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Existence of Nash equilibria with $\mathscr{C}$-convexity

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Abstract

The purpose of this paper is to introduce general $\mathscr{C}$-convex and $\mathscr{C}$-concave conditions, and then to prove two existence theorems of Nash equilibria in generalized games with $\mathscr{C}$-concavity. Our results generalize the corresponding results due to Nash, Forgó, Takahashi and Kim–Lee in several ways.

Keywords: Nash equilibrium; $\mathscr{C}$-convex; $\mathscr{C}$-concave; Strongly $\mathscr{C}$-convex; Reflexive; Fixed point

1. Introduction

In 1951, Nash [19] established the well-known equilibrium existence theorem. Since then, the classical results of Nash [19], Debreu [4,5], Nikaido–Isoda [20] and Friedman [11] have served as basic references for the existence of Nash equilibrium for non-cooperative generalized games. In all of them, convexity of strategy spaces, continuity and concavity/quasi-concavity of the payoff functions were assumed.

On the other hand, the convexity and concavity are very essential natures in numerous applications in nonlinear analysis, and in particular, those concepts have been extensively studied and generalized by several authors in the last five decades, e.g., see [1,2,7–9,12–16]. Two important concepts for relaxing the convexity/quasi-convexity assumptions of functions are marked by the seminal papers of Fan [8,9]. In fact, the concept of convex-like condition due to Fan [8] does not require any linear structure on the strategy space and using this concept he gave a new minimax theorem and its applications. Also, in a recent

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paper [16], Kim and Lee introduced the $\mathscr{C}$-convexity which generalizes both convex condition and CF-convexity without assuming the linear structure, and as applications, they proved existence theorems of Nash equilibria and a minimax theorem.

In this paper, we will introduce more general $\mathscr{C}$-convex and $\mathscr{C}$-concave conditions. Using these notions and the partition of unity argument, we prove two existence theorems of Nash equilibria in generalized N-person games with $\mathscr{C}$-concavity. The crucial step in proofs is to use Cauty's celebrated fixed point theorem [3] which resolved the long-standing Schauder conjecture and its multi-valued modification due to Dobrowolski [6]. For these fixed point theorems, readers can refer to a very useful survey paper by Park [22]. Our results extend the corresponding results due to Nash, Forgó, Takahashi and Kim–Lee in several ways.

2. Preliminaries

Let $I=\{1,2,\ldots,n\}$ be an index set. For each $i\in I$, X_i be a topological space and denote $X_{\hat{i}}:=\prod_{j\in I\setminus\{i\}}X_j$. If $x=(x_1,\ldots,x_n)\in X$, we shall write $x_{\hat{i}}=(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)\in X_{\hat{i}}$. If $x_i\in X_i$ and $x_{\hat{i}}\in X_{\hat{i}}$, we shall use the notation $(x_i,x_{\hat{i}}):=(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n)=x\in X$. Denote by $[0,1]^n$ the Cartesian product space of unit intervals $\underbrace{[0,1]\times\cdots\times[0,1]}_{n\text{ times}}$. Denote the unit simplex in $[0,1]^n$ by Δ_n. Let X be a topological space and for a correspondence $T:X\to 2^X$, denote the fixed point set $\mathscr{F}(T):=\{x\in X\mid x\in T(x)\}$. Let $I=\{1,\ldots,n\}$ be a set of players. A *non-cooperative generalized N-person game* is an ordered $3n$-tuple $\Gamma:=(X_1,\ldots,X_n;T_1,\ldots,T_n;f_1,\ldots,f_n)$, where for each player $i\in I$, the non-empty set X_i is the strategy set, $T_i:X=\prod_{i=1}^n X_i\to 2^{X_i}$ is the player's constraint correspondence, and $f_i:X\to\mathbb{R}$ is the player's payoff function. The set X, *joint strategy space*, is the Cartesian product of the individual strategy sets, and an element of X is called a *strategy*. A strategy $(\bar{x}_1,\ldots,\bar{x}_n)\in X$ is called a *Nash equilibrium* for the generalized game Γ if for each $i=1,\ldots,n$, the following system of inequalities holds:

$$\bar{x}_i\in T_i(\bar{x})\quad\text{and}\quad f_i(\bar{x}_i,\bar{x}_{\hat{i}})\geqslant f_i(x_i,\bar{x}_{\hat{i}})\quad\text{for all } x_i\in T_i(\bar{x}).$$

A correspondence $T_i:X\to 2^{X_i}$ satisfies the *reflexivity* when $x_i\in T_i(x)$ for each $x\in X$. When $T_i(x)=X_i$ for each $x\in X$ and $i\in I$, then T_i clearly satisfies the reflexivity condition. In this case, the non-cooperative generalized N-person game reduces to the normal form of N-person game and the Nash equilibrium is exactly the same concept as in [16]. Here we note that the model of a game in this paper is a non-cooperative game.

When X and Y are any arbitrary sets, recall that $f:X\times Y\to\mathbb{R}$ is *concave-like* on X [9] if for any $x_1,x_2\in X$ and $\lambda\in[0,1]$, there exists an $x_0\in X$ such that $f(x_0,y)\geqslant\lambda f(x_1,y)+(1-\lambda)f(x_2,y)$ for every $y\in Y$. In [10], adding the continuity to concave-like functions, Forgó introduced the CF-concavity as follows: Let X be a topological space, Y an arbitrary set. Then $f:X\times Y\to\mathbb{R}$ is said to be *CF-concave* on X with respect to Y if there exists a continuous function $\Psi:X\times X\times[0,1]\to X$ such that for any $x_1,x_2\in X$ and $\lambda\in[0,1]$,

$$f(\Psi(x_1,x_2,\lambda),y)\geqslant\lambda f(x_1,y)+(1-\lambda)f(x_2,y)\quad\text{for all } y\in Y.$$

Next, we will introduce the following general convexity.

Definition 1. Let X be a topological space, $T : X \to 2^X$ a correspondence. Let D be a subset of X. Then $f : X \times X \to \mathbb{R}$ is called *$\mathscr{C}$-convex on D with respect to T* if for every $n \geqslant 2$, whenever n points $x_1, \ldots, x_n \in X$ are arbitrarily given, there exists a continuous function $\phi_n : \Delta_n \to D$ such that

$$f(\phi_n(\lambda_1, \ldots, \lambda_n), y) \leqslant \lambda_1 f(x_1, y) + \cdots + \lambda_n f(x_n, y) \tag{1}$$

for all $(\lambda_1, \ldots, \lambda_n) \in \Delta_n$ and for all $y \in T(\phi_n(\lambda_1, \ldots, \lambda_n))$; and f is called *$\mathscr{C}$-concave on D with respect to T* if $-f$ is $\mathscr{C}$-convex on D with respect to T.

Remarks. (1) When $T(x) = X$ for each $x \in X$, Definition 1 reduces to the definition in Kim–Lee [16].

(2) Note that the continuous function ϕ_n need not be globally defined on $X_1 \times \cdots \times X_n \times [0, 1]^n$, but defined only on Δ_n for each $n \geqslant 2$ as in the Definition 1. In fact, by defining $\phi_n(\lambda_1, \ldots, \lambda_n) := \Psi_n(x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_n)$, for any given n points $x_1, \ldots, x_n \in X$, we can see that the CF-convexity due to Forgó [10] implies the $\mathscr{C}$-convexity.

3. Existence of Nash equilibria

By following the skew-symmetrization method of Nikaido–Isoda [20], let us define the total sum of payoff functions $H : X \times X \to \mathbb{R}$ associated with the non-cooperative game Γ, as follows:

$$H(x, y) := \sum_{i=1}^{n} f_i(y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n) \tag{2}$$

for each $x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n) \in X = \prod_{i=1}^n X_i$.

Lemma 1. *Let Γ be a generalized N-person game. If there exists a point $\bar{x} \in X$ such that for each $i = 1, \ldots, n$,*

$$\bar{x}_i \in T_i(\bar{x}) \quad \text{and} \quad H(\bar{x}, \bar{x}) \geqslant H(x, \bar{x}) \quad \text{for each } x \in \prod_{i=1}^{n} T_i(\bar{x}),$$

then $\bar{x}$ is a Nash equilibrium for the generalized game Γ.

Proof. For each $i \in I$, we take any $x = (\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_n) \in \prod_{i=1}^n T_i(\bar{x})$. Then, by substitution, we can see that

$$\begin{aligned} H(\bar{x}, \bar{x}) &= \sum_{j \in I \setminus \{i\}} f_j(\bar{x}_1, \ldots, \bar{x}_i, \ldots \bar{x}_n) + f_i(\bar{x}_1, \ldots, \bar{x}_i, \ldots \bar{x}_n) \\ &\geqslant H(x, \bar{x}) = \sum_{j \in I \setminus \{i\}} f_j(\bar{x}_1, \ldots, \bar{x}_i, \ldots \bar{x}_n) + f_i(\bar{x}_1, \ldots, x_i, \ldots \bar{x}_n) \end{aligned}$$

for all $x_i \in T_i(\bar{x})$. Therefore we have $f_i(\bar{x}_i, \bar{x}_{\hat{i}}) \geqslant f_i(x_i, \bar{x}_{\hat{i}})$ for all $x_i \in T_i(x)$; hence $\bar{x}$ is a Nash equilibrium. $\square$

Using the general $\mathscr{C}$-concavity, we now prove the following new existence theorem of Nash equilibrium:

Theorem 1. *Let $I = \{1, 2, \ldots, n\}$ be a finite set of players, and let Γ be a non-cooperative generalized game satisfying the following*:

(1) *the strategy space $X := \prod_{i=1}^{n} X_i$ is homeomorphic to a non-empty compact convex subset of a Hausdorff topological vector space*;
(2) *the constraint correspondence $T_i : X \to 2^{X_i}$ satisfies the reflexivity condition*, *i.e.*, *$x_i \in T_i(x)$ for each $x \in X$ and for each $i \in I$*;
(3) *the function $H(x, y)$ is continuous on $X \times X$*;
(4) *the function $x \mapsto H(x, y)$ is $\mathscr{C}$-concave on X with respect to $T := \prod_{i=1}^{n} T_i$.*

Then there exists a Nash equilibrium $\bar{x} \in X$ for the generalized game Γ.

Proof. Suppose the contrary. Then, by Lemma 1, for all $x \in X$, either of the following holds:

(i) there exists $i \in I$ such that $x_i \notin T_i(x)$;
(ii) there exists an $y \in T(x) := \prod_{i \in I} T_i(x)$ such that $H(x, x) < H(y, x)$.

Since each T_i satisfies the reflexivity condition, case (i) cannot happen. For any $z \in X$, we let $U(z) := \{x \in X \mid H(x, x) < H(z, x)\}$. Then, by assumptions (2) and (3), each $U(z)$ is (possibly empty) open in X; and also $\bigcup_{z \in X} U(z) = X$. Since X is homeomorphic to a non-empty compact set, X is also compact, and hence there exists a finite number of non-empty open sets $U(z_1), \ldots, U(z_m)$ such that $\bigcup_{i=1}^{m} U(z_i) = X$. Note that $m \geqslant 2$ because $z \notin U(z)$ for all $z \in X$. Let $\{\alpha_i \mid 1 \leqslant i \leqslant m\}$ be the partition of unity subordinate to the open covering $\{U(z_i) \mid 1 \leqslant i \leqslant m\}$ of X, i.e., $0 \leqslant \alpha_i(x) \leqslant 1$, $\sum_{i=1}^{m} \alpha_i(x) = 1$ for all $x \in X$, $i = 1, \ldots, m$, and if $x \notin U(z_j)$, for some j, then $\alpha_j(x) = 0$. For such $\{z_1, \ldots, z_m\} \subset X$, since H is $\mathscr{C}$-concave, there exists a continuous mapping $\phi_m : \Delta_m \to X$ satisfying the condition $H(\phi_m(\lambda_1, \ldots, \lambda_m), y) \geqslant \lambda_1 H(z_1, y) + \cdots + \lambda_m H(z_m, y)$, for all $(\lambda_1, \ldots, \lambda_m) \in \Delta_m$ and $y \in T(\phi_m(\lambda_1, \ldots, \lambda_m))$.

