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QUADRATIC HYPERBOLOIDS IN MINKOWSKI GEOMETRIES

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Abstract. A Minkowski plane is Euclidean if and only if at least one hyperbola is a quadric. We discuss the higher dimensional consequences too.

1. Introduction

Let $\mathcal{I}$ be an open, strictly convex, bounded domain in $\mathbb{R}^n$, (centrally) symmetric to the origin. Then a function $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$d(\boldsymbol{x}, \boldsymbol{y}) = \inf\{\lambda > 0 : (\boldsymbol{y} - \boldsymbol{x})/\lambda \in \mathcal{I}\}$$

is a metric on $\mathbb{R}^n$ [1, IV.24], and is called *Minkowski metric on* $\mathbb{R}^n$. It satisfies the strict triangle inequality, i.e. $d(A, B) + d(B, C) = d(A, C)$ is valid if and only if $B \in \overline{AC}$. A pair $(\mathbb{R}^n, d)$, where d is a Minkowski metric, is called *Minkowski geometry*, and $\mathcal{I}$ is called the *indicatrix* of it. In a Minkowski geometry $(\mathbb{R}^n, d)$,

(D_1) a set $\mathcal{H}^a_{d;F_1,F_2} := \{X : 2a = |d(F_1, X) - d(F_2, X)|\}$, where $a < d(F_1, F_2)/2$, is called a *hyperbola* if $n = 2$, and a *hyperboloid* in higher dimensions,

where $F_1, F_2 \in \mathbb{R}^n$ are called the *focuses*, and $a > 0$ is called the *radius*.

A hypersurface in $\mathbb{R}^n$ is called a *quadric* if it is the zero set of an irreducible polynomial of degree two in n variables. We call a hypersurface *quadratic* if it is part of a quadric. Since every isometric mapping between two Minkowski geometries is a restriction of an affinity, and every affinity maps quadrics to quadrics, the quadraticity of a metrically defined hypersurface is a geometric property in each Minkowski geometry. Thus the question arises whether the metrically defined hypersurfaces are quadrics. This question is answered for conics in [6].

We prove that (Theorem 4.3) a Minkowski plane is a model of the Euclidean plane, which means that the indicatrix is a bounded quadric [1, IV.25.4], if and only if at least one of the hyperbolas is a quadric, and that (Theorem 4.4) a Minkowski plane is analytic if and only if at least one of the hyperbolas is analytic.

As for higher dimensions, we prove (Theorem 5.1) that a Minkowski geometry is a model of the Euclidean geometry if and only if every central planar section of at least one quadric is either a hyperbola or an ellipse.

Similar problems for the ellipsoids were solved in [7].

2. Notations and preliminaries

Points of $\mathbb{R}^n$ are labeled as $A, B, \dots$, vectors are denoted by $\overrightarrow{AB}$ or $\boldsymbol{a}, \boldsymbol{b}, \dots$, but we use these latter notations also for points if the origin is fixed. The open

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segment with endpoints A and B is denoted by $\overline{AB}$, while $\overline{A}B$ denotes the open ray starting from A passing through B, finally, $AB = \overline{A}B \cup A\overline{B}$.

On an affine plane the *affine ratio* $(A,B;C)$ of collinear points A,B and C satisfies $(A,B;C)\overrightarrow{BC} = \overrightarrow{AC}$ [1, III.15.10], and the *cross ratio* of the collinear points A,B and C,D is $(A,B;C,D) = (A,B;C)/(A,B;D)$ [1, VI.40.17].

It is easy to observe in (D_1) that a hyperboloid intersects line F_1F_2, the *main axis*, in exactly two points, whose distance is twice the radius. Further notions are the *(linear) eccentricity* $c = d(F_1,F_2)/2$, the *numerical eccentricity* $\varepsilon = c/a$. The metric midpoint of the segment $\overline{F_1F_2}$ is called the *center*.

Notations $\boldsymbol{u}_\varphi = (\cos\varphi, \sin\varphi)$ and $\boldsymbol{u}_\varphi^\perp := (\cos(\varphi+\pi/2), \sin(\varphi+\pi/2))$ are frequently used. It is worth noting that, by these, we have $\frac{\mathrm{d}}{\mathrm{d}\varphi}\boldsymbol{u}_\varphi = \boldsymbol{u}_\varphi^\perp$.

A *quadric* in the plane has the equation of the form

$$\mathcal{Q}_{\mathfrak{s}}^{\sigma} := \left\{(x,y) : \begin{array}{ll} 1 = x^2 + \sigma y^2 & \text{if } \sigma \in \{-1,1\}, \\ x = y^2 & \text{if } \sigma = 0, \end{array}\right\} \tag{D_q}$$

in a suitable affine coordinate system $\mathfrak{s}$, and we call it *elliptic*, *parabolic*, or *hyperbolic*, if $\sigma = 1$, $\sigma = 0$, or $\sigma = -1$, respectively.

We usually *polar parameterize* the boundary $\partial\mathcal{D}$ of a non-empty domain $\mathcal{D}$ in $\mathbb{R}^2$ starlike with respect to a point $P \in \mathcal{D}$ so that $\boldsymbol{r}\colon [-\pi,\pi) \to \mathbb{R}^2$ is defined by $\boldsymbol{r}(\varphi) = r(\varphi)\boldsymbol{u}_\varphi$, where r is the *radial function* of $\mathcal{D}$ with *base point* P.

We call a curve *analytic* if the coordinates of its points depend on its arc length analytically.

3. Utilities

In this section, the underlying plane is Euclidean.

Lemma 3.1. *The border of a convex domain is an analytic curve if and only if any one of its radial functions is analytic.*

Proof. Let $\mathcal{D}$ be an open convex domain containing the origin $O = (0,0)$. Let $s \mapsto \boldsymbol{p}(s)$ be an arc length parametrization of $\partial\mathcal{D}$, where $s \geq 0$, and let $\varphi \mapsto \boldsymbol{r}(\varphi) = r(\varphi)\boldsymbol{u}_\varphi$ be a polar parametrization of $\partial\mathcal{D}$ on $[-\pi,\pi)$ such that $\boldsymbol{p}(0) = \boldsymbol{r}(-\pi)$. Then

$$s(\xi) = \int_{-\pi}^{\xi} |\dot{\boldsymbol{r}}(\varphi)| d\varphi = \int_{-\pi}^{\xi} \sqrt{\dot{r}^2(\varphi) + r^2(\varphi)}\, d\varphi, \tag{3.1}$$ ⟨2, 3⟩

hence the function $s\colon \xi \mapsto s(\xi)$ is strictly monotonously increasing, and therefore its inverse function $\sigma\colon s(\xi) \mapsto \xi$ exists and is strictly monotonously increasing.

First, assume the analyticity of r. Then, as r is bounded from below by a positive number, the integrand on the right-hand side of (3.1) is analytic, and therefore s is analytic. As $\dot{s}(\xi)$ is positive by (3.1), the analyticity of σ follows from the analytic inverse function theorem [3, Theorem 4.2], and this implies the analyticity of $\boldsymbol{p}(s) = \boldsymbol{r}(\sigma(s)) = r(\sigma(s))\boldsymbol{u}_{\sigma(s)}$.

Conversely, assume that $\boldsymbol{p}$ is analytic. As the derivatives of the cosine and sine functions do not vanish simultaneously, $\boldsymbol{u}_{\sigma(s)} = \boldsymbol{p}(s)/|\boldsymbol{p}(s)|$ proves that σ is analytic. As the derivative $\dot\sigma(t) = 1/\dot s(\sigma(t))$ vanishes nowhere, the analyticity of s follows again by the analytic inverse function theorem [3, Theorem 4.2]. Then the analyticity of $r(\xi) = \langle \boldsymbol{p}(s(\xi)), \boldsymbol{u}_\xi\rangle$ follows.

The lemma is proved. □

Notice that the differentiation of the last formula in the proof and then the substitution of the derivative of (3.1) give

$$\dot r(\xi) = \langle \dot{\boldsymbol{p}}(s(\xi)), \boldsymbol{u}_\xi\rangle \sqrt{\dot r^2(\xi) + r^2(\xi)}. \tag{3.2}$$ ⟨9⟩

Let $\mathcal{H}$ be a hyperbola with center O and focuses F_1 and F_2. Let us label the intersection points of F_1F_2 and $\mathcal{H}$ so that $A \in \overline{F_1B}$. We clearly have $O \in \overline{AB} \subset \overline{F_1F_2}$, so we can choose a point W on F_1F_2 such that $F_2 \in \overline{BW}$.

There exists an angle $\Phi \in (0, \pi/2)$ such that a unique point H exists on $\mathcal{H}$ for every $\varphi \in [0, \Phi) \cup (\pi - \Phi, \pi)$, such that $\angle WOH = \varphi$.

Given $\varphi_0 \in (0, \Phi)$, let $H_0 = H(\varphi_0)$, $\alpha_0 = \angle WF_1H_0$ and $\beta_0 = \angle WF_2H_0$. Assuming that H_{2i} is defined for an $i \in \mathbb{N}$, we define sequences recursively as follows (see Figure 3.1): $H_{2i+1} := \overline{F_1H_{2i}} \cap \mathcal{H}$, $\alpha_{2i+1} := \alpha_{2i}$, and $\beta_{2i+1} := \angle WF_2H_{2i+1}$; then $H_{2i+2} := \overline{F_2H_{2i+1}} \cap \mathcal{H}$, $\alpha_{2i+2} = \angle WF_1H_{2i+2}$, and $\beta_{2i+2} := \beta_{2i+1}$. We clearly have $\varphi_{2i} \in (0, \Phi)$ and $\varphi_{2i+1} \in (\pi - \Phi, \pi)$ for every $i \in \mathbb{N}$.

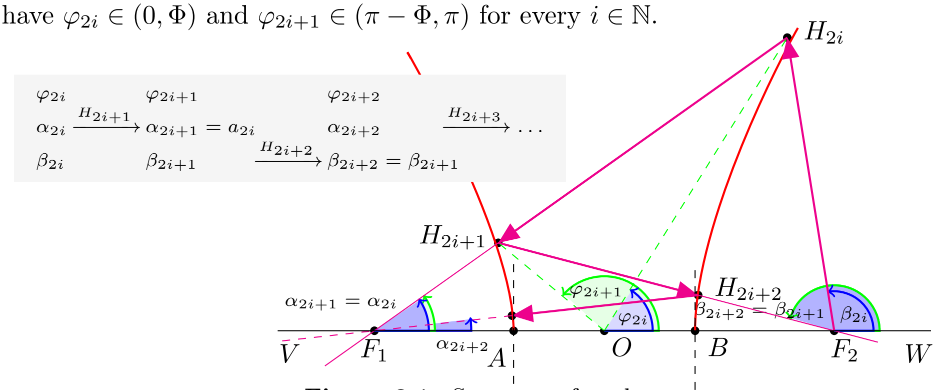

Figure 3.1. Sequence of angles

Lemma 3.2. *If $i \to \infty$, then α_{2i} and φ_{2i} tend to zero, β_{2i}, β_{2i+1}, and φ_{2i+1} tend to π, and $\alpha_{2i+2}/\alpha_{2i}$ tends to $(F_1, F_2; A, B)$.*

Proof. We clearly have $\varphi_{2i} < \Phi < \pi/2$ and $\varphi_{2i+1} > \pi - \Phi > \pi/2$, and therefore

$$\begin{aligned} \alpha_{2i} < \pi - \beta_{2i} \quad &\text{and} \quad \pi - \beta_{2i+1} < \alpha_{2i+1} \quad (\text{or} \quad \pi - \beta_{2i+2} < \alpha_{2i}),\\ \alpha_{2i+2} < \pi - \beta_{2i+2} \quad &\text{and} \quad \pi - \beta_{2i+1} < \alpha_{2i}, \end{aligned}$$

hence $\beta_{2i+2} > \beta_{2i}$, $\alpha_{2i+2} < \alpha_{2i}$, and $\pi - \beta_{2i+2} < \alpha_{2i} < \pi - \beta_{2i}$.

Thus, the sequences β_{2i}, β_{2i+1} increase monotonously, while the sequences α_{2i}, α_{2i+1} decrease monotonously. As these sequences are bounded, they are convergent.

Assuming $\lim_{i\to\infty}\beta_{2i}<\pi$, i.e. $\lim_{i\to\infty}(\pi-\beta_{2i})>0$, $\lim_{i\to\infty}\frac{\pi-\beta_{2i+2}}{\pi-\beta_{2i}}=1$, and $\lim_{i\to\infty}\frac{\alpha_{2i}}{\pi-\beta_{2i}}=1$ follow, hence the sinus law for triangle $\triangle F_1F_2H_{2i}$ implies

$$\lim_{i\to\infty}\frac{d(F_2,H_{2i})}{d(H_{2i},F_1)}=\lim_{i\to\infty}\frac{\sin\alpha_{2i}}{\sin(\pi-\beta_{2i})}\cdot\lim_{i\to\infty}\frac{\pi-\beta_{2i}}{\alpha_{2i}}=1,$$

which, by the continuity of d, gives $d(F_2,B)=d(B,F_1)$, a contradiction.

Thus $\lim_{i\to\infty}\beta_{2i}=\pi$, hence β_{2i+1}, and φ_{2i+1} also tend to π, and α_{2i}, α_{2i+1}, and φ_{2i} tend to zero.

So, observing Figure 3.1, we see that

$$\begin{aligned} h_1(\alpha_{2i})&:=d(F_1,H_{2i})\to d(F_1,B), & h_1(\alpha_{2i+1})&:=d(F_1,H_{2i+1})\to d(F_1,A),\\ h_2(\beta_{2i})&:=d(F_2,H_{2i})\to d(F_2,B), & h_2(\beta_{2i+1})&:=d(F_2,H_{2i+1})\to d(F_2,A). \end{aligned}\tag{3.3}$$ ⟨4⟩

The sine law for triangles $\triangle F_1F_2H_{2i}$ and $\triangle F_1F_2H_{2i+1}$ gives

$$\frac{h_2(\beta_{2i+1})}{h_1(\alpha_{2i+1})}=\frac{\sin\alpha_{2i+1}}{\sin(\pi-\beta_{2i+1})}\quad\text{and}\quad\frac{h_2(\beta_{2i+2})}{h_1(\alpha_{2i+2})}=\frac{\sin\alpha_{2i+2}}{\sin(\pi-\beta_{2i+2})},$$

respectively. Multiplying these by $\cos\beta_{2i+1}/\cos\alpha_{2i+1}$ and $\cos\beta_{2i+2}/\cos\alpha_{2i+2}$, respectively, and taking the ratio of the resulting fractions, we obtain

$$\frac{\tan\alpha_{2i+2}}{\tan\alpha_{2i}}=\frac{h_2(\beta_{2i+2})\cos\beta_{2i+2}}{h_1(\alpha_{2i+2})\cos\alpha_{2i+2}}\frac{h_1(\alpha_{2i+1})\cos\alpha_{2i+1}}{h_2(\beta_{2i+1})\cos\beta_{2i+1}}.$$

By (3.3), the right-hand side tends to $(F_1,F_2;A,B)$, so the proof is complete. □

Let $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ be curves in the plane with analytic arc length parametrizations on $[-1,1]$ such that $\boldsymbol{r}_1(0)=\boldsymbol{r}_2(0)$ and $\dot{\boldsymbol{r}}_1(0)=\dot{\boldsymbol{r}}_2(0)$. Let ℓ be the line through $\boldsymbol{r}_1(0)$ that is orthogonal to $\dot{\boldsymbol{r}}_1(0)$, and let F_1,F_2, and B be different points on ℓ such that $B\in\overline{F_1F_2}$ and $\boldsymbol{r}_1(0)\notin\{B,F_1,F_2\}$. Let $\boldsymbol{h}$ be an analytic arc length parameterization of a curve such that $B=\boldsymbol{h}(0)$ and $\dot{\boldsymbol{h}}(0)=\boldsymbol{u}_{\pi/2}$. Every point $H=\boldsymbol{h}(s)$ determines two straight lines $\ell_1:=F_1H$ and $\ell_2:=F_2H$ forming small angles α and $\gamma=\pi-\beta$ with ℓ, respectively. Let the straight line $\bar{\ell}_j$ $(j=1,2)$ through the midpoint O of the segment $\overline{F_1F_2}$ be parallel to ℓ_j. See Figure 3.2.

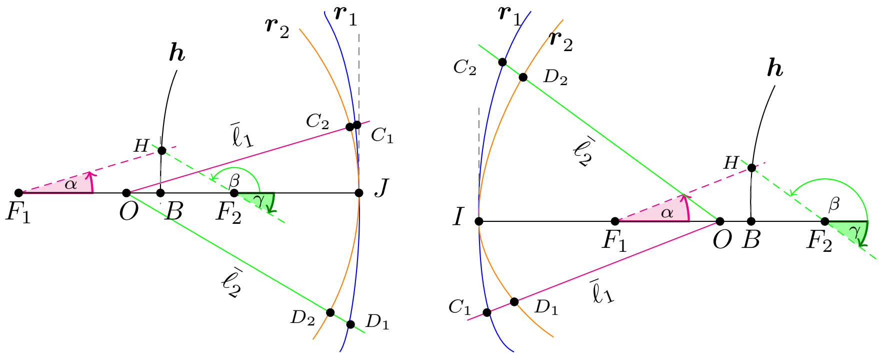

Figure 3.2. Specially placed curves with different lines

Denote the intersections of ℓ_1 and ℓ_2 with $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ by $\bar{C}_1$, $\bar{D}_1$ and $\bar{C}_2$, $\bar{D}_2$, respectively. Let s_i be the arc length parameters of $\boldsymbol{r}_i$ $(i = 1, 2)$, and define $\delta(\alpha) = \langle C_1 - D_1, \boldsymbol{u}_\alpha\rangle$ and $\delta(\gamma) = \langle C_2 - D_2, \boldsymbol{u}_\gamma\rangle$, where $\gamma = \beta - \pi$.

Lemma 3.3. *If the curves $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are different in every neighborhood of the point $K := \boldsymbol{r}_1(0)$, and H tends to B on the curve $\boldsymbol{h}$, then*

$$\frac{\delta(\alpha)}{\delta(\gamma)} \to (F_2, F_1; B)^k, \quad \text{for an integer } k \geq 2. \tag{3.4}$$ ⟨8⟩

Proof. If $\boldsymbol{r}_1^{(i)}(0) = \boldsymbol{r}_2^{(i)}(0)$ for every $i \in \mathbb{N}$, then, by the analyticity of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, $\boldsymbol{r}_1 = \boldsymbol{r}_2$ in a neighborhood of K, so $k := \min\{i \in \mathbb{N} : \boldsymbol{r}_1^{(i)}(0) \neq \boldsymbol{r}_2^{(i)}(0)\}$ is well defined and is at least two.

Letting $H^\perp$ be the orthogonal projection of H onto ℓ, L'Hôpital's rule gives

$$\frac{|F_2 - B|}{|F_1 - B|} = \lim_{s\to 0} \frac{|F_2 - H^\perp|}{|F_1 - H^\perp|} = \lim_{s\to 0} \frac{\tan\alpha}{-\tan\gamma} = -\lim_{s\to 0} \frac{\dot{\alpha}}{\dot{\gamma}}. \tag{3.5}$$ ⟨6⟩

If $\lim_{s\to 0} \frac{\delta(\alpha)}{\delta(\gamma)}$ exists, then L'Hôpital's rule can be used, so we get

$$\lim_{s\to 0} \frac{\delta(\alpha)}{\delta(\gamma)} = \lim_{s\to 0} \frac{\dot{\delta}(\alpha)\dot{\alpha}}{\dot{\delta}(\gamma)\dot{\gamma}} = \lim_{s\to 0} \frac{\dot{\delta}(\alpha)}{\dot{\delta}(\gamma)} \lim_{s\to 0} \frac{\dot{\alpha}}{\dot{\gamma}} