Limited domain Radon transform

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Abstract. The problem in this article is to recover a function on $\mathbb{R}^n$ from its integrals known only on hyperplanes intersecting the unit ball.

1. Introduction

There is a number of papers concerning the reconstruction of a function from only a partial knowledge of the function's Radon transform. The two most known examples are the exterior Radon transform [6] and the limited angle Radon transform [4].

The classical Radon transform is defined for an integrable function f on $\mathbb{R}^n$ by

$$Rf(\omega, p) = \int_{H(\omega,p)} f(x) dx_h,$$

where $\omega \in S^{n-1}$ is a unit vector, $p \in \mathbb{R}_+$ and $Rf(\omega, p)$ is just the integral of f over the hyperplane $H(\omega, p) = \{x \in \mathbb{R}^n : \langle x, \omega\rangle = p\}$ by the surface measure dx_h on it.

In the limited angle case $Rf(\omega, p)$ is restricted in ω to a subset of S^{n-1}. The exterior Radon transform is the restriction of Rf to the set $p > 1$.

We define the *limited domain Radon transform* $R_L f$ of a function as the restriction of Rf onto the set $p \le 1$. In the next section we show its continuity on a weighted class $L^2_{\alpha,\beta}(\mathbb{E}^n)$ of square integrable functions that are zero in a neighborhood of the origin. In Section 3 we give the null space and range of R_L acting on $L^2_{\alpha,\beta}(\mathbb{E}^n)$ for the odd dimensional spaces. In Section 4 we do the same for even dimensional spaces, where the injectivity of R_L on $L^2_{\alpha,\beta}(\mathbb{E}^n)$ turns out.

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2. Preliminarities

The limited domain Radon transform of a function f integrable on the hyperplanes is defined in means of the formula $R_L f = \chi_{[0,1)} Rf$, where $\chi_{[0,1)}$ is the characteristic function of the interval $[0,1)$.

From the well known inversion formulas [3] we know that a square integrable function is determined by the limited domain Radon transform in the unit open ball B^n if the dimension n is odd. In even dimensions the situation is more complicated, but still there is an approximate possibility, as C. Berenstein and D. Walnut proved in [1] that to recover f to a given accuracy in a ball of radius $R > 0$ it is sufficient to know $Rf(\omega, p)$ only for $p < R + \alpha$ for some $\alpha > 0$ that depends on the accuracy desired (the greater the accuracy desired, the greater α must be).

Motivated by these observations, in the next two sections we carry out investigations on the following spaces. $L^2_{\alpha,\beta}(\mathbb{E}^n)$ is the Hilbert space of functions supported in $\mathbb{E}^n = \mathbb{R}^n \setminus \operatorname{Cl} B^n$ equipped with the inner product

$$\langle f, g\rangle_{\alpha,\beta} = \int_{\mathbb{E}^n} f(x)g(x)|x|^\alpha (1 - |x|^{-2})^\beta dx.$$

Estimate on the weight shows immediately, that $L^2_{\alpha,\beta}(\mathbb{E}^n) \supseteq L^2_{\alpha',\beta'}(\mathbb{E}^n)$ for $\alpha \le \alpha'$ and $\beta \le \beta'$. Denote $\mathbb{B}^n = \{H(\omega, p) : 0 \le p \le 1,\ \omega \in S^{n-1}\}$. Then the Hilbert space of functions on $\mathbb{B}^n$ with the inner product

$$\langle F, G\rangle_{\gamma,\delta} = \int_{S^{n-1}} \int_0^1 F(\omega, p)G(\omega, p)p^\gamma (1 - p^2)^\delta dp d\omega$$

is $L^2_{\gamma,\delta}(\mathbb{B}^n)$. Here $L^2_{\gamma,\delta}(\mathbb{B}^n) \supseteq L^2_{\gamma',\delta'}([0,1])$ for $\gamma' \le \gamma$ and $\delta' \le \delta$. Finally we need the Hilbert space $L^2_{\gamma,\delta}([0,1])$ of functions on $[0,1]$ with the inner product

$$\langle f, g\rangle_{\gamma,\delta} = \int_0^1 f(p)g(p)p^\gamma (1 - p^2)^\delta dp.$$

For $-1 < \gamma \le 0$, $-1 < \delta \le 0$, the polynomials are dense in this space, and $L^2_{\gamma,\delta}([0,1]) \supseteq L^2_{\gamma',\delta'}([0,1])$ for $\gamma' \le \gamma$ and $\delta' \le \delta$.

