

## GLOBAL STABILITY FOR SIR AND SIRS MODELS WITH NONLINEAR INCIDENCE AND REMOVAL TERMS VIA DULAC FUNCTIONS

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**ABSTRACT.** We prove the global asymptotic stability of the disease-free and the endemic equilibrium for general SIR and SIRS models with nonlinear incidence. Instead of the popular Volterra-type Lyapunov functions, we use the method of Dulac functions, which allows us to extend the previous global stability results to a wider class of SIR and SIRS systems, including nonlinear (density-dependent) removal terms as well. We show that this method is useful in cases that cannot be covered by Lyapunov functions, such as bistable situations. We completely describe the global attractor even in the scenario of a backward bifurcation, when multiple endemic equilibria coexist.

**1. Introduction.** One of the key questions in the analysis of compartmental models in epidemiology with demographic turnover is whether the basic reproduction number  $\mathcal{R}_0$  completely determines the global dynamics of the system. Typically, in the case  $\mathcal{R}_0 \leq 1$  the global asymptotic stability of the disease-free equilibrium can be shown relatively easily (provided it is the unique equilibrium). The question is usually more challenging if  $\mathcal{R}_0 > 1$ . Korobeinikov [7] applied Lyapunov functions of Volterra type to prove the global asymptotic stability of the endemic equilibrium for a class of SIR and SIRS models with nonlinear transmission functions. In this work we use the method of Dulac functions, and show that we can not only recover the previous global stability results, but generalize to a wider class of systems with nonlinear incidence and nonlinear removal terms, which cannot be treated by the usual Lyapunov functions. By means of Dulac functions and dynamical systems theory, we can completely describe the global attractor of SIR models with multiple stable equilibria as well. The structure of the paper is the following. In Section 2, we show how simple Dulac functions and Poincaré–Bendixson type arguments can be used to prove the global asymptotic stability of the disease-free, resp. endemic equilibrium for general SIR- and SIRS-type models. Our results can be extended to models for which global stability has not been proved in the literature, such as models with density-dependent nonlinear removal terms (Section 3). In Section 4 we study an SIR model describing the dynamics of an infectious disease against which individuals can acquire resistance that is only temporary and only partially protective. In [11] the authors identified the basic reproduction number and an exact condition for the occurrence of a backward bifurcation. In the present paper

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we recall these results and give a complete characterization of the global dynamics of the system describing the structure of the global attractor in all possible cases, depending on the reproduction number and the presence of a backward bifurcation.

## 2. Global stability for SIR and SIRS models with nonlinear incidence.

In this section we show that it is possible to use simple Dulac functions in the proof of global asymptotic stability for a wide class of SIR- and SIRS-type models following [7], where Korobeinikov proves global stability via Lyapunov functions. As in [7], we consider an SIR model where a population of constant size (assumed to be equal to 1) is divided into three compartments: susceptibles (denoted by  $S$ ), infected ( $I$ ) and recovered ( $R$ ). Once getting infected, an individual moves from the class  $S$  to the class  $I$ , and then to the recovered compartment. It is assumed that after recovery, individuals obtain permanent immunity. The transmission of the infection is governed by the incidence rate  $f(S, I)$ . Using the notations  $\mu$  for birth rates of the susceptible and recovered classes, as well as for the death rate of the susceptible class and  $\delta$  for the sum of the death rate of the infected compartment (which we now assume to be equal to the death rate of susceptibles) and the recovery rate, we obtain the basic SIR model

$$\begin{aligned} S' &= \mu - f(S, I) - \mu S, \\ I' &= f(S, I) - \delta I, \\ R' &= (\delta - \mu)I - \mu R. \end{aligned} \tag{1}$$

Note that  $\delta - \mu$  is the recovery rate of the infected compartment, following the notations of [7]. The equation for  $R'(t)$  can be omitted, as the population size is constant and we consider only the two-dimensional system

$$\begin{aligned} S' &= \mu - f(S, I) - \mu S, \\ I' &= f(S, I) - \delta I. \end{aligned} \tag{2}$$

Due to the biological meaning, we assume that  $f(S, I)$  is a positive and monotonically growing function for all  $S, I > 0$  and  $f(S, 0) = f(0, I) = 0$  holds. The nonnegative quadrant of the  $SI$  plane is invariant with respect to system (2). Depending on the parameters, the system might have two equilibria, the disease-free equilibrium  $Q_0 = (S_0, I_0) = (1, 0)$  and the endemic equilibrium  $Q^* = (S^*, I^*)$  such that  $\mu = f(S^*, I^*) + \mu S^*$  and  $\delta I^* = f(S^*, I^*)$  hold.

For model (2), the basic reproduction number (i.e. the average number of secondary cases produced by a single infective individual introduced into an entirely susceptible population) is [7, 12]

$$\mathcal{R}_0 = \frac{1}{\delta} \frac{\partial f(S_0, I_0)}{\partial I}. \tag{3}$$

It is shown in [7, Theorem 2.1] that if the function  $f(S, I)$  monotonically grows with respect to both variables and it is concave with respect to  $I$  (i.e.  $\frac{\partial^2 f}{\partial I^2} \leq 0$ ) and if  $\mathcal{R}_0 > 1$ , then system (2) has a unique positive endemic equilibrium state  $Q^*$  which is globally asymptotically stable, while if  $\mathcal{R}_0 \leq 1$ , then there is no endemic equilibrium and the infection-free equilibrium  $Q_0$  is globally asymptotically stable.

The proof of the theorem is based on the construction of a Lyapunov function and LaSalle's invariance principle.

In the following, we will show that the above results on the global stability properties of (2) can be shown for a more general class of incidence functions by proving the existence of an appropriate Dulac function.

Let us consider a differentiable function  $F(I)$  with the property

$$F'(I) \leq \frac{F(I)}{I}. \tag{L}$$

For a function  $F(I)$  with  $F(0) = 0$ , this property means that the slope of the secant line connecting any point of the graph of the function with the origin is greater than that of the tangent line at the same point of the graph. Clearly, this property holds for any concave function  $h(I)$  with  $h(0) = 0$ . It is also easy to see that this property is more general than concavity.

**Example 1.** The function  $F(I) = I^3 - \alpha I^2$  with  $\alpha \in (2, 3)$  possesses the property (L) on the interval  $[0, 1]$ . The function is concave on the interval  $[0, \alpha/3]$ , but convex on the interval  $[\alpha/3, 1]$ .

Property (L) can also be characterized the following way.

**Lemma 2.1.** *Let  $F(I)$  be such that  $F(0) = 0$  and  $F(I) > 0$  for  $I > 0$ . Then  $F(I)$  possesses the property (L) if and only if  $F(I)$  is sublinear (in the sense of Krasnoselskii [9]), i.e.*

$$cF(I) \leq F(cI) \quad \text{for } c \in [0, 1].$$

*Proof.* Let us first suppose that  $F(I)$  is sublinear, i.e.  $cF(I) \leq F(cI)$  for  $c \in [0, 1]$  and let  $c \in (0, 1)$  be arbitrary. Then  $cF(I) \leq F(cI)$  implies

$$\frac{F(I) - F(cI)}{I - cI} \leq \frac{F(I)}{I}.$$

By letting  $c \rightarrow 1$ , the left-hand side of this inequality tends to  $F'(I)$ , from which we obtain that  $F(I)$  has the property (L).

Now let us assume that  $F(I)$  possesses the property (L). We have to show that  $cF(I) \leq F(cI)$  for  $c \in [0, 1]$ . Let us suppose that this does not hold, i.e. there exists an  $I_1$  and  $c \in (0, 1)$  such that  $cF(I_1) > F(cI_1)$ . This implies

$$\frac{F(cI_1)}{cI_1} < \frac{F(I_1)}{I_1},$$

which means that the secant line connecting the origin with the point  $(cI_1, F(cI_1))$  lies below the secant line connecting the origin with the point  $(I_1, F(I_1))$ . As the function  $F$  possesses the property (L), we have

$$F'(cI_1) \leq \frac{F(cI_1)}{cI_1}.$$

This means that there exists a neighbourhood of  $cI_1$  such that the graph of the function  $F(I)$  lies below the secant line connecting the origin with the point  $(cI_1, F(cI_1))$ . As the point  $(I_1, F(I_1))$  lies above this secant line, the function  $F(I)$  has to cross this line at least once. Let us suppose that the function  $F(I)$  crosses this secant line in the point  $(I_2, F(I_2))$ . In this point, the derivative  $F'(I_2)$  is greater than the slope of the secant line connecting the origin with the point  $(cI_1, F(cI_1))$  and with the point  $(I_2, F(I_2))$ . However, from the property (L) it follows that  $F'(I_2) \leq \frac{F(I_2)}{I_2}$ , which is a contradiction.  $\square$

Let us introduce the notation

$$f_S(I) := f(S, I) \quad \text{for } 0 \leq S, I \leq 1.$$

**Definition 2.2.** A function  $f(S, I)$  which is partially differentiable with respect to  $I$  will be called *uniformly sublinear* if  $f_S(I)$  is sublinear for every  $0 \leq S \leq 1$ .

**Lemma 2.3.** *If the function  $f(S, I)$  is uniformly sublinear and monotonically grows with respect to  $S$  and  $I$  then equation (2) has a unique endemic equilibrium if  $\mathcal{R}_0 > 1$  and there is no endemic equilibrium if  $\mathcal{R}_0 \leq 1$ .*

*Proof.* The first step of our proof (i.e. that an endemic equilibrium exists if and only if  $\mathcal{R}_0 > 1$ ) is identical to the first part of the proof of [7, Theorem 1], while the proof of the uniqueness is different. For the readers' convenience, here we include the complete proof.

At a fixed point, the equalities  $\delta I + \mu S = \mu$  and  $\delta I = f(S, I)$  hold. These two equalities define a negatively sloped straight line  $q_1$ , resp. a curve  $q_2$  in the  $IS$  plane. The equality  $\delta I + \mu S = \mu$  also defines a function  $S = h(I)$ . If  $\frac{\partial f(S, I)}{\partial S}$  is strictly positive, then the implicit function theorem implies that the function  $h(I)$  is defined and continuous for  $I > 0$ . It is easy to see that if  $h(0) = S_* \leq S_0 = 1$  then there exists at least one intersection point of the line  $q_1$  and the curve  $q_2$  defined by the two equalities. As the function  $f(S, I)$  grows monotonically with respect to both of its variables, we have  $S_0/S_* > 1$  if

$$\lim_{I \rightarrow 0} \frac{f(S_0, I)}{f(S_*, I)} = \lim_{I \rightarrow 0} \frac{f(S_0, I)}{\delta I} = \frac{1}{\delta} \frac{\partial f(S_0, 0)}{\partial I} = \mathcal{R}_0 > 1.$$

Now we turn to the proof of the uniqueness of the endemic equilibrium. In [7], this is proved using the Lyapunov function, so the method used there does not apply here. At an equilibrium of system (2), the equalities  $\delta I = \mu - \mu S$  and  $\delta I = f(S, I)$  hold. Let us suppose that there exist two endemic equilibria,  $(S^*, I^*)$  and  $(S_1, I_1)$ . Let us suppose that  $I_1 > I^*$ , or, equivalently,  $I^* = cI_1$  for some  $0 < c < 1$ . Then  $\delta I^* = \mu - \mu S^*$  and  $\delta I_1 = \mu - \mu S_1$  are satisfied, so  $S^* > S_1$  must hold. Using the fact that  $f(S, I)$  has the property (L) and that it is monotonically growing in both variables, by Lemma 2.1 we have

$$c\delta I_1 = cf_{S_1}(I_1) \leq f_{S_1}(I^*) < f_{S^*}(I^*) = \delta I^*,$$

from which we would obtain  $I^* > cI_1$ , which contradicts  $I^* = cI_1$ , and this contradiction implies the uniqueness of the endemic equilibrium.  $\square$

Before proving our first theorem, we recall the notions of  $\omega$ - and  $\alpha$ -limit sets.

**Definition 2.4.** Consider a flow  $x' = G(x)$  on a metric space  $X$ , and a point  $x_0 \in X$ . We call a point  $y \in X$  an  $\omega$ -limit point of  $x_0$  if there exists a sequence  $\{t_n\}$  in  $\mathbb{R}$  such that  $\lim_{n \rightarrow \infty} t_n = \infty$  and  $\lim_{n \rightarrow \infty} x(t_n; x_0) = y$ . An  $\alpha$ -limit point is defined similarly with  $\lim_{n \rightarrow \infty} t_n = -\infty$ . The set of all  $\omega$ -limit points of  $x_0$  (resp.  $\alpha$ -limit points) for a given orbit is called  $\omega$ -limit set (resp.  $\alpha$ -limit set) and is denoted by  $\omega(x_0)$  (resp.  $\alpha(x_0)$ ).

In the proofs of our theorems on global asymptotic stability, we will use the following result.