Now consider a continuous mapping $\Psi : X \to X$, defined by

$$\Psi(x) := \phi_m(\alpha_1(x), \ldots, \alpha_m(x)) \quad \text{for all } x \in X.$$

Since ϕ_m and each α_i are continuous, Ψ is continuous on X. Moreover, Ψ maps X, which is homeomorphic to a non-empty compact convex subset of a Hausdorff topological vector space, into itself. Therefore, by Cauty's fixed point theorem [3], there exists an $\bar{x} \in X$ such that $\Psi(\bar{x}) = \bar{x}$. On the while, by the $\mathscr{C}$-concavity of H, we have

$$H(\Psi(x), y) \geqslant \alpha_1(x) H(z_1, y) + \cdots + \alpha_m(x) H(z_m, y) \quad \text{for all } y \in T(\Psi(x));$$

and so by using $\Psi(\bar{x}) = \bar{x}$, we have

$$H(\bar{x}, y) \geqslant \alpha_1(\bar{x}) H(z_1, y) + \cdots + \alpha_m(\bar{x}) H(z_m, y) \quad \text{for all } y \in T(\bar{x}) \subseteq X.$$

Since $\bar{x} \in T(\bar{x})$ by assumption (2), we have

$$H(\bar{x}, \bar{x}) \geqslant \alpha_1(\bar{x})H(z_1, \bar{x}) + \cdots + \alpha_m(\bar{x})H(z_m, \bar{x}). \tag{$*$}$$

However, if $\bar{x} \in U(z_j)$ for some $1 \leqslant j \leqslant m$, then we have $H(\bar{x}, \bar{x}) < H(z_j, \bar{x})$, and $\alpha_j(\bar{x}) > 0$; and if $\bar{x} \notin U(z_k)$ for some $1 \leqslant k \leqslant m$, then $\alpha_k(\bar{x}) = 0$. Thus we have

$$H(\bar{x}, \bar{x}) = \sum_{i=1}^{m} \alpha_i(\bar{x})H(\bar{x}, \bar{x}) < \sum_{i=1}^{m} \alpha_i(\bar{x})H(z_i, \bar{x});$$

which contradicts to the fact $(*)$. This completes the proof. □

Remarks. (1) Theorem 1 generalizes the previous equilibrium existence theorems due to Nash [19], Forgó [10] and Kim–Lee [16] in the following aspects:

(a) the strategy sets $X_1, \ldots, X_n$ need not be convex but $\prod_{i=1}^{n} X_i$ has the fixed point property (in fact, if X_i is homeomorphic to a compact convex subset of finite dimensional Euclidean space as in Forgó [10], then $\prod_{i=1}^{n} X_i$ is clearly a fixed point space);
(b) all payoff functions $f_1, \ldots, f_n$ need not be continuous nor concave, and also H need not be CF-concave on X with respect to T.

(2) When $T_i(x) = X_i$ for each $x \in X$ and $i \in I$, then T_i clearly satisfies assumption (2), and in this case, Theorem 1 is reduced to Theorem 1 of Kim–Lee [16].

(3) Note that the constraint correspondence T_i does not assume any continuity assumption, but each T_i should satisfy the reflexivity condition.

