $\mathcal{E}_{aj_m,q_m b}$ is not bounded below on $\mathcal{C}(E_m)$. From the above cited general fact we infer that $\mathcal{E}_{AJ_m,Q_m B}$ also fails to be bounded below on $K(E_m)$, and we may pick a normalized $S_0 \in K(E_m)$ so that

$$\|\mathcal{E}_{AJ_m,Q_m B}(S_0)\| = \|\widetilde{\mathcal{E}}_{A,B}(J_m S_0 Q_m)\| < \varepsilon.$$

Finally, $S = J_m S_0 Q_m \in K(\ell^p)$ is the desired operator. We refer the reader to [ST98, pp. 221-222] for a precise version of the above partly heuristic argument

Step 2. By a gliding hump argument we next obtain an increasing sequence $(m_k) \subset \mathbb{N}$ and a normalized sequence $(S_k) \subset K(\ell^p)$ of finite rank operators so that

$$P_{(m_k,m_{k+1}]} S_k P_{(m_k,m_{k+1}]} = S_k, \tag{4.20}$$

$$\|\widetilde{\mathcal{E}}_{A,B}(S_k)\| < 1/2^k \tag{4.21}$$

for all $k \in \mathbb{N}$. Above $P_{(r,s]} \equiv P_s - P_r \in L(\ell^p)$ is the natural projection onto $[e_{r+1}, \dots, e_s]$ for $r, s \in \mathbb{N}$ and $r < s$. Here (4.20) states that (S_k) is a block-diagonal sequence on ℓ^p.

We outline the general step of the induction for completeness. Suppose that we have already found $m_1 = 1 < m_2 < \ldots < m_{n+1}$ and $S_1, \ldots, S_n \in K(\ell^p)$ that satisfy (4.20) - (4.21). From Step 1 we get $S \in K(\ell^p)$ so that

$$S = Q_{m_{n+1}} S Q_{m_{n+1}}, \ \|S\| = 1 \text{ and } \|\widetilde{\mathcal{E}}_{A,B}(S)\| < 1/2^{n+2}.$$

For $r > m_{n+1}$ put $S = P_r S P_r + Z_r$, where $Z_r \equiv Q_r S + S Q_r - Q_r S Q_r \to 0$ as $r \to \infty$, since S is a compact operator on ℓ^p (and $1 < p < \infty$). By continuity we may then pick $r = m_{n+2} > m_{n+1}$ so that

$$\|P_{m_{n+2}} S P_{m_{n+2}}\| \approx 1 \quad \text{and} \quad \|\widetilde{\mathcal{E}}_{A,B}(P_{m_{n+2}} S P_{m_{n+2}})\| < 1/2^{n+2}.$$

Then $S_{n+1} = \|P_{m_{n+2}} S P_{m_{n+2}}\|^{-1} P_{m_{n+2}} S P_{m_{n+2}}$ satisfies (4.20) and (4.21).

Step 3. We next explain how to build an isometric copy of ℓ^∞ inside the kernel $\mathrm{Ker}(\mathcal{E}_{a,b})$. First fix a countable partition $\mathbb{N} = \bigcup_r N_r$ into infinite sets N_r and consider the operators $U_r = \sum_{k \in N_r} S_k \in L(\ell^p)$ for $r \in \mathbb{N}$ (the U_r are defined in the strong operator topology). One must verify the following facts for any $(c_r) \in \ell^\infty$:

$$\|\sum_{r=1}^{\infty} c_r U_r\|_e = \|\sum_{r=1}^{\infty} c_r U_r\| = \sup_r |c_r|, \tag{4.22}$$

$$\sum_{r=1}^{\infty} c_r u_r \in \mathrm{Ker}(\mathcal{E}_{a,b}). \tag{4.23}$$

The equality (4.22) is quite easy to verify on ℓ^p as one may essentially treat the disjoint normalized blocks S_k as diagonal elements (cf. the proof of (4.10)). The inclusion (4.23) in turn follows by observing that $\sum_{r=1}^{\infty} c_r \mathcal{E}_{A,B}(U_r) \in K(\ell^p)$, since the sum in question can formally be rewritten as $\sum_{k=1}^{\infty} a_k \mathcal{E}_{A,B}(S_k)$, where $|a_k| \le \|(c_r)\|_\infty$ for each k and where we have norm convergence thanks to (4.21). We leave the details to the reader. □

We mention that the strategy of the proof of Thm. 4.10 is to embed ℓ^∞ isomorphically into the quotient $\mathcal{C}(\ell^p)/\overline{Im(\mathcal{E}_{a,b})}$ whenever $\mathcal{E}_{a,b}$ is not surjective on $\mathcal{C}(\ell^p)$. For this purpose one builds certain block diagonal sequences in the nuclear operators $N(\ell^{p'})$, where p' is the dual exponent of p, as well as a related sequence $(\phi_r) \subset K(\ell^p)^\perp = \mathcal{C}(\ell^p)^*$ of norm-1 functionals which are used to norm the desired ℓ^∞-copy. We refer to [ST98, Thm. 4] for the details.

To illustrate Theorem 4.10 in a simple special case recall that $1 = I_{\ell^2} + K(\ell^2) \notin Im(L_a - R_a)$ for any $A \in L(\ell^2)$ by a well-known commutator fact. It follows from Theorem 4.10 that the quotient $\mathcal{C}(\ell^2)/\overline{Im(L_a - R_a)}$ is non-separable, so that

$$L(\ell^2)/\overline{Im(L_A - R_A) + K(\ell^2)}$$

is also non-separable for any $A \in L(\ell^2)$. (Apparently Stampfli [St73] first noticed that $L(\ell^2)/\overline{Im(L_A - R_A)}$ is non-separable for any $A \in L(\ell^2)$). This fact should be contrasted with the following remarkable result of Anderson [A73], which is based on C*-algebraic tools.

Theorem 4.11. *There are $A \in L(\ell^2)$ for which*

$$I_{\ell^2} \in \overline{Im(L_A - R_A)},$$

that is, there is $(S_n) \subset L(\ell^2)$ so that $\|I_{\ell^2} - (AS_n - S_n A)\| \to 0$ as $n \to \infty$.

It remains unclear how far the techniques behind Theorems 4.9 and 4.10 can be pushed beyond ℓ^p. We state this as a problem, where the case $X = L^p(0,1)$ for $p \neq 2$ would be particularly interesting.

Problem 4.12. *Do Theorems 4.9 and 4.10 hold in the class of reflexive Banach spaces X having an unconditional basis?*

It was pointed out in [ST98, p. 226] that Theorem 4.9 remains valid for $X = \ell^p \oplus \ell^q$, where $1 < p < q < \infty$, and also for $X = c_0$. The technical obstruction in the argument of Theorem 4.9 is the following: given a block diagonal sequence (U_r) as in Step 3 above the sum $\sum_k c_k U_k$ does not always define a bounded operator on X for $(c_k) \in c_0$. (There is an analogous obstruction related to Theorem 4.10.) This phenomenon already occurs for $\ell^p \oplus \ell^q$, but in that case it can be circumvented.

Other developments. Magajna [M95] obtained a surprising formula for the norm of an arbitrary elementary operator $\mathcal{E}_{a,b}$ on the Calkin algebra $\mathcal{C}(\ell^2)$ in terms of the completely bounded norm. (Clearly this result is also closely related to Section 3.) We refer e.g. to [P03, Section 1] for a discussion of the cb-norm.

Theorem 4.13. *Let $A, B \in L(\ell^2)^n$ be arbitrary n-tuples. Then*

$$\|\mathcal{E}_{a,b} \colon \mathcal{C}(\ell^2) \to \mathcal{C}(\ell^2)\| = \|\mathcal{E}_{a,b}\|_{cb}.$$

Subsequently Archbold, Mathieu and Somerset [AMS99, Thm. 6] characterized the precise class of C*-algebras where the preceding identity holds. Recall for this that the C*-algebra $\mathcal{A}$ is *antiliminal* if no non-zero positive element $x \in \mathcal{A}$ generates an abelian hereditary C*-subalgebra. (The C*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ is hereditary if $x \in \mathcal{B}$ whenever $y \in \mathcal{B}_+$, $0 \leq x \leq y$ and $x \in \mathcal{A}$. Here $\mathcal{B}_+$ is the positive part of $\mathcal{B}$.) The Calkin algebra $\mathcal{C}(\ell^2)$ is an antiliminal algebra, see [AM03, pp. 34-35]. The C*-algebra $\mathcal{A}$ is called *antiliminal-by-abelian*, if there are C*-algebras $\mathcal{J}$ and $\mathcal{B}$ so that $\mathcal{J}$ is abelian, $\mathcal{B}$ is antiliminal and

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow 0$$

is a short exact sequence. Finally, recall that the map $L_a R_b; s \mapsto asb$ is a bounded linear operator $\mathcal{A} \to \mathcal{A}$ for any $a, b \in M(\mathcal{A})$, the multiplier algebra of $\mathcal{A}$. (Roughly speaking, $M(A)$ is the maximal unital C*-algebra which contains $\mathcal{A}$ as a closed 2-sided ideal, cf. [AM03, pp. 27-28].) Thus $\mathcal{E}_{a,b} = \sum_{j=1}^n L_{a_j} R_{b_j}$ defines a bounded elementary operator $\mathcal{A} \to \mathcal{A}$ for any $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in M(\mathcal{A})^n$.

Theorem 4.14. *Let $\mathcal{A}$ be a C*-algebra. Then the following conditions are equivalent.*

(i) $\|\mathcal{E}_{a,b}\| = \|\mathcal{E}_{a,b}\|_{cb}$ for any n-tuples $a, b \in M(\mathcal{A})^n$,
(ii) $\mathcal{A}$ is antiliminal-by-abelian.

We refer to [M95] and [AMS99], or [AM03, Sect. 5.4] for the proofs of Theorems 4.13 and 4.14.

5. Concluding remarks

There is a quite rich and well-developed structural theory of special elementary operators on $L(\ell^2)$ to which we have paid less attention because of the restraints of this survey. A good introduction is found in the survey [Fi92].

The following topic may have some relevance for Banach spaces. Let X be a Banach space and $A, B \in L(X)$. It is well-known that a commutator $AB - BA$ cannot be of the form $\lambda I_X + K$, where $\lambda \neq 0$ and $K \in K(X)$. The commutator theorem due to Brown and Pearcy completely identifies the set of commutators on $L(\ell^2)$:

$$\{AB - BA : A, B \in L(\ell^2)\} = \{S \in L(\ell^2) : S \neq \lambda I_X + K \text{ for } K \in K(\ell^2), \lambda \neq 0\}.$$

This characterization was subsequently extended by Apostol [Ap72], [Ap73] to the case of $X = \ell^p$ for $1 \leq p \leq \infty$ and $X = c_0$. The following question, which is related to the classical spaces, appears neglected.

Problem 5.1. *Characterize the commutators on $L^p(0,1)$ for $1 < p < \infty$, $p \neq 2$.*

Some partial results are due to Schneeberger [Sch71] and Apostol.

References

[AW80] C. Akemann and S. Wright, *Compact actions on C^*-algebras*, Glasgow Math. J. **21** (1980), 143-149.

[A73] J.H. Anderson, *Derivation ranges and the identity*, Bull. Amer. Math. Soc. **79** (1973), 705-708.

[Ap72] C. Apostol, *Commutators on ℓ^p-spaces*, Rev. Roum. Math. Pures Appl. **42** (1972), 1513-1534.

[Ap73] C. Apostol, *Commutators on c_0-spaces and ℓ^∞-spaces*, Rev. Roum. Math. Pures Appl. **43** (1973), 1025-1032.

[AF86] C. Apostol and L. Fialkow, *Structural properties of elementary operators*, Canad. J. Math. **38** (1986), 1485-1524.

[AM03] P. Ara and M. Mathieu, *Local multipliers of C^*-algebras*, Springer Monographs in Mathematics (Springer-Verlag, 2003).

[AMS99] R.J. Archbold, M. Mathieu and D.W.B Somerset, *Elementary operators on antiliminal C^*-algebras*, Math. Ann. **313** (1999), 609-616.

[AST05] R.J. Archbold, D.W.B Somerset and R.M. Timoney, *On the central Haagerup tensor product and completely bounded mappings on a C^*-algebra*, J. Funct. Anal. (to appear).

[ARS94] W. Arendt, F. Räbiger and A. Sourour, *Spectral properties of the operator equation $AX + XB = Y$*, Quart. J. Math. Oxford **45** (1994), 133-149.

[AT86] K. Astala and H.-O. Tylli, *On semiFredholm operators and the Calkin algebra*, J. London Math. Soc. **34** (1986), 541-551.