Our main tool is the spherical harmonic expansion in these spaces. Briefly, the spherical harmonics, $Y_{\ell,m}$ constitute a complete polynomial orthonormal system in the Hilbert space $L^2(S^{n-1})$. If $f \in C^\infty(S^{n-1} \times \mathbb{R}_+)$ and $f_{\ell,m}(p)$ is the corresponding coefficient of $Y_{\ell,m}(\omega)$ in the expansion of $f(\omega, p)$, ie. $f_{\ell,m}(p) = \int_{S^{n-1}} f(\omega, p)\overline{Y_{\ell,m}(\omega)} d\omega$, then the series $\sum_{\ell,m}^\infty f_{\ell,m}(p)Y_{\ell,m}(\omega)$ converges uniformly absolutely on compact subsets of $S^{n-1}\times\mathbb{R}$ to $f(\omega,p)$. For further references we refer to [7]. Below we shall use the expansions

$$g(\varphi,p)=\sum_{m=-\infty}^{\infty} g_m(p)\exp(im\varphi) \quad\text{and}\quad g(\omega,p)=\sum_{\ell,m}^{\infty} g_{\ell,m}(p)Y_{\ell,m}(\omega)$$

in dimension two and in higher dimensions, respectively. In dimension two, φ will mean the angle of the respective unit vector to a fixed direction.

The spherical expansions of the Radon transforms are well known [6]. Applying these to the functions in $L^2_{\alpha,\beta}(\mathbb{E}^n)$ we obtain

$$(R_L f)_m(p)=2\int_1^\infty f_m(q)\frac{\cos(m\arccos(p/q))}{\sqrt{1-p^2/q^2}}dq \tag{2.1}$$

for dimension two and

$$(Rf)_{l,m}(p)=\frac{|S^{n-2}|}{C_m^\lambda(1)}\int_1^\infty f_{\ell,m}(q)q^{n-2}C_m^\lambda\left(\frac{p}{q}\right)\left(1-\frac{p^2}{q^2}\right)^{\frac{n-3}{2}}dq \tag{2.2}$$

for higher dimensions, where C_m^λ is the Gegenbauer polynomial of degree m, $\lambda=(n-2)/2$ and $p\le 1$.

An important consequence of these expansions is the continuity of R_L.

Theorem 2.1. *R_L maps $L^2_{\alpha,\beta}(\mathbb{E}^n)$ continuously into $L^2_{\gamma,\delta}(\mathbb{B}^n)$, where $\alpha>n-2$, $\beta<1$, $\gamma>-1$ and* $\delta>\begin{cases}-1 & \text{if } n\ge 3\\ -1/2 & \text{if } n=2.\end{cases}$

Proof. First observe that

$$\|R_L f\|^2_{\gamma,\delta}=\sum_{\ell,m}\|Y_{\ell,m}\|_2^2\int_0^1 (R_L f)^2_{\ell,m}(p)p^\gamma(1-p^2)^\delta dp. \tag{2.3}$$

Using (2.1), (2.2) and that $|C_m^\lambda(x)|\le|C_m^\lambda(1)|$ for $|x|\le 1$ we can over estimate (2.3) by

$$c_1\int_0^1\left|\int_1^\infty f_{\ell,m}(q)q^{n-2}\left(1-\frac{p^2}{q^2}\right)^{\frac{n-3}{2}}dq\right|^2 p^\gamma(1-p^2)^\delta dp,$$

where c_1 is a suitable constant independent from m. For $n\ge 3$ this is less than

$$c_1\left|\int_1^\infty f_{\ell,m}(q)q^{n-2}dq\right|^2\int_0^1 p^\gamma(1-p^2)^\delta dp. \tag{2.4}$$

For $n=2$ we estimate with $|1-p^2/q^2|\geq 1-p^2$ and obtain

$$c_1\left|\int_1^\infty f_{\ell,m}(q)dq\right|^2\int_0^1 p^\gamma(1-p^2)^{\delta-1/2}dp. \tag{2.5}$$

The second integrals in (2.4) and (2.5), respectively, explain the restrictions on γ and δ given in the theorem.

The restrictions on α and β come from the following estimate of the first integrals in (2.4) and (2.5).

(2.6)

$$\begin{aligned}\left|\int_1^\infty f_{\ell,m}(q)q^{n-2}dq\right|^2 &= \left|\int_1^\infty f_{\ell,m}(q)q^{-1-\alpha}(1-q^{-2})^{-\beta}q^{\alpha+n-1}(1-q^{-2})^{\beta}dq\right|^2\\ &\leq \int_1^\infty f_{\ell,m}^2(q)q^{\alpha+n-1}(1-q^{-2})^{\beta}dq\times\\ &\quad\times\int_1^\infty q^{-2-2\alpha}(1-q^{-2})^{-2\beta}q^{\alpha+n-1}(1-q^{-2})^{\beta}dq.\end{aligned}$$

The last two integrals need to be finite to ensure the finiteness of (2.3). The first one is finite because $f\in L_{\alpha,\beta}(\mathbb{E}^n)$, the other is finite if $\alpha>n-2$ and $\beta<1$. ■

Throughout this paper we shall assume that α, β, γ and δ satisfy the conditions given in Theorem 2.1.

The weight $\left(1-p^2/q^2\right)^{(n-3)/2}$ is substantially different in odd and in even dimensions; it is polynomial in odd dimensions.

3. Odd dimensions

Let $n=2d+3$ and $d\geq 0$. Then $\lambda=d+1/2$ and

$$(R_Lf)_{l,m}(p)=\frac{|S^{n-2}|}{C_m^\lambda(1)}\int_1^\infty f_{\ell,m}(q)q^{n-2}D_m^\lambda\left(\frac{p}{q}\right)dq, \tag{3.1}$$

where $D_m^\lambda(x)=C_m^\lambda(x)(1-x^2)^d$.

Lemma 3.1. *D_m^λ is a polynomial of the form*

$$D_m^\lambda(x) = \sum_{i=0}^{d+[m/2]} d_{m,i}^\lambda x^{m+2d-2i},$$

where the coefficients $d_{m,i}^\lambda \neq 0$.

The proof is a simple consequence of the formulas (8.932.1), (9.136.1) and (9.131.1) in [2].

Substituting the polynomial form of D_m^λ into (3.1) we see that $(R_L f)_{l,m}$ is also a polynomial of the form

$$(R_L f)_{l,m}(p) = \frac{|S^{n-2}|}{C_m^\lambda(1)} \sum_{i=0}^{d+[m/2]} p^{m+2d-2i} d_{m,i}^\lambda \int_1^\infty f_{\ell,m}(q) q^{1-m+2i} dq. \tag{3.2}$$

Let $2\phi_{\ell,m}(x^2) = f_{\ell,m}(1/x)/x^n$ and change the variable $q = 1/\sqrt{x}$. Then the integral in (3.2) becomes

$$c_{\ell,m,i} = \int_1^\infty f_{\ell,m}(q) q^{1-m+2i} dq = \int_0^1 \phi_{\ell,m}(x) x^{d-i+m/2} \frac{dx}{\sqrt{x}}. \tag{3.3}$$

Theorem 3.2. *The null space of R_L in $L^2_{\alpha,\beta}(\mathbb{E}^n)$ is the closure of the span of functions*

$$g_{k,\ell,m}(\omega, q) = P_k^{(0,\varepsilon_m)}(2q^{-2} - 1) q^{-n} Y_{\ell,m}(\omega),$$

where $P_k^{(.,.)}$ is the Jacobi polynomial of degree k, $n-1 \leq \alpha < n$, $-1 < \beta \leq 0$,

$$k \geq \left[\frac{m+n-1}{2}\right] \quad \text{and} \quad \varepsilon_m = \begin{cases} -1/2 & \text{if } m \text{ even} \\ 0 & \text{if } m \text{ odd.} \end{cases}$$

Proof. The fact that $g_{k,\ell,m} \in L^2_{\alpha,\beta}(\mathbb{E}^n)$ follows from $\alpha < n$ and $-1 < \beta$. Since also $n-1 \leq \alpha$ and $\beta \leq 0$ it follows that $\phi_{\ell,m} \in L^2_{-1/2,0}([0,1])$.

According to (3.2) and (3.3) the function $R_L f$ vanishes if and only if $c_{\ell,m,i} = 0$ for all ℓ, m, and $0 \leq i \leq d + [m/2]$.