**Theorem 2.5.** *Let us consider a system  $x' = G(x)$  on a forward invariant domain  $D \subset \mathbb{R}^2$ . Let us suppose that there exists an  $x^* \in \text{int } D$  such that for any  $y \in D$ ,  $\omega(y) = \{x^*\}$  and there are no homoclinic orbits in  $D$ . Then  $x^*$  is a stable equilibrium of  $x' = G(x)$ .*

*Proof.* Let us suppose that  $x^*$ , which is the only equilibrium in  $D$ , is not stable. We use the notation  $S(x, \varepsilon)$  for the set  $\{\bar{x} \in \mathbb{R}^n, |\bar{x} - x| = \varepsilon\}$  and  $B(x, \varepsilon)$  for the set  $\{\bar{x} \in \mathbb{R}^n, |\bar{x} - x| < \varepsilon\}$ . Because of the instability of  $x^*$ , there exists an  $\varepsilon > 0$  and a sequence  $\{x_n\} \subset D, x_n \rightarrow x^* (n \rightarrow \infty)$  such that the solutions  $x(t; x_n)$  will leave  $B(x^*, \varepsilon)$ . We can suppose that  $\varepsilon$  is such that  $B(x^*, \varepsilon) \subset \text{int } D$ . Let  $z_n := x(t_n; x_n)$ , where  $t_n := \min\{t > 0 : x(t; x_n) \in S(x^*, \varepsilon)\}$ . The sequence  $\{z_n\}$  has a convergent subsequence, let us denote the limit of this subsequence by  $z^*$ . We will show that  $\alpha(z^*) = \{x^*\}$ . Consider the negative orbit  $\gamma^-(z^*)$  started from  $z^*$  and let  $\hat{\varepsilon} > \varepsilon$  be such that  $B(x^*, \hat{\varepsilon}) \subset \text{int } D$ . There are two possible cases: either there exists a  $t^* < 0$  such that  $x(t^*; z^*) \in S(x^*, \hat{\varepsilon})$ , or the negative orbit started from  $z^*$  stays in  $B(x^*, \hat{\varepsilon})$  for all  $t < 0$ .

Let us suppose the first case holds, in this case there exists a  $\delta > 0$  such that  $|x(t; z^*) - x^*| > \delta$  for  $t^* < t < 0$ . From the continuous dependence of the solutions on the initial conditions we obtain that there exists an  $N \in \mathbb{N}$  such that if  $n > N$ , then  $|x_n - x^*| < \frac{\delta}{2}$  and the solution  $x(t; z_n)$  reaches  $S(x^*, \varepsilon)$  at some time  $t_n^*$  while  $|x(t; z_n) - x^*| > \frac{\delta}{2}$  for  $t_n^* < t < 0$ . However, we defined  $z_n$  as the first exit point of the solution  $x(t; x_n)$  from  $B(x^*, \varepsilon)$ , which leads to a contradiction.

In the second case, the negative orbit started from  $z^*$  stays inside  $B(x^*, \hat{\varepsilon})$  for all  $t < 0$  and thus the negative limit set  $\alpha(z^*) \subset \overline{B(x^*, \varepsilon)}$  exists. Using the Poincaré–Bendixson theorem, we obtain that  $\alpha(z^*) = \{x^*\}$ . By assumption,  $\omega(z^*) = \{x^*\}$ , contradicting the non-existence of homoclinic orbits.  $\square$

**Theorem 2.6.** *Let the function  $f(S, I)$  be uniformly sublinear and monotonically growing with respect to  $S$  and  $I$ . Then the following assertions hold.*

- (i) *If  $\mathcal{R}_0 \leq 1$  then the disease-free equilibrium  $Q_0$  is globally asymptotically stable on the state space*

$$X := \{(S, I) \in \mathbb{R}_+^2 \mid 0 \leq S + I \leq 1\}.$$

- (ii) *If  $\mathcal{R}_0 > 1$  then the endemic equilibrium  $Q^*$  is globally asymptotically stable on the phase space  $X$  with the exception of the disease-free subspace*

$$X_I := \{(S, 0) \in \mathbb{R}_+^2 \mid 0 \leq S \leq 1\}.$$

*On the subspace  $X_I$  the disease-free equilibrium  $Q_0$  is globally asymptotically stable.*

*Proof.* First we will show that if the function  $f(S, I)$  is uniformly sublinear and monotonically grows with respect to  $S$  and  $I$  then equation (2) does not have any periodic solutions in the positive quadrant of the  $SI$  plane.

According to the Bendixson–Dulac theorem [4] we have to construct a continuous function  $\Psi(S, I)$  for the system (2) such that the expression

$$\frac{\partial(\Psi(S, I)(\mu - f(S, I) - \mu S))}{\partial S} + \frac{\partial(\Psi(S, I)(f(S, I) - \delta I))}{\partial I}$$

has the same sign almost everywhere in the positive quadrant of the  $SI$  plane. We define the Dulac function as  $\Psi(S, I) = 1/I$ . With this choice, the above expression takes the form

$$\begin{aligned} & \frac{\partial}{\partial S} \left( \frac{\mu}{I} - \frac{f(S, I)}{I} - \frac{\mu S}{I} \right) + \frac{\partial}{\partial I} \left( \frac{f(S, I)}{I} - \delta \right) \\ &= -\frac{1}{I} \frac{\partial}{\partial S} f(S, I) - \frac{\mu}{I} + \frac{\frac{\partial}{\partial I} f(S, I) I - f(S, I)}{I^2}. \end{aligned}$$

Using the assumption  $\frac{\partial}{\partial S}f(S, I) > 0$ , we obtain that the first two terms of this expression are negative, thus, if we can show that the last term is also negative, then the assertion of the theorem holds. The negativity of this term is equivalent to the relation

$$\frac{\partial}{\partial I}f(S, I) < \frac{f(S, I)}{I},$$

which holds for any uniformly sublinear function  $f(S, I)$ .

Assertion (i) follows immediately from the Poincaré–Bendixson theorem using the above result and Theorem 2.5.

To show assertion (ii), we can again apply our previous result and the Poincaré–Bendixson theorem to conclude that all solutions with positive initial data converge to one of the two equilibria  $Q_0$  and  $Q^*$ . Let us suppose that there exists a solution which converges to the disease-free equilibrium, i.e.  $\lim_{t \rightarrow \infty} S(t) = 1$  and  $\lim_{t \rightarrow \infty} I(t) = 0$ . We can write the equation for  $I'(t)$  in the form

$$I'(t) = f(S, I) - \delta I = \left( \frac{f(S, I) - f(S, 0)}{I} - \delta \right) I(t).$$

However, for  $t$  sufficiently large, the multiplier of  $I(t)$  is positive if  $\mathcal{R}_0 > 1$ , which contradicts  $\lim_{t \rightarrow \infty} I(t) = 0$ . Thus, all solutions with positive initial data converge to the endemic equilibrium  $Q^*$ , while stability follows from Theorem 2.5. The statement concerning solutions started from the disease-free subspace is obvious.  $\square$

**Corollary 1.** *If the function  $f(S, I)$  monotonically grows with respect to  $S$  and  $I$  and it is concave with respect to the variable  $I$  (i.e.  $\frac{\partial^2 f}{\partial I^2} \leq 0$ ) then the endemic equilibrium  $Q^*$  is globally asymptotically stable for  $\mathcal{R}_0 > 1$ . If  $\mathcal{R}_0 \leq 1$  then the disease-free equilibrium  $Q_0$  is globally asymptotically stable.*

*This means that [7, Theorem 1] follows from our Theorem 2.6. However, let us point out that the proof of [7, Theorem 1] also applies in the case of a uniformly sublinear incidence function  $f(S, I)$  instead of concavity in the second variable.*

*Proof.* As we have already seen, uniform sublinearity follows from the concavity of  $f(S, I)$  with respect to the variable  $I$ .  $\square$

In the following corollary we show that the results of Theorem 2.6 can be extended from SIR-type models to SIRS-type models.