Next, without assuming the reflexivity assumption on T_i in Theorem 1, we can obtain the following:

Theorem 2. *Let $I = \{1, 2, \ldots, n\}$ be a finite set of players, and let Γ be a non-cooperative generalized game satisfying the following*:

(1) *the strategy space $X := \prod_{i=1}^{n} X_i$ is linearly homeomorphic to a non-empty compact convex subset of a Hausdorff topological vector space*;
(2) *the constraint correspondence $T_i : X \to 2^{X_i}$ is upper semicontinuous such that $T_i(x)$ is non-empty closed convex for each $x \in X$, and $T := \prod_{i=1}^{n} T_i : X \to 2^X$ is a correspondence*;
(3) *the function $H(x, y)$ is continuous on $X \times X$*;
(4) *the function $x \mapsto H(x, y)$ is $\mathscr{C}$-concave on $\mathscr{F}(T)$ with respect to T*;
(5) *for each $x \in \mathscr{F}(T)$, $H(x, x) \geqslant H(y, x)$ for all $y \in X \backslash \mathscr{F}(T)$.*

Then there exists a Nash equilibrium $\bar{x} \in X$ for the generalized game Γ.

Proof. Since $T : X \to 2^X$ is upper semicontinuous such that each $T(x)$ is non-empty closed convex and X is homeomorphic to a non-empty compact convex subset of a Hausdorff topological vector space, by Dobrowolski's fixed point theorem [6], there exists an $x \in X$ such that $x \in T(x)$. Denote $D := \mathscr{F}(T) = \{x \in X \mid x \in T(x)\}$; then D is non-empty. Since

T is upper semicontinuous, D is a non-empty compact subset of X. Suppose the conclusion were false. Then, by Lemma 1, for each $x \in X$, either of the following holds:

(i) there exists $i \in I$ such that $x_i \notin T_i(x)$;
(ii) there exists an $y \in T(x) := \prod_{i\in I} T_i(x)$ such that $H(x,x) < H(y,x)$.

In fact, case (i) implies that $x \notin D$. For each $z \in X$, we let

$$U(z) := \{x \in X \mid H(x,x) < H(z,x)\}.$$

Then, by assumption (3), each $U(z)$ is (possibly empty) open in X; and also $(\bigcup_{z\in X} U(z)) \cup (X\backslash D) = X$. By assumption (5), for each $z \in X\backslash D$, we have that $U(z) \subset X\backslash D$. Since

$$X = \left(\bigcup_{z\in D} U(z)\right) \cup \left(\bigcup_{z\in X\backslash D} U(z)\right) \cup (X\backslash D),$$

we obtain that $D \subset \bigcup_{z\in D} U(z)$. Since D is compact and each $U(z)$ is open, there exists a finite number of non-empty open sets $U(z_1), \ldots, U(z_m)$ such that $D \subset \bigcup_{i=1}^m U(z_i)$, where $\{z_1, \ldots, z_m\} \subset X$. If $\bigcup_{i=1}^m U(z_i) \subsetneqq X$, we can choose $z_{m+1} \in X\backslash D$ satisfying that $z_{m+1} \notin U(z_i)$ for each $i \in \{1, \ldots, m\}$. In this case, we denote a non-empty open set $U(z_{m+1}) := X\backslash D$. Then $\{U(z_1), \ldots, U(z_{m+1})\}$ is a finite open covering of X. Here we note that if $X = \bigcup_{i=1}^m U(z_i)$, we do not need an extra open set $U(z_{m+1})$. Since X is homeomorphic to a non-empty compact set, X is also compact, and hence there exists a partition of unity $\{\alpha_1, \ldots, \alpha_{m+1}\}$ subordinate to the open covering $\{U(z_1), \ldots, U(z_{m+1})\}$, i.e., $0 \leqslant \alpha_i(x) \leqslant 1$, $\sum_{i=1}^{m+1} \alpha_i(x) = 1$ for all $x \in X$, $i = 1, \ldots, m+1$; and if $x \notin U(z_j)$ for some j, then $\alpha_j(x) = 0$. For such $\{z_1, \ldots, z_{m+1}\} \subset X$, since $x \mapsto H(x,y)$ is $\mathscr{C}$-concave on D with respect to T, there exists a continuous mapping $\phi_{m+1} : \Delta_{m+1} \to D$ satisfying the condition

$$H(\phi_{m+1}(\lambda_1, \ldots, \lambda_{m+1}), y) \geqslant \lambda_1 H(z_1, y) + \cdots + \lambda_{m+1} H(z_{m+1}, y)$$

for all $(\lambda_1, \ldots, \lambda_{m+1}) \in \Delta_{m+1}$ and $y \in T(\phi_{m+1}(\lambda_1, \ldots, \lambda_{m+1}))$.

Next, we consider a continuous mapping $\Psi : X \to D$, defined by

$$\Psi(z) := \phi_{m+1}(\alpha_1(z), \ldots, \alpha_{m+1}(z)) \quad \text{for all } z \in X.$$

Since ϕ_{m+1} and each α_i are continuous, Ψ is continuous on X. Moreover, Ψ maps X, which is homeomorphic to a non-empty compact convex subset of a Hausdorff topological vector space, into a compact subset C of X. Therefore, by Cauty's theorem [3], there exists an $\bar{x} \in D$ such that $\Psi(\bar{x}) = \bar{x}$. Furthermore, we have

$$H(\Psi(\bar{x}), y) \geqslant \alpha_1(\bar{x}) H(z_1, y) + \cdots + \alpha_m(\bar{x}) H(z_m, y) + \alpha_{m+1}(\bar{x}) H(z_{m+1}, y),$$

for all $y \in T(\phi_{m+1}(\alpha_1(\bar{x}), \ldots, \alpha_{m+1}(\bar{x}))) = T(\Psi(\bar{x})) = T(\bar{x})$. Since $\bar{x} \in D$, $\bar{x} \in T(\bar{x})$; and so by putting $y := \bar{x}$, we have

$$H(\bar{x}, \bar{x}) \geqslant \alpha_1(\bar{x}) H(z_1, \bar{x}) + \cdots + \alpha_m(\bar{x}) H(z_m, \bar{x}) + \alpha_{m+1}(\bar{x}) H(z_{m+1}, \bar{x}). \qquad (**)$$

However, if $\bar{x} \in U(z_j)$ for some $1 \leqslant j \leqslant m$, then $H(\bar{x}, \bar{x}) < H(z_j, \bar{x})$ and $\alpha_j(\bar{x}) > 0$; and if $\bar{x} \notin U(z_k)$ for some $1 \leqslant k \leqslant m$, $\alpha_k(\bar{x}) = 0$. Also note that since $\bar{x} \in D$, $\bar{x} \notin X \setminus D = U(z_{m+1})$; and so $\alpha_{m+1}(\bar{x}) = 0$. Therefore, we have

$$\sum_{i=1}^{m+1} \alpha_i(\bar{x}) H(z_i, \bar{x}) > \sum_{i=1}^{m+1} \alpha_i(\bar{x}) H(\bar{x}, \bar{x}) = H(\bar{x}, \bar{x});$$

which contradicts $(**)$. This completes the proof. □

Remark. When $T_i(x) = X_i$ for each $x \in X$ and $i \in I$, then T_i clearly satisfies assumptions (2) and (5), and in this case, assumption (4) implies that $x \mapsto H(x, y)$ is $\mathscr{C}$-concave on X; hence Theorem 2 is reduced to Theorem 1 of Kim–Lee [16]. Therefore, Theorem 2 also generalizes the previous equilibrium existence theorems due to Nash [19], Forgó [10] and Kim–Lee [16] in several aspects.

4. Examples of generalized game

First we give an example where Theorem 1 can be applied but the previous results due to Nash [19], Nikaido–Isoda [20], Friedman [11], Kim–Lee [16] and Theorem 2 are not available.

Example 1. Let $\Gamma = \{X_1, X_2; T_1, T_2; f_1, f_2\}$ be a generalized 2-person game where $X_1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1, -1 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$, $X_2 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1, 0 \leqslant x, y \leqslant 1\}$, respectively. Let $T_i : X = X_1 \times X_2 \to 2^{X_i}$ and payoff functions $f_i : X \to \mathbb{R}$ be given as follows:

$$\begin{aligned}
T_1((x, y), (u, v)) &:= X_1 \cap \{(x, z) \mid z \geqslant y\} \quad \text{for each } ((x, y), (u, v)) \in X; \\
T_2((x, y), (u, v)) &:= X_2 \cap \{(u, w) \mid w \leqslant v\} \quad \text{for each } ((x, y), (u, v)) \in X; \\
f_1((x, y), (u, v)) &:= x^2 u + v^2 \quad \text{for each } ((x, y), (u, v)) \in X = X_1 \times X_2; \\
f_2((x, y), (u, v)) &:= -x v^2 + y \quad \text{for each } ((x, y), (u, v)) \in X = X_1 \times X_2.
\end{aligned}$$

Then T_1 and T_2 clearly satisfy the reflexive condition, and note that each T_i is upper semicontinuous and closed valued but not convex valued; hence Theorem 2 cannot be applied. Also note that $f_1((x, y), (u, v))$ is not quasi-concave for any fixed $(u, v) \in X_2$. Thus Theorems due to Nash [19] and Friedman [11] cannot be applied. The strategy sets X_1 and X_2 are not convex but homeomorphic to a compact convex set $\{(x, 0) \mid 0 \leqslant x \leqslant 1\}$. The total sum of payoff function $H : X \times X \to \mathbb{R}$ is given by

$$\begin{aligned}
H(((x, y), (u, v)), ((t_1, t_2), (t_1', t_2'))) &= f_1((x, y), (t_1', t_2')) + f_2((t_1, t_2), (u, v)) \\
&= x^2 t_1' + t_2'^2 - t_1 v^2 + t_2 \text{ for each } (((x, y), (u, v)), ((t_1, t_2), (t_1', t_2'))) \in X \times X.
\end{aligned}$$

For arbitrarily given two points $((x_1, y_1), (u_1, v_1)), ((x_2, y_2), (u_2, v_2)) \in X$, we now define a continuous function $\phi_2 : [0, 1]^2 \to X$ by

$$\phi_2(\lambda, \mu) = \left(\left(\sqrt{\lambda x_1^2 + \mu x_2^2}, \sqrt{1 - \lambda x_1^2 - \mu x_2^2} \right), \left(\sqrt{1 - \lambda v_1^2 - \mu v_2^2}, \sqrt{\lambda v_1^2 + \mu v_2^2} \right) \right)$$

for each $(\lambda, \mu) \in [0,1]^2$, where $\lambda + \mu = 1$. Then ϕ_2 is a continuous function depending on the given points $((x_1, y_1), (u_1, v_1)), ((x_2, y_2), (u_2, v_2)) \in X$. Moreover, for any $((t_1, t_2), (t_1', t_2')) \in X$ and $(\lambda, \mu) \in [0,1]^2$ with $\lambda + \mu = 1$, we have

$$
\begin{aligned}
&H(\phi_2(\lambda, \mu), ((t_1, t_2), (t_1', t_2')))\\
&\quad = H\left(\left(\left(\sqrt{\lambda x_1^2 + \mu x_2^2}, \sqrt{1 - \lambda x_1^2 - \mu x_2^2}\right), \left(\sqrt{1 - \lambda v_1^2 - \mu v_2^2}\right), \sqrt{\lambda v_1^2 + \mu v_2^2}\right)\right),\\
&\qquad ((t_1, t_2), (t_1', t_2'))\\
&\quad \geqslant (\lambda x_1^2 + \mu x_2^2) t_1' + t_2'^2 - t_1(\lambda v_1^2 + \mu v_2^2) + t_2\\
&\quad = \lambda(x_1^2 t_1' + t_2'^2 - t_1 v_1^2 + t_2) + \mu(x_2^2 t_1' + t_2'^2 - t_1 v_2^2 + t_2)\\
&\quad = \lambda H(((x_1, y_1), (u_1, v_1)), ((t_1, t_2), (t_1', t_2')))\\
&\qquad + \mu H(((x_2, y_2), (u_2, v_2)), ((t_1, t_2), (t_1', t_2'))).
\end{aligned}
$$

For arbitrarily given n points $((x_1, y_1), (u_1, v_1)), \ldots, ((x_n, y_n), (u_n, v_n)) \in X$, we can similarly define a continuous function ϕ_n, and hence H is $\mathscr{C}$-concave on X. Therefore, we can apply Theorem 1 to the game Γ; and we can easily check that $((1, 0), (1, 0))$ is a Nash equilibrium for Γ.

Even in 1-person game, the following simple example shows that Theorem 2 can be applied but Theorem 1 is not applicable.

Example 2. Let $\Gamma = (X, T, f)$ be a generalized 1-person game where $X = [-1, 1]$, and the constraint correspondence $T : X \to 2^X$ and the payoff function $f : X \to \mathbb{R}$ be given as follows:

$$
T(x) := \begin{cases} [x, 1], & \text{if } 0 \leqslant x \leqslant 1;\\ \{0\}, & \text{if } -1 \leqslant x < 0; \end{cases}
$$

$$
f(x) := x^3 \quad \text{for each } x \in [-1, 1].
$$

Then T satisfies the assumptions in Theorem 2. Clearly, the fixed point set $\mathscr{F}(T)$ is $[0, 1]$, and note that $x \notin T(x)$ for each $x \in [-1, 0]$ so that the reflexivity assumption (2) of Theorem 1 is not satisfied. For each $x \in \mathscr{F}(T) \equiv [0, 1]$, $H(x, x) = f(x) = x^3 \geqslant H(y, x) = f(y) = y^3$ for all $y \in [-1, 0] \equiv X \backslash \mathscr{F}(T)$, so that assumption (5) of Theorem 2 is satisfied.

For any given two points $x_1, x_2 \in \mathscr{F}(T) \equiv [0, 1]$, we define a continuous function $\phi_2 : \Delta_2 \to X \times X$ by

$$
\phi_2(\lambda, 1 - \lambda) := \sqrt[3]{\lambda x_1 + (1 - \lambda) x_2} \quad \text{for all } \lambda \in [0, 1].
$$

Then it is easy to see that ϕ_2 is a continuous function on Δ_2. Also, for every $\lambda \in [0, 1]$ and $y \in T(\phi_2(\lambda, 1 - \lambda))$, we have

$$
\begin{aligned}
H(\phi_2(\lambda, 1 - \lambda), y) &= H\left(\sqrt[3]{\lambda x_1 + (1 - \lambda) x_2}, y\right) = \lambda x_1 + (1 - \lambda) x_2\\
&\geqslant \lambda H(x_1, y) + (1 - \lambda)\, H(x_2, y) = \lambda x_1^3 + (1 - \lambda) x_2^3,
\end{aligned}
$$

so that the related total sum of payoff function $H: X \times X \to \mathbb{R}$ satisfies assumption (4) of Theorem 2. Also, note that if $x_1, x_2 \in [-1, 0]$, then assumption (4) of Theorem 2 cannot be satisfied. Therefore, we can apply Theorem 2 to the 1-person game Γ; and clearly, 1 is an equilibrium for Γ. In fact, we have $1 \in T(1)$, and

$$1 = f(1) \geqslant f(x) = x^3 \quad \text{for every } x \in T(1) = [0, 1].$$

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Further Reading

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