For m even this gives that

$$0 = \int_0^1 \phi_{\ell,m}(x) x^j \frac{dx}{\sqrt{x}} \quad \text{for all} \quad 0 \leq j \leq d + m/2. \tag{3.4}$$

The shifted Jacobi polynomials $P_k^{(0,-1/2)}(2x-1)$ constitute a complete orthogonal system on $[0,1]$ with respect to the weight $1/\sqrt{x}$ [2(8.904)], therefore $\phi_{\ell,m}$ must be in the closure of the span of $\{P_k^{(0,-1/2)}(2x-1)\}_{k=d+1+[m/2]}^{\infty}$.

For m odd we have

$$0 = \int_0^1 \phi_{\ell,m}(x)x^j\,dx \quad \text{for all} \quad 0 \le j \le d+(m-1)/2. \tag{3.5}$$

The Jacobi polynomials $P_k^{(0,0)}(2x-1)$ constitute a complete orthogonal system on $[0,1]$ [2(8.904)], so $\phi_{\ell,m}$ is in the closure of the span of $\{P_k^{(0,0)}(2x-1)\}_{k=d+1+[m/2]}^{\infty}$.

The results of (3.4) and (3.5) give the theorem.

∎

In the following we determine the range of the limited domain Radon transform.

Theorem 3.3. *R_L maps the $L^2_{\alpha,\beta}(\mathbb{E}^n)$ closure of the span of the functions*

$$h_{k,\ell,m}(\omega,q) = q^{-n-2k}Y_{\ell,m}(\omega), \qquad 0 \le k \le d+[m/2]$$

where $n-1 \le \alpha < n$ and $-1 < \beta < 0$, onto the $L^2_{\gamma,\delta}(\mathbb{B}^n)$ closure of the span of functions

$$F_{\ell,m}(\omega,p) = Y_{\ell,m}(\omega)\sum_{i=0}^{d+[m/2]} p^{m+2d-2i}b_{\ell,m,i}$$

continuously and bijectively.

Proof. The easy verification of $h_{k,\ell,m} \in L^2_{\alpha,\beta}(\mathbb{E}^n)$ and $F_{\ell,m} \in L^2_{\gamma,\delta}(\mathbb{B}^n)$ is left to the reader.

Since $R_L\colon L^2_{\alpha,\beta}(\mathbb{E}^n) \to L^2_{\gamma,\delta}(\mathbb{B}^n)$ is continuous by Theorem 2.1, it takes closed set to closed set. Further it is injective on the given functions by Theorem 3.2, therefore we only have to give coefficients $e'_{k,\ell,m} \in \mathbb{R}$ so that

$$F_{\ell,m} = R_L(f_{\ell,m}Y_{\ell,m}), \quad \text{where} \quad f_{\ell,m}Y_{\ell,m} = \sum_{k=0}^{d+[m/2]} e'_{k,\ell,m}h_{k,\ell,m}.$$

Eliminating $Y_{\ell,m}$ and reordering the summation we can search for $f_{\ell,m}$ in the form

$$f_{\ell,m}(q) = \sum_{k=0}^{d+[m/2]} e_{k,\ell,m}q^{-n}P_k^{(0,\varepsilon_m)}(2q^{-2}-1), \tag{3.6}$$

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where the coefficients $e_{k,\ell,m}$ are to be determined.

In $F_{\ell,m} = R_L(f_{\ell,m}Y_{\ell,m})$ the equality of two polynomials appears, that is equivalent to the coincidence of their corresponding coefficients. By (3.2) and (3.3) this gives

$$b_{\ell,m,i} = \frac{|S^{n-2}|}{C_m^\lambda(1)} d_{m,i}^\lambda \int_1^\infty f_{\ell,m}(q) q^{1-m+2i} dq \tag{3.7}$$

for $0 \le i \le d + [m/2]$. Substituting $f_{\ell,m}$ according to (3.6) this becomes a system of linear equations for $e_{k,\ell,m}$ with coefficients

$$\begin{aligned} a_{i,k,m} &= \int_1^\infty q^{1+2i-m-n} P_k^{(0,\varepsilon_m)}(2q^{-2}-1) dq \\ &= \frac{1}{2} \int_0^1 x^{d+[m/2]-i} P_k^{(0,\varepsilon_m)}(2x-1) x^{\varepsilon_m} dx, \end{aligned}$$

where $0 \le k \le d + [m/2]$. The Jacobi polynomials $P_k^{(0,\varepsilon_m)}(2x-1)$ constitute orthogonal polynomial system with respect to the weight x^{ε_m} on $[0,1]$, therefore $a_{i,k,m} = 0$ for $k + i > d + [m/2]$ and $a_{i,d+[m/2]-i,m} \neq 0$ for $0 \le i \le d + [m/2]$. By these properties the equations

$$b_{\ell,m,i} = \frac{|S^{n-2}|}{C_m^\lambda(1)} d_{m,i}^\lambda \sum_{k=0}^{d+[m/2]} e_{k,\ell,m} a_{i,k,m},$$

where $0 \le i,k \le d + [m/2]$, determine uniquely $e_{k,\ell,m}$. This completes the proof. ∎

4. Even dimensions

The nature of the problem changes considerable in even dimensions. On one hand we do not have easy to handle polynomials, but on the other hand just this sole inconvenience gives uniqueness on these spaces.

We have

$$(R_L f)_{l,m}(p) = |S^{n-2}| \int_1^\infty f_{\ell,m}(q) q^{2\lambda} \frac{D_m^\lambda(p/q)}{\sqrt{1-p^2/q^2}} dq, \tag{4.1}$$

where the dimension $n = 2\lambda + 2$ is even, $\lambda \ge 0$, $|S^0| = 2$ and

$$D_m^\lambda(x) = \begin{cases} \frac{C_m^\lambda(x)}{C_m^\lambda(1)}(1-x^2)^\lambda & \text{if } \lambda \ge 1 \\ \cos(m \arccos x) & \text{if } \lambda = 0 \end{cases} \tag{4.2}$$

is a polynomial of degree $m + 2\lambda$.

Lemma 4.1. *In the Taylor expansion*

$$\frac{D_m^\lambda(x)}{\sqrt{1-x^2}} = \sum_{i=0}^{\infty} d_{m,i}^\lambda x^{2i+m-2[m/2]},$$

the coefficients $d_{m,i}^\lambda$ *are not zero. The series is convergent absolutely on* $(-1,1)$ *and uniformly on* $[-1+\varepsilon, 1-\varepsilon]$ *for any* $1>\varepsilon>0$.

Proof. For $\lambda>0$, (8.932.1), (9.136.1) and (9.131.1) of [2] give

$$\frac{C_m^\lambda(x)}{(1-x^2)^{1/2-\lambda}} =$$
$$K_{m,\lambda} F\Big(\frac{1-m}{2}-\lambda, \frac{1+m}{2}, \frac{1}{2}; x^2\Big) + xK'_{m,\lambda} F\Big(\frac{2-m}{2}+\lambda, \frac{2+m}{2}, \frac{3}{2}; x^2\Big),$$

where F is the hypergeometric function [2(9.1)], $K'_{m,\lambda}=0$ for m even and $K_{m,\lambda}=0$ for m odd.

For $\lambda=0$ we prove

$$\frac{\cos(m \arccos x)}{\sqrt{1-x^2}} = K_m F\Big(\frac{1-m}{2}, \frac{1+m}{2}, \frac{1}{2}; x^2\Big) + xK'_m F\Big(\frac{2-m}{2}, \frac{2+m}{2}, \frac{3}{2}; x^2\Big)$$

with (8.942.1), (9.136.1) and (9.131.1) of [2], where $K'_m=0$ for m even and $K_m=0$ for m odd.

The well known properties of the hypergeometric functions and the formulas above prove the statement.

■

Let $b_{m,i}^\lambda = |S^{n-2}| d_{m,i}^\lambda$. According to Lemma 4.1 the Taylor expansion of (4.1) is

$$(R_L f)_{l,m}(p) = \sum_{i=0}^{\infty} b_{m,i}^\lambda p^{2i+m-2[m/2]} \int_1^\infty f_{\ell,m}(q) q^{2\lambda-2i+2[m/2]-m} dq, \tag{4.3}$$

Let $2\phi_{\ell,m}(x^2) = f_{\ell,m}(1/x)/x^n$. Changing the variable $q=1/\sqrt{x}$ the integral in (4.3) becomes

$$c_{\ell,m,i} = \int_0^1 \phi_{\ell,m}(x) x^i x^{\varepsilon_m} dx. \tag{4.4}$$

Theorem 4.2. *The limited domain Radon transform* R_L *is injective on* $L^2_{\alpha,\beta}(\mathbb{E}^n)$ *if* $n-1 \le \alpha < n$ *and* $-1 < \beta < 0$.