**Corollary 2.** *Let us consider the SIRS-type model*

$$\begin{aligned} S' &= \mu - f(S, I) + rR - \mu S, \\ I' &= f(S, I) - \delta I, \\ R' &= (\delta - \mu)I - rR - \mu R. \end{aligned} \tag{4}$$

*Let the function  $f(S, I)$  be uniformly sublinear and monotonically growing with respect to  $S$  and  $I$ , then the following assertions hold.*

- (i) *If  $\mathcal{R}_0 \leq 1$  then there exists a unique disease-free equilibrium which is globally asymptotically stable on the state space*

$$X := \{(S, I, R) \in \mathbb{R}_+^3 \mid S + I + R = 1\}.$$

- (ii) *If  $\mathcal{R}_0 > 1$  then there exists a disease-free equilibrium and a unique endemic equilibrium which is globally asymptotically stable on the phase space  $X$  with the exception of the disease-free subspace*

$$X_I := \{(S, 0, R) \in \mathbb{R}_+^3 \mid S + R = 1\}.$$

On the subspace  $X_I$ , the disease-free equilibrium is globally asymptotically stable.

*Proof.* It is easy to see that system (4) can be transformed and reduced to the form (2). After substituting  $1 - S - I$  into the place of  $R$ , we obtain the two-dimensional reduced system

$$\begin{aligned} S' &= \mu - f(S, I) + r(1 - S - I) - \mu S, \\ I' &= f(S, I) - \delta I. \end{aligned}$$

By introducing the notations  $\tilde{f}(S, I) := f(S, I) + rI$ ,  $\tilde{\mu} := \mu + r$  and  $\tilde{\delta} := \delta + r$ , we obtain the system

$$\begin{aligned} S' &= \tilde{\mu} - \tilde{f}(S, I) - \tilde{\mu} S, \\ I' &= \tilde{f}(S, I) - \tilde{\delta} I. \end{aligned} \tag{5}$$

It is easy to see that  $\tilde{f}(S, I)$  is also uniformly sublinear, i.e. system (5) is equivalent to system (2), from which the assertions of the corollary follow.  $\square$

**3. Global stability for SIR and SIRS models with nonlinear (density-dependent) removal terms.** In this subsection we will show that the method seen above can also be applied for more general systems. For example, we can consider the system

$$\begin{aligned} S' &= \mu - f(S, I) - \mu S, \\ I' &= f(S, I) - g(I), \end{aligned} \tag{6}$$

where, in comparison to (2), instead of the removal term  $\delta I$  we use the nonlinear function  $g(I)$  for the sum of the death rate and the recovery rate for the infected individuals, where the function  $g(I)$  satisfies  $g(0) = 0$  and  $g(I) > 0$  for  $I > 0$ . Such a nonlinear  $g(I)$  term appears in various models, for example when recovery is facilitated by treatment. When the health care resources are constrained, the recovery rate will be naturally dependent on the number of infected individuals (see [13]). Clearly, equation (6) always has the disease-free equilibrium  $Q_0 = (S_0, I_0) = (1, 0)$ . The basic reproduction number can be calculated as

$$\mathcal{R}_0 = \frac{\frac{\partial}{\partial I} f(S_0, I_0)}{g'(I_0)}.$$

We can state the following theorem for system (6).

**Theorem 3.1.** *Let the function  $f(S, I)$  be such that  $f(S, 0) = f(0, I) = 0$  for  $0 \leq S, I \leq 1$  and  $f(S, I) > 0$  for  $S, I > 0$ , let  $g(I)$  satisfy  $g(0) = 0$  and  $g(I) > 0$  for  $I > 0$  and let us suppose that*

$$\frac{d}{dI} \left( \log \frac{g(I)}{f_S(I)} \right) \geq 0 \quad \text{holds for all } 0 < S, I \leq 1. \tag{7}$$

Then the following assertions hold for equation (6).

- (i) *If  $\mathcal{R}_0 < 1$  then the disease-free equilibrium  $Q_0$  is globally asymptotically stable on the state space*

$$X := \{(S, I) \in \mathbb{R}_+^2 \mid 0 \leq S + I \leq 1\}.$$

(ii) Let us suppose that there exists a unique endemic equilibrium  $Q^*$  if  $\mathcal{R}_0 > 1$ . Then the endemic equilibrium is globally asymptotically stable on the phase space  $X$  with the exception of the disease-free subspace

$$X_I := \{(S, 0) \in \mathbb{R}_+^2 \mid 0 \leq S \leq 1\}.$$

On the subspace  $X_I$ , the disease-free equilibrium  $Q_0$  is globally asymptotically stable.

*Proof.* The proof is similar to that of Theorem 2.6. We will use the Dulac function  $1/g(I)$  to show that equation (6) does not have any periodic solutions:

$$\begin{aligned} & \frac{\partial}{\partial S} \frac{\mu - f(S, I) - \mu S}{g(I)} + \frac{\partial}{\partial I} \frac{f(S, I) - g(I)}{g(I)} \\ &= \frac{-\frac{\partial}{\partial S} f(S, I)}{g(I)} - \frac{\mu}{g(I)} + \frac{\frac{\partial}{\partial I} f(S, I)g(I) - f(S, I)g'(I)}{g^2(I)}, \end{aligned} \tag{8}$$

which is negative if

$$\frac{\partial}{\partial I} f(S, I)g(I) \leq f(S, I) \frac{d}{dI} g(I).$$

This is equivalent to condition (7). From here we can proceed just as in the proof of Theorem 2.6 using the Poincaré–Bendixson theorem and Theorem 2.5.  $\square$

**Example 2.** Epidemic models with nonlinear incidence rates of the form

$$\frac{\beta S^p I^s}{1 + \alpha I^q}$$

with  $p, q, s > 0$  have been investigated by several authors (see e.g. [1, 3, 6, 10, 14]). Let us now consider the system with  $s = 1$ ,

$$\begin{aligned} S' &= \mu - \frac{\beta S^p I}{1 + \alpha I^q} - \mu S, \\ I' &= \frac{\beta S^p I}{1 + \alpha I^q} - \mu I - I^r \end{aligned} \tag{9}$$

where  $q < 1$  and  $r > 1$ . The basic reproduction number for system (9) can easily be calculated as  $\mathcal{R}_0 = \beta/\mu$ . For any parameter setting, the system has the disease-free equilibrium  $Q_0 = (1, 0)$ . To find an endemic equilibrium, let us note that for any  $\tilde{I} \in [0, 1]$ , there exists a unique  $\tilde{S} \in [0, 1]$  such that

$$0 = \mu - \frac{\beta \tilde{S}^p \tilde{I}}{1 + \alpha \tilde{I}^q} - \mu \tilde{S}$$

holds for  $\tilde{S}$  and  $\tilde{I}$ . This can easily be seen as the right-hand side of this equation takes  $\mu > 0$  in  $\tilde{S} = 0$  and takes a negative value in  $\tilde{S} = 1$  and it is strictly decreasing in  $\tilde{S}$ . Let us introduce the function  $u(I): [0, 1] \rightarrow [0, 1]$  such that  $u(\tilde{I})$  is this unique  $\tilde{S}$  for any  $\tilde{I} \in [0, 1]$ . One can see that  $u(I)$  is strictly decreasing if  $q < 1$ . As for the equation

$$0 = \frac{\beta \tilde{S}^p \tilde{I}}{1 + \alpha \tilde{I}^q} - \mu \tilde{I} - \tilde{I}^r,$$

for  $\tilde{I} \neq 0$  we can express  $\tilde{S}$  as  $((\mu + \alpha\mu\tilde{I}^q + \tilde{I}^{r-1} + \alpha\tilde{I}^{q+r-1})/\beta)^{1/p} =: v(\tilde{I})$ . The function  $v(\tilde{I})$  is strictly increasing and  $\lim_{\tilde{I} \rightarrow 0} v(\tilde{I}) = \mu/\beta$ . Clearly,  $v(1) > u(1)$ , thus, if  $\mathcal{R}_0 > 1$ , then there exists a unique  $I \in [0, 1]$  such that  $u(I) = v(I)$ , i.e. there exists a unique endemic equilibrium of (9).

If condition (7) holds, we can apply Theorem 3.1 to obtain that the disease-free equilibrium  $Q_0$  is globally asymptotically stable if  $\mathcal{R}_0 \leq 1$ , while if  $\mathcal{R}_0 > 1$ , then the endemic equilibrium  $Q^*$  is globally asymptotically stable. In the special case of this example, this condition takes the form

$$\frac{d}{dI} \left( \log \frac{g(I)}{f_S(I)} \right) = \frac{\alpha\mu q I^{q+1} + (r-1)I^r + \alpha(q+r-1)I^{q+r}}{I(1+\alpha I^q)(\mu I + I^r)},$$

which is nonnegative if  $r > 1$ , thus, condition (7) is satisfied.

Similarly as in Corollary 2.3, we can extend the result of Theorem 3.1 from SIR-type models to SIRS-type models in the case of a nonlinear removal rate  $g(I)$  instead of  $\delta I$ . Thus we obtain the following corollary.

**Corollary 3.** *Let us consider the following SIRS-type model with nonlinear removal term.*

$$\begin{aligned} S' &= \mu - f(S, I) + rR - \mu S, \\ I' &= f(S, I) - g(I), \\ R' &= g(I) - \mu I - \mu R. \end{aligned} \tag{10}$$

Assume that condition (7) holds. Then the following assertions hold for equation (10).

- (i) If  $\mathcal{R}_0 < 1$  then the disease-free equilibrium  $Q_0 = (1, 0, 0)$  is globally asymptotically stable on the state space

$$X := \{(S, I, R) \in \mathbb{R}_+^3 \mid S + I + R = 1\}.$$

- (ii) Let us suppose that there exists a unique endemic equilibrium  $Q^*$  if  $\mathcal{R}_0 > 1$ . Then the endemic equilibrium is globally asymptotically stable on the phase space  $X$  with the exception of the disease-free subspace

$$X_I := \{(S, 0, R) \in \mathbb{R}_+^3 \mid S + R = 1\}.$$

On the subspace  $X_I$  the disease-free equilibrium  $Q_0$  is globally asymptotically stable.

#### 4. Global dynamics of SIR models with bistability.

**4.1. Formulation of the model.** In this section we will show that Dulac functions similarly simple to those used in the previous section can be applied even in the case of multiple endemic equilibria, in which case the method of Lyapunov functions described in [7] cannot be used. As an example, we will perform a complete global stability analysis of a model by Reluga and Medlock [11] which describes the dynamics of an infectious disease against which individuals can acquire resistance that is only temporary and only partially protective. The population, which is divided into susceptible ( $S$ ), infected ( $I$ ), and recovered and resistant ( $R$ ) classes, is assumed to be a constant:  $S(t) + I(t) + R(t) = N$ . The disease transmission rate is denoted by  $\beta$ , recovery rate by  $\gamma$ , and  $\mu$  stands for birth and death rate. Individuals in the resistant class have a reduced risk: they become infected at a fraction  $\sigma$  of the rate of susceptible individuals. Susceptible individuals directly acquire resistance at rate  $v$ , presumably through some public health intervention, but resistant individuals revert to the susceptible class at rate  $a$ . Of those individuals recovering

from infection, the fraction  $1 - f$  enter the resistant class and the fraction  $f$  enter the susceptible class. Using these notations, one obtains the system

$$\begin{aligned} S'(t) &= \mu N - \beta \frac{S(t)I(t)}{N} + f\gamma I(t) + aR(t) - \nu S(t) - \mu S(t), \\ I'(t) &= \beta \frac{(S(t) + \sigma R(t))I(t)}{N} - \gamma I(t) - \mu I(t), \\ R'(t) &= -\sigma\beta \frac{R(t)I(t)}{N} + (1 - f)\gamma I(t) - aR(t) + \nu S(t) - \mu R(t). \end{aligned} \quad (11)$$

In the following, we will present some basic properties of this model. The formula for the basic reproduction number and the condition for the occurrence of a backward bifurcation (with no details on the proof) can also be found in [11], however, to make our paper self-contained, we will also present these results and we also prove the condition for backward bifurcation in details (see Theorem 4.2).

The basic reproduction number can easily be obtained as

$$\mathcal{R}_0 = \frac{\beta(a + \mu + \nu\sigma)}{(\gamma + \mu)(a + \mu + \nu)}. \quad (12)$$

The system has one disease-free equilibrium given by

$$\begin{aligned} S_0 &= N \frac{a + \mu}{a + \mu + \nu}, \\ I_0 &= 0, \\ R_0 &= N \frac{\nu}{a + \mu + \nu}, \end{aligned} \quad (13)$$

which is locally stable for  $\mathcal{R}_0 < 1$  and locally unstable for  $\mathcal{R}_0 > 1$ , as can easily be seen by calculating the equilibria of the Jacobian of system (11) at the disease-free equilibrium.

As the sum of the three compartments is constant, we might express  $S(t)$  as  $S(t) = N - I(t) - R(t)$  to obtain the two-dimensional system

$$\begin{aligned} I'(t) &= \frac{\beta I(t)(N - I(t) - R(t) + \sigma R(t))}{N} - \gamma I(t) - \mu I(t), \\ R'(t) &= -\sigma\beta \frac{R(t)I(t)}{N} + (1 - f)\gamma I(t) - aR(t) \\ &\quad + \nu(N - I(t) - R(t)) - \mu R(t). \end{aligned} \quad (14)$$

To find the endemic equilibria of this system, we have to solve the algebraic system of equations

$$\begin{aligned} 0 &= \frac{\beta \hat{I}(N - \hat{I} - \hat{R} + \sigma \hat{R})}{N} - \gamma \hat{I} - \mu \hat{I}, \\ 0 &= -\beta\sigma \frac{\hat{R}\hat{I}}{N} + (1 - f)\gamma \hat{I} - a\hat{R} + \nu(N - \hat{I} - \hat{R}) - \mu \hat{R}. \end{aligned}$$

We can express  $\hat{R}$  from the first equation as

$$\hat{R} = \frac{\beta \hat{I} - \beta N + \gamma N + \mu N}{\beta(\sigma - 1)}$$

and by substituting this into the second equation we obtain the quadratic equation

$$A\hat{I}^2 + B\hat{I} + C = 0 \quad (15)$$

with

$$A = \frac{\beta\sigma}{N}, \quad B = a + \gamma - f\gamma + \mu + \sigma(f\gamma + \mu + \nu - \beta)$$

and

$$C = \frac{N((\gamma + \mu)(a + \mu + \nu) - \beta(a + \mu + \nu\sigma))}{\beta}.$$

It is easy to see that  $C = \frac{N}{\beta}(1 - \mathcal{R}_0)(\mu + \gamma)(a + \mu + \nu)$ . From this we obtain the following lemma.

**Lemma 4.1.** *If  $\mathcal{R}_0 > 1$  then there exists a unique positive equilibrium of system (14), given by*

$$\hat{I} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}.$$

At  $\mathcal{R}_0 = 1$  we have  $C = 0$  while  $A$  is always positive. This means that to have a positive solution of (15),  $B$  has to be negative. Let us suppose that this is in fact the case, i.e. equation (15) has the unique solution  $\hat{I} = -B/A > 0$  at  $\mathcal{R}_0 = 1$ . Because of the continuous dependence on the parameters, we have  $B < 0$  and  $B^2 - 4AC > 0$  on the interval  $\mathcal{R}_c < \mathcal{R}_0 < 1$  for some  $\mathcal{R}_c < 1$ . As  $C > 0$  holds for  $\mathcal{R}_0 < 1$ , equation (15) has exactly two positive roots on this interval implying that there exist two endemic equilibria of system (14) for  $\mathcal{R}_0 < 1$ , i.e. a backward bifurcation occurs at  $\mathcal{R}_0 = 1$ . We give the condition (can also be found in [11]) on the parameters for this backward bifurcation in the following theorem.

**Theorem 4.2.** *If the condition*

$$1 + \frac{(a + \sigma\nu)^2 + \mu\nu\sigma(1 + \sigma) + 2a\mu + \mu^2}{\gamma(1 - \sigma)(a + \mu)} < \left(1 + \frac{\sigma\nu}{a + \mu}\right) f \tag{16}$$

*holds then a backward bifurcation occurs at  $\mathcal{R}_0 = 1$ . Otherwise, the bifurcation is forward.*

*Proof.* The condition  $B < 0$  can be written as

$$a + \gamma + \mu + \sigma(f\gamma + \mu + \nu) < f\gamma + \beta\sigma,$$

while the condition  $C = 0$  (which holds at  $\mathcal{R}_0 = 1$ ) is equivalent to

$$\beta = \frac{(a + \mu + \nu)(\gamma + \mu)}{a + \mu + \nu\sigma}.$$

If we substitute this into the previous condition, we obtain

$$a + \gamma + \mu + \mu\sigma + \nu\sigma - \frac{\sigma(a + \mu + \nu)(\gamma + \mu)}{a + \mu + \nu\sigma} < f\gamma(1 - \sigma).$$

Multiplying by  $(a + \mu + \nu\sigma)/(\gamma(1 - \sigma)(a + \mu))$ , we obtain

$$\frac{(a + \gamma + \mu + \mu\sigma + \nu\sigma)(a + \mu + \nu\sigma) - \sigma(a + \mu + \nu)(\gamma + \mu)}{\gamma(1 - \sigma)(a + \mu + \nu\sigma)} < \frac{a + \mu + \nu\sigma}{a + \mu} f.$$

By rearranging the terms in the numerator on the left-hand side, we obtain the condition (16). □

Now we will calculate the value  $\mathcal{R}_c$ , i.e. the value for which the two endemic equilibria appear at  $\mathcal{R}_0 = \mathcal{R}_c$  if the backward bifurcation condition (16) holds. It is obvious from our calculations so far that the condition  $B^2 - 4AC > 0$  has to be fulfilled for (15) having two positive solutions. We have already seen that

$C = \frac{N}{\beta}(1 - \mathcal{R}_0)(\mu + \gamma)(a + \mu + \nu)$ , thus, we obtain that  $B^2 - 4AC = 0$  is equivalent to

$$B^2 = 4\sigma(1 - \mathcal{R}_0)(\gamma + \mu)(a + \mu + \nu),$$

$$\frac{B^2}{4\sigma(\gamma + \mu)(a + \mu + \nu)} = 1 - \mathcal{R}_0,$$

$$\mathcal{R}_0 = 1 - \frac{B^2}{4\sigma(\gamma + \mu)(a + \mu + \nu)},$$

where  $B = a + \gamma - f\gamma + \mu + \sigma(f\gamma + \mu + \nu - \beta)$ . Thus, we have proved the following lemma.

**Lemma 4.3.** *If there is a backward bifurcation taking place at  $\mathcal{R}_0 = 1$ , then system (14) has two endemic equilibria if  $\mathcal{R}_c < \mathcal{R}_0 < 1$ , where*

$$\mathcal{R}_c = 1 - \frac{(a + \gamma - \gamma f + \mu + \sigma(f\gamma + \mu + \nu - \beta))^2}{4\sigma(\gamma + \mu)(a + \mu + \nu)}.$$

**4.2. Local stability of the endemic equilibria.** As we have seen earlier, the disease-free equilibrium  $(S_0, I_0, R_0)$  is locally stable for  $\mathcal{R}_0 < 1$  and locally unstable for  $\mathcal{R}_0 > 1$ . In this subsection we will discuss the local stability of the endemic equilibria. The calculations performed in a similar way as in [8]. The Jacobian of system (14) evaluated at  $(\hat{I}, \hat{R})$  is given by

$$J = \begin{pmatrix} \beta - \gamma - \mu - \frac{2\beta\hat{I}}{N} + \frac{\beta(\sigma-1)\hat{R}}{N} & \frac{\beta(\sigma-1)\hat{I}}{N} \\ (1-f)\gamma - \nu - \frac{\hat{R}\beta\sigma}{N} & -(a + \mu + \nu) - \frac{\beta\sigma\hat{I}}{N} \end{pmatrix},$$

which – using that in the endemic equilibria  $\gamma + \mu = -\beta\hat{I}/N + \beta + \beta\hat{R}(\sigma - 1)/N$  holds – can be written in the simpler form

$$J = \begin{pmatrix} -\frac{\beta\hat{I}}{N} & \frac{\beta(\sigma-1)\hat{I}}{N} \\ (1-f)\gamma - \nu - \frac{\hat{R}\beta\sigma}{N} & -(a + \mu + \nu) - \frac{\beta\sigma\hat{I}}{N} \end{pmatrix}.$$

The characteristic polynomial has the form  $\lambda^2 + b_1\lambda + c_1$ , where

$$b_1 = a + \mu + \nu + \frac{\beta(1 + \sigma)\hat{I}}{N}$$

and

$$c_1 = \frac{\beta\hat{I}}{N^2}(aN + \beta\sigma\hat{I} + \beta\sigma(\sigma - 1)\hat{R} + \mu N + N(f - 1)\gamma(\sigma - 1) + \nu\sigma N).$$

From the Routh–Hurwitz stability criterion (see, e.g. [5]) we know that for all solutions of the characteristic equation to have negative real parts, all coefficients have to be of the same sign. Clearly, the leading coefficient and  $b_1$  are positive for a positive  $\hat{I}$ . As for the third coefficient, we can use the fact that at an equilibrium  $(\hat{I}, \hat{R})$ , the equality  $\beta(\hat{I} + (1 - \sigma)\hat{R})/N = \beta - \gamma - \mu$  holds, which allows us to rewrite  $c_1$  as

$$c_1 = \frac{\beta\hat{I}}{N} \left( \frac{2\beta\sigma\hat{I}}{N} + a + \sigma(\mu + \gamma - \beta) + \mu + f\gamma\sigma - f\gamma - \gamma\sigma + \gamma + \nu\sigma \right)$$

$$= \frac{\beta\hat{I}}{N}(2A\hat{I} + B).$$

From this, we easily obtain the following lemma.

**Lemma 4.4.** *The endemic equilibrium  $(I^+, R^+)$  with  $I^+ = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$  is always locally asymptotically stable when it exists, i.e. when  $\mathcal{R}_0 > 0$  as well as when  $\mathcal{R}_c < \mathcal{R}_0 < 1$  and there is a backward bifurcation. The endemic equilibrium  $(I^-, R^-)$  with  $I^- = \frac{-B - \sqrt{B^2 - 4AC}}{2A}$  is always unstable when it exists, i.e. when there is a backward bifurcation and  $\mathcal{R}_c < \mathcal{R}_0 < 1$ .*

*Proof.* The assertion follows from the previous observation, i.e. that the sign of  $c_1$  depends on the sign of  $2A\hat{I} + B$ . It is easy to see that this expression is positive for  $\hat{I} = I^+$  and negative for  $\hat{I} = I^-$ , from which we obtain the assertion of the lemma using the Routh–Hurwitz criterion. Let us also note that in the latter case, as  $c_1 < 0$ , both roots of the characteristic equation are real with one of them being positive and the other one negative.  $\square$

**4.3. Global dynamics.** In this subsection, we will describe the global behaviour of the solutions of the system (11). First, we prove that all solutions of the system tend to one of the equilibria. Similarly to the previous subsection, we reduce the system to the two-dimensional system (14).

According to the Bendixson–Dulac theorem [4] we have to construct a continuous function  $\Phi(I, R)$  for the system (14) such that the expression

$$\begin{aligned} & \frac{\partial}{\partial I} \left[ \Phi(I, R) \left( \frac{\beta I(N - I - (1 - \sigma)R)}{N} - (\gamma + \mu)I \right) \right] \\ & + \frac{\partial}{\partial R} \left[ \Phi(I, R) \left( -\sigma\beta\frac{RI}{N} + (1 - f)\gamma I + \nu(N - I - R) - (\mu + a)R \right) \right] \end{aligned} \quad (17)$$

has the same sign almost everywhere in the positive quadrant of the  $SI$  plane for an appropriate Dulac function  $\Phi(I, R)$ . By choosing the Dulac function  $\Phi(I, R) = 1/I$ , expression (17) takes the form

$$-\frac{\beta}{N} - \frac{\beta\sigma}{N} - a - \mu - \nu N,$$

which is clearly negative. Thus, all solutions of system (14) tend to one of the equilibria.

**Theorem 4.5.** *The following assertions hold for system (14).*

- (i) *If no endemic equilibrium exists (i.e. if  $\mathcal{R}_0 \leq 1$  in the case of a forward bifurcation, resp. if  $\mathcal{R}_0 < \mathcal{R}_c$  in the case of a backward bifurcation), then the disease-free equilibrium is globally asymptotically stable.*
- (ii) *If there is a backward bifurcation taking place, then for  $\mathcal{R}_c < \mathcal{R}_0 < 1$ , all solutions converge to one of the three equilibria.*
- (iii) *If  $\mathcal{R}_0 > 1$ , then the unique endemic equilibrium is globally asymptotically stable.*

*Proof.* The first assertion of the theorem follows directly from the above result and Theorem 2.5.

In the case of a backward bifurcation, on the interval  $\mathcal{R}_c < \mathcal{R}_0 < 1$  there exist three equilibria, the disease-free equilibrium and two endemic equilibria, one of which being unstable and the other locally asymptotically stable, as seen in the previous subsection. As  $\mathcal{R}_0 < 1$ , the disease-free equilibrium is locally stable, so on the interval  $\mathcal{R}_c < \mathcal{R}_0 < 1$  we have bistability.

In the case  $\mathcal{R}_0 > 1$ , only one endemic equilibrium exists. We have to prove that no solution can converge to the disease-free equilibrium. Let us suppose this is not

true, i.e. there exists a solution converging to the disease-free equilibrium. If this holds, i.e.

$$\lim_{t \rightarrow \infty} I(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} R(t) = N \frac{\nu}{a + \mu + \nu},$$

then there exists a  $T > 0$  such that for all  $\varepsilon > 0$

$$I(t) < \varepsilon \quad \text{and} \quad R(t) < N \frac{\nu}{a + \mu + \nu} + \varepsilon$$

holds for all  $t > T$ . Thus, for  $t > T$ , we can estimate  $I'(t)$  as follows:

$$\begin{aligned} I'(t) &= (N - I(t) - R(t)(1 - \sigma)) \frac{\beta I(t)}{N} - (\gamma + \mu)I(t) \\ &> \left( N - \beta\varepsilon - \left( \frac{N\nu}{a + \mu + \nu} + \varepsilon \right) (1 - \sigma) \right) \frac{\beta I(t)}{N} - (\gamma + \mu)I(t) \\ &= \left( \frac{\beta(a + \mu + \nu\sigma)}{a + \mu + \nu} - (\gamma + \mu) \right) I(t) - \frac{\beta\varepsilon(2 - \sigma)I(t)}{N} > 0 \end{aligned}$$

for  $\varepsilon$  small enough, as  $\mathcal{R}_0 > 1$ . This implies that  $I(t)$  cannot converge to 0 if  $\mathcal{R}_0 > 1$ , which means that all solutions converge to the endemic equilibrium, while stability follows from Theorem 2.5.  $\square$

**4.4. Structure of the global attractor.** Let  $X$  be a metric space and  $M \subset X$ . Following the notation of [2, 1.1.7], by  $Mt$  we denote the set consisting of the states at time  $t$  of the solutions started from all of the points  $x \in M$  and we use standard terminology for global attractors (see the following definition).

**Definition 4.6.** Let  $X$  be a metric space and  $\mathcal{A}$  be a compact invariant subset of  $X$ . If  $\mathcal{A}$  attracts each bounded subset of  $X$ , i.e. for any bounded subset  $M \subset X$  and any neighbourhood  $U$  of  $\mathcal{A}$  there exists a  $T < \infty$  such that  $Mt \subset U$  for all  $t > T$ , then  $\mathcal{A}$  is called the *global attractor*.

**Theorem 4.7.** *If there is no backward bifurcation and  $\mathcal{R}_0 \leq 1$  then the global attractor consists of the disease-free equilibrium. If  $\mathcal{R}_0 > 1$  then the global attractor consists of the disease-free equilibrium, the endemic equilibrium and a connecting orbit from the disease-free equilibrium to the endemic equilibrium. If there is a backward bifurcation taking place, then for  $\mathcal{R}_c < \mathcal{R}_0 < 1$ , the global attractor consists of the three equilibria and two orbits: one connecting the unstable endemic equilibrium and the disease-free equilibrium, the other connecting the unstable endemic equilibrium and the stable endemic equilibrium.*

*Proof.* The first assertion follows from the fact that if  $\mathcal{R}_0 < 1$  and there is no backward bifurcation taking place, then the disease-free equilibrium is globally asymptotically stable on the whole phase space.

In the case of a backward bifurcation ( $\mathcal{R}_c < \mathcal{R}_0 < 1$ ), we have two endemic equilibria. In Subsection 4.2 we showed that one of the two endemic equilibria is stable, while the other one is unstable with one stable and one unstable eigenvector. Thus, the unstable manifold of the unstable equilibrium is one-dimensional. From our results so far, it is clear that the  $\omega$ -limit set of any solution started from the unstable manifold is one of the two stable equilibria, while the  $\alpha$ -limit set is the unstable equilibrium, as the existence of a homoclinic orbit is ruled out by the Bendixson–Dulac criterion. We need to show that there exists a connecting orbit from the unstable equilibrium to both stable equilibria. Let us suppose that this is not true, i.e. solutions started from both branches of the unstable manifold of

the unstable equilibrium tend to the same stable equilibrium. Let us now start a solution from the stable manifold of the unstable equilibrium. The  $\omega$ -limit set of this solution is the unstable equilibrium, while – according to the Bendixson–Dulac criterion – its  $\alpha$ -limit set is one of the equilibria. Because of the stability of the other two equilibria, the  $\alpha$ -limit set can only be the unstable equilibrium, which means that this is a homoclinic orbit. However, the existence of such an orbit is ruled out by the Bendixson–Dulac criterion. Thus, there exists a connecting orbit from the unstable equilibrium to both stable equilibria. Since the unstable manifold of the unstable equilibrium is one-dimensional, no other connections exist.

We have seen that if  $\mathcal{R}_0 > 1$ , then the unique endemic equilibrium is globally asymptotically stable on the whole phase space with the exception of the extinction space of the infected compartment, where the disease-free equilibrium is globally asymptotically stable. By standard linearization, we calculate the eigenvalues and eigenvectors of the Jacobian of the reduced two-dimensional system

$$\begin{aligned} S'(t) &= \mu N - \beta \frac{S(t)I(t)}{N} + f\gamma I(t) + a(N - S(t) - I(t)) \\ &\quad - \nu S(t) - \mu S(t), \\ I'(t) &= \beta \frac{(S(t) + \sigma(N - S(t) - I(t)))I(t)}{N} - \gamma I(t) - \mu I(t) \end{aligned} \tag{18}$$

in the disease-free equilibrium: the eigenvalues are  $\lambda_1 = -a - \mu - \nu$  with corresponding eigenvector  $(1, 0)$  and  $\lambda_2 = (\gamma + \mu)(\mathcal{R}_0 - 1)$  with corresponding eigenvector

$$\left( -\frac{a^2 + a\beta - a\gamma f + a\mu + a\nu + \beta\mu - \gamma f\mu - \gamma f\nu}{a^2 + a\beta - a\gamma + a\mu + 2a\nu + \beta\mu + \beta\nu\sigma - \gamma\mu - \gamma\nu + \mu\nu + \nu^2}, 1 \right).$$

It is easy to see that the first of these eigenvectors is always stable, while the second is stable for  $\mathcal{R}_0 < 1$  and unstable for  $\mathcal{R}_0 > 1$ . Thus for  $\mathcal{R}_0 > 1$ , the disease-free equilibrium has a one-dimensional unstable manifold. If we start a solution from this unstable manifold, according to our results, that solution converges to the endemic equilibrium, from which we can conclude the existence of a heteroclinic orbit connecting the disease-free equilibrium and the endemic equilibrium.  $\square$

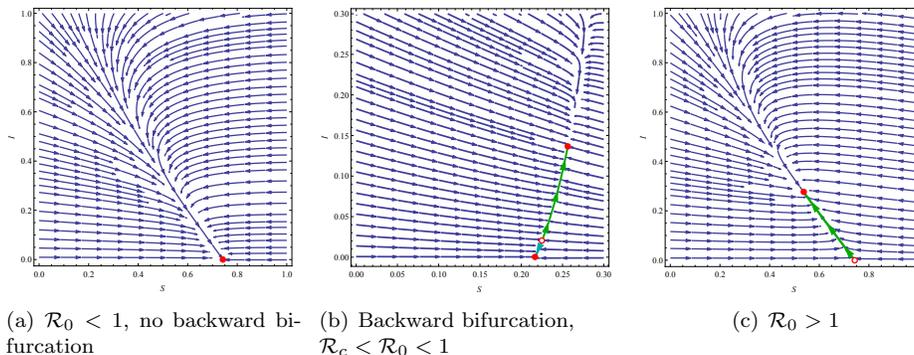


FIGURE 1. The structure of the global attractor for different parameter values

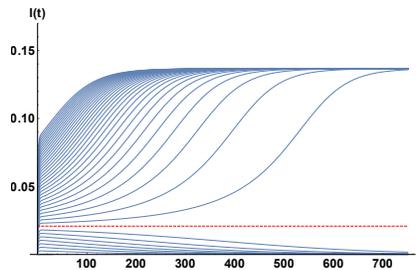


FIGURE 2.  $I(t)$  coordinate of solutions with different initial values converging to two different equilibria. The dashed line denotes the unstable equilibrium. Parameter values are  $a = 0.001$ ,  $f = 1$ ,  $n = 1$ ,  $\beta = 2.75$ ,  $\gamma = 0.909$ ,  $\mu = 0.15$ ,  $\nu = 0.546$ ,  $\sigma = 0.212$

The three possible scenarios characterized by Theorem 4.7 are depicted in Figure 1. The bistable case is illustrated by numerical simulation of solutions of (11) converging to two different equilibria in Figure 2.

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