[AT87] K. Astala and H.-O. Tylli, *On the bounded compact approximation property and measures of noncompactness*, J. Funct. Anal. **70** (1987), 388-401.

[BB01] M. Barraa and M. Boumazgour, *Inner derivations and norm equality*, Proc. Amer. Math. Soc. **130** (2001), 471-476.

[BR97] R. Bhatia and P. Rosenthal, *How and why to solve the operator equation $AX - XB = Y$*, Bull. London Math. Soc. **29** (1997), 1-21.

[BBR04] A. Blanco, M. Boumazgour and T.J. Ransford, *On the norm of elementary operators*, J. London Math. Soc. **70** (2004), 479-498.

[BDL01] J. Bonet, P. Domański and M. Lindström, *Weakly compact composition operators on analytic vector-valued function spaces*, Ann. Acad. Sci. Fennica Math. **26** (2001), 233-248.

[B81] J. Bourgain, *On the Dunford-Pettis property*, Proc. Amer. Math. Soc. **81** (1981), 265-272.

[CDR89] B. Carl, A. Defant and M. Ramanujan, *On tensor stable operator ideals*, Michigan J. Math. **36** (1989), 63-75.

[CS00] B. Carl and C. Schiebold, *Ein direkter Ansatz zur Untersuchung von Solitongleichungen*, Jahresber. Deutsch. Math.-.Ver. **102** (2000), 102-148. [English translation: *A direct approach to the study of soliton equations*, Jenaer Schriften zur Mathematik und Informatik Math/Inf/18/01 (Friedrich-Schiller-Universität Jena, 2001)]

[Cu83] R.E. Curto, *The spectra of elementary operators*, Indiana Univ. J. Math. **32** (1983), 193-197.

[Cu86] R.E. Curto, *Connections between Harte and Taylor spectra*, Rev. Roum. Math. Pures Appl. **32** (1986), 203-215.

[Cu88] R.E. Curto, *Applications of several complex variables to multiparameter spectral theory*, in *Surveys of some recent results in operator theory, vol. II* (J.B. Conway and B.B. Morrel, eds.), pp. 25-90. Pitman Research Notes in Mathematics vol. 192 (Longman, 1988).

[Cu92] R.E. Curto, *Spectral theory of elementary operators*, in *Elementary operators and Applications* (M. Mathieu, ed.) (Proc. Int. Conf. Blaubeuren 1991), pp. 3-52. (World Scientific, 1992)

[CuF87] R.E. Curto and L.A. Fialkow, *The spectral picture of $[L_A, R_B]$*, J. Funct. Anal. **71** (1987), 371-392.

[Da96] K.R. Davidson, *C^*-algebras by Example*, Fields Institute Monographs vol. 6 (American Mathematical Society, 1996).

[DeF93] A. Defant and K. Floret, *Tensor norms and operator ideals*, (North-Holland, 1993).

[Di80] J. Diestel, *A survey of results related to the Dunford-Pettis property*, Contemp. Math. **2** (1980), 15-60.

[DiF76] J. Diestel and B. Faires, *Remarks on the classical Banach operator ideals*, Proc. Amer. Math. Soc. **58** (1976), 189-196.

[DJT95] J. Diestel, H. Jarchow and A. Tonge, *Absolutely summing operators.* Cambridge Studies in advanced mathematics vol. **43** (Cambridge University Press, 1995).

[DiU77] J. Diestel and J.J. Uhl, jr., *Vector measures*. Mathematical Surveys **15** (American Mathematical Society, 1977).

[E88] J. Eschmeier, *Tensor products and elementary operators*, J. reine angew. Math. **390** (1988), 47-66.

[F84] A.S. Fainshtein, *The Fredholm property and the index of a function of left and right multiplication operators*, Akad. Nauk Azerbaidzhan. SSR Dokl. **40** (1984), 3-7. [in Russian]

[FeS75] M. Feder and P. Saphar, *Spaces of compact operators and their dual spaces*, Israel J. Math. **21** (1975), 38-49.

[Fi79] L. A. Fialkow, *A note on norm ideals and the operator* $X \to AX - XB$, Israel J. Math. **32** (1979), 331-348.

[Fi92] L. A. Fialkow, *Structural properties of elementary operators*, in *Elementary operators and Applications* (M. Mathieu, ed.) (Proc. Int. Conf. Blaubeuren 1991), pp. 55-113. (World Scientific, 1992).

[FS79] C.K. Fong and A. Sourour, *On the operator identity* $\sum A_k X B_k = 0$, Canad. J. Math. **31** (1979), 845-857.

[GST95] M. González, E. Saksman and H.-O. Tylli, *Representing non-weakly compact operators*, Studia Math. **113** (1995), 265-282.

[G86] J. Gravner, *A note on elementary operators on the Calkin algebra*, Proc. Amer. Math. Soc. **97** (1986), 79-86.

[Ho70] J. Holub, *Tensor product mappings*, Math. Ann. **188** (1970), 1-12.

[Ho74] J. Holub, *Tensor product mappings. II*, Proc. Amer. Math. Soc. **42** (1974), 437-441.

[J71] B.E. Johnson, *Norms of derivations on* $L(X)$, Pacific J. Math. **38** (1971), 465-469.

[JL03] W.B. Johnson and J. Lindenstrauss, *Basic concepts in the geometry of Banach spaces*, in *Handbook in the geometry of Banach spaces, vol. 1* (W.B. Johnson and J. Lindenstrauss, eds.), pp. 1- 83 (North-Holland, 2003).

[KP62] M.I. Kadec and A. Pelczynski, *Bases, lacunary sequences and complemented subspaces in the spaces* L_p, Studia Math. **21** (1962), 161-176.

[Ky77] J. Kyle, *Norms of derivations*, J. London Math. Soc. **16** (1977), 297-312.

[LeS71] A. Lebow and M. Schechter, *Measures of noncompactness and semigroups of operators*, J. Funct. Anal. **7** (1971), 1 - 26.

[LT77] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I. Sequence spaces*, Ergebnisse der Mathematik vol. **92** (Springer-Verlag, 1977).

[LST05] M. Lindström, E. Saksman and H.-O. Tylli, *Strictly singular and cosingular multiplications*, Canad. J. Math. (to appear).

[LSch99] M. Lindström and G. Schlüchtermann, *Composition of operator ideals*, Math. Scand. **84** (1999), 284-296.

[LR59] G. Lumer and M. Rosenblum, *Linear operator equations*, Proc. Amer. Math. Soc. **10** (1959), 32-41.

[M87] B. Magajna, *A system of operator equations*, Canad. Math. Bull. **30** (1987), 200-209.

[M93] B. Magajna, *A transitivity theorem for algebras of elementary operators*, Proc. Amer. Math. Soc. **118** (1993), 119-127.

[M95] B. Magajna, *The Haagerup norm on the tensor product of operator modules*, J. Funct. Anal. **129** (1995), 325-348.

[Ma88] M. Mathieu, *Elementary operators on prime* C^**-algebras, II*, Glasgow Math. J. **30** (1988), 275-284.

[Ma89] M. Mathieu, *Properties of the product of two derivations of a* C^**-algebra*, Canad. Math. Bull. **32** (1989), 490-497.

[Ma01a] M. Mathieu, *The norm problem for elementary operators*, in *Recent progress in functional analysis (Valencia, 2000)*, (K.D. Bierstedt, J. Bonét, M. Maestre, J. Schmets, eds.) pp. 363-368 (North-Holland, 2001).

[Ma01b] M. Mathieu, *Elementary operators on Calkin algebras*, Irish Math. Soc. Bull. **46** (2001), 33-42.

[MPR88] A. McIntosh, A. Pryde and W. Ricker, *Systems of operator equations and perturbation of spectral subspaces of commuting operators*, Michigan Math. J. **35** (1988), 43-65.

[Mi70] V.D. Milman, *Operators of class* C_0 *and* C_0^*, Funkcii Funkcional. Anal. i Prilozen. **10** (1970), 15-26. [in Russian]

[Mu02] V. Müller, *On the Taylor functional calculus*, Studia Math. **159** (2002), 79-97.

[Og54] T. Ogasawara, *Finite-dimensionality of certain Banach algebras*, J. Sci. Hiroshima Univ. Ser. A **17** (1954), 359-364.

[Pi80] A. Pietsch, *Operator ideals*, (North-Holland, 1980).

[Pi87] A. Pietsch, *Eigenvalues and s-numbers*, (Geest and Portig K-G; Cambridge Univ.Press, 1987).

[P03] G. Pisier, *Introduction to Operator Space Theory*, London Mathematical Society Lecture Notes vol. **294** (Cambridge University Press, 2003).
[Ra92] G. Racher, *On the tensor product of weakly compact operators*, Math. Ann. **294** (1992), 267-275.
[Ro56] M. Rosenblum, *On the operator equation $BX - XA = Q$*, Duke Math. J. **23** (1956), 263-269.
[Ru84] W. Ruess, *Duality and geometry of spaces of compact operators*, in *Functional Analysis: Surveys and recent results III* (North-Holland, 1984) pp. 59-79.
[Run00] V. Runde, *The flip is often discontinuous*, J. Operator Theory **48** (2002), 447-451.
[S95] E. Saksman, *Weak compactness and weak essential spectra of elementary operators*, Indiana Univ. Math. J. **44** (1995), 165-188.
[ST92] E. Saksman and H.-O. Tylli, *Weak compactness of multiplication operators on spaces of bounded linear operators*, Math. Scand. **70** (1992), 91-111.
[ST94] E. Saksman and H.-O. Tylli, *Weak essential spectra of multiplication operators on spaces of bounded linear operators*, Math. Ann. **299** (1994), 299-309.
[ST98] E. Saksman and H.-O. Tylli, *Rigidity of commutators and elementary operators on Calkin algebras*, Israel J. Math. **108** (1998), 217-236.
[ST99] E. Saksman and H.-O. Tylli, *The Apostol-Fialkow formula for elementary operators on Banach spaces*, J. Funct. Anal. **161** (1999), 1-26.
[ST] E. Saksman and H.-O. Tylli, *Approximating inverses of elementary operators.* (unpublished manuscript)
[Sch71] C. Schneeberger, *Commutators on a separable L^p-space*, Proc. Amer. Math. Soc. **28** (1971), 464-472.
[ShT05] V. Shulman and L. Turowska, *Operator synthesis II: Individual synthesis and linear operator equations*, J. reine angew. Math. (to appear).
[SZ96] L.L. Stacho and B. Zalar, *On the norm of Jordan elementary operators in standard operator algebras*, Publ. Math. Debrecen **49** (1996), 127-134.
[St70] J. Stampfli, *The norm of a derivation*, Pacific J. Math. **33** (1970), 737-747.
[St73] J. Stampfli, *Derivations on $B(H)$: The range*, Illinois J. Math. **17** (1973), 518-524.
[Sy1884] J. Sylvester, C.R. Acad. Sci Paris **99** (1884), pp. 67-71, 115-118, 409-412, 432-436, 527-529.
[Ta70] J.L. Taylor, *A joint spectrum for several operators*, J. Funct. Anal. **6** (1970), 172-191.
[Ti01] R. M. Timoney, *Norms of elementary operators*, Irish Math. Soc. Bulletin **46** (2001), 13-17.
[Ti03a] R. M. Timoney, *Norms and CB norms of Jordan elementary operators*, Bull. Sci. Math. **127** (2003), 597-609.
[Ti03b] R. M. Timoney, *Computing the norm of elementary operators*, Illinois J. Math. **47** (2003), 1207-1226.
[Ti05] R. M. Timoney, *Some formulae for norms of elementary operators*, J. Operator Theory (to appear).
[T94] H.-O. Tylli, *Lifting non-topological divisors of zero modulo the compact operators*, J. Funct. Anal. **125** (1994), 389-415.
[Va64] K. Vala, *On compact sets of compact operators*, Ann. Acad. Sci. Fenn. A I Math. **351** (1964).
[We77] L. Weis, *On perturbations of Fredholm operators in $L_p(\nu)$ spaces*, Proc. Amer. Math. Soc. **67** (1977), 287-292.
[Wo91] P. Wojtaszczyk, *Banach spaces for analysts*, Cambridge studies in advanced mathematics vol. **25** (Cambridge University Press, 1991).
[Y75] K. Ylinen, *Weakly completely continuous elements of C^*-algebras*, Proc. Amer. Math. Soc. **52** (1975), 323-326.

Department of Mathematics and Statistics, P.O. Box 35 (MaD), FIN-40014 University of Jyväskylä, Finland
E-mail address: saksman@maths.jyu.fi

Department of Mathematics and Statistics, P.O. Box 68 (Gustaf Hällströmin katu 2b), FIN-0014 University of Helsinki, Finland
E-mail address: hojtylli@cc.helsinki.fi