Proof. Easy calculation shows $\phi_{\ell,m} \in L^2_{-1/2,0}([0,1])$. Then $R_L f(\omega,p) = 0$ for $p<1$ implies $c_{\ell,m,i} = 0$ for all $i,m \ge 0$, hence $\phi_{\ell,m}$ must be zero. ∎

Because (4.3) is an infinite series, we need more sophisticated tools to determine the range of R_L.

Theorem 4.3. R_L *maps the* $L^2_{n-1,0}(\mathbb{E}^n)$ *closure of the span of the functions*

$$f_{k,\ell,m}(\omega,q) = q^{-n-2k}Y_{\ell,m}(\omega), \qquad 0 \le k$$

onto the $L^2_{\gamma,\delta}(\mathbb{B}^n)$ *closure of the span of functions*

$$F_{j,\ell,m}(\omega,p) = p^{2j+m-2[m/2]}Y_{\ell,m}(\omega) \qquad 0 \le j$$

continuously and bijectively.

Proof. The verification of $f_{k,\ell,m} \in L^2_{n-1,0}(\mathbb{E}^n)$ and $F_{j,\ell,m} \in L^2_{\gamma,\delta}(\mathbb{B}^n)$ is left to the reader. Since $R_L\colon L^2_{n-1,0}(\mathbb{E}^n) \to L^2_{\gamma,\delta}(\mathbb{B}^n)$ is continuous by Theorem 2.1, it takes closed set to closed set. Further it is injective by Theorem 4.2, therefore we only have to give coefficients $e'_{k,\ell,m} \in \mathbb{R}$ so that

$$F_{j,\ell,m} = R_L(f_{\ell,m}Y_{\ell,m}), \quad \text{where} \quad f_{\ell,m}Y_{\ell,m} = \sum_{k=0}^{\infty} e'_{k,\ell,m} f_{k,\ell,m}.$$

The functions $f_{\ell,m}$ of this form can be written in the form

$$f_{\ell,m}(q) = \sum_{k=0}^{\infty} e_{k,\ell,m} q^{-n} P_k^{(0,\varepsilon_m)}(2q^{-2}-1) \tag{4.5}$$

with unique coefficients $e_{k,\ell,m}$. To find these coefficients we consider (4.5) and (4.3), where the left hand side is substituted with $F_{j,\ell,m}$, as a system of linear equations with infinite dimensional matrix of entries

$$\begin{aligned} a_{i,k,m} &= \int_1^{\infty} q^{-2-2i+2[m/2]-m} P_k^{(0,\varepsilon_m)}(2q^{-2}-1)\,dq \\ &= \frac{1}{2}\int_0^1 x^i P_k^{(0,\varepsilon_m)}(2x-1)x^{\varepsilon_m}\,dx. \end{aligned}$$

By the orthogonality of $P_k^{(0,\varepsilon_m)}$ these are zero for $k > i$. For $i \geq k$

$$a_{i,k,m} = (k!)^2 \binom{i}{k} \prod_{h=0}^{k} \frac{1}{h+i+1+\varepsilon_m}$$

by [2(7.391.3)]. In virtue of the Stirling formula for the factorials we find the (not sharp) inequality

$$16 < \lim_{i\to\infty} \frac{a_{i,i,m}}{\left(\frac{i}{4e}\right)^i}. \tag{4.6}$$

The system of equations

$$\delta_{i,j} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} = b_{m,i}^{\lambda} \sum_{k=0}^{i} e_{k,\ell,m} a_{i,k,m}, \quad i, j \geq 0 \tag{4.7}$$

obtained from (4.3), by substituting $F_{j,\ell,m}$ into the left hand side and substituting $f_{\ell,m}$ into the right hand side according to (4.5), determine the coefficients $e_{k,\ell,m}$.

What remains to prove is only that the function defined by (4.5) with these coefficients is in $L^2_{n-1,0}(\mathbb{E}^n)$, that is equivalent to

$$\int_1^\infty \left(\sum_{k=0}^{\infty} e_{k,\ell,m} P_k^{(0,\varepsilon_m)}(2q^{-2}-1) \right)^2 q^{-2} dq < \infty. \tag{4.8}$$

Using Cauchy's inequality we can over estimate the left hand side by

$$\sum_{k=0}^{\infty} (k+1)^2 e_{k,\ell,m}^2 \sum_{k=0}^{\infty} \int_1^\infty \left(\frac{P_k^{(0,\varepsilon_m)}(2q^{-2}-1)}{k+1} \right)^2 q^{-2} dq. \tag{4.9}$$

First we estimate the second multiplier in (4.9). With a change in the variable we get

$$\int_1^\infty \left(P_k^{(0,\varepsilon_m)}(2q^{-2}-1) \right)^2 q^{-2} dq = \frac{\sqrt{2}}{4} \int_{-1}^{1} \left(P_k^{(0,\varepsilon_m)}(x) \right)^2 (1+x)^{-1/2} dx. \tag{4.10}$$

For m even, the right hand side evaluates explicitly $1/(4k+1)$ by [2(7.391.1)]. For m odd, $P_k^{(0,\varepsilon_m)}(x) = C_k^{1/2}(x)$ by [2(8.962.4)], which is less than $C_k^{1/2}(1) = 1$. Therefore the second multiplier in (4.9) is less then $\sum_{k=0}^{\infty} (k+1)^{-2} = \pi^2/6$.

To estimate the first multiplier in (4.9) we introduce for $N > j$ the N-dimensional vectors

$$\begin{aligned}\delta^N &= (b^\lambda_{m,j})^{-1}(\delta_{0,j},\dots,\delta_{i,j},\dots,\delta_{N,j})\\ e^N &= (e_{0,\ell,m}, 2e_{1,\ell,m},\dots,(k+1)e_{k,\ell,m},\dots,(N+1)e_{N,\ell,m})\end{aligned}$$

and the $N\times N$ quadratic matrix $A^N = (\frac{a_{i,k,m}}{k+1})^N_{i,k=0}$. With these notations, (4.7) can be rewritten as $\delta^N = A^N e^N$ and therefore

$$\sum_{k=0}^{N}(k+1)^2 e^2_{k,\ell,m} = \|e^N\|^2_2 \le \|\delta^N\|^2_2 \cdot \|\left(A^N\right)^{-1}\|^2_2. \tag{4.11}$$

Obviously $\|\delta^N\|^2_2 = (b^\lambda_{m,j})^{-2}$ does not depend on N. Since the matrix A^N is triangular so is its inverse $\left(A^N\right)^{-1}$, and the diagonal elements in $\left(A^N\right)^{-1}$ are $(i+1)/a_{i,i,m}$, that are also the eigenvalues of $\left(A^N\right)^{-1}$. In virtue of (4.6) the estimate

$$\|\left(A^N\right)^{-1}\|^2_2 \le \max_{0\le i\le M}\frac{i+1}{a_{i,i,m}}$$

is valid for M big enough. Together with (4.11) this completes the proof. ∎

As a consequence we obtain the following.

Theorem 4.4. *The range of R_L is dense in $L^2_{\gamma,\delta}(\mathbb{B}^n)$, where $-1 < \gamma \le 0$ and* $0 > \delta > \begin{cases} -1 & \textit{if } \lambda > 0 \\ -1/2 & \textit{if } \lambda = 0. \end{cases}$

Proof. According to Theorem 4.3 it is enough to prove that $\langle F, F_{j,\ell,m}\rangle_{\gamma,\delta} = 0$ for all $j,\ell,m \ge 0$ implies $F = 0$ for $F \in L^2_{\gamma,\delta}(\mathbb{B}^n)$.

Let $\Phi(\omega,p^2) = F(\omega,p)$. Then $F \in L^2_{\gamma,\delta}(\mathbb{B}^n)$ implies $\Phi \in L^2_{(\gamma-1)/2,\delta}(\mathbb{B}^n)$, hence

$$0 = \langle F, F_{j,\ell,m}\rangle_{\gamma,\delta} = \int_0^1 \Phi_{\ell,m}(p)p^j p^{\frac{\gamma-1}{2}+\frac{m}{2}-[\frac{m}{2}]}(1-p)^\delta dp$$

for all $j \ge 0$ implies $\Phi_{\ell,m} \equiv 0$ for all $\ell, m \ge 0$, i.e. $F = 0$. ∎

Note that we used the special values $\alpha = n-1$ and $\beta = 0$ in Theorem 4.3 only to simplify the estimate of (4.10) that would need tedious calculations in general.

However, the results can be extended to the spaces $L^2_{\alpha,\beta}$ for $n-1 \leq \alpha < n$ and $-1 < \beta \leq 0$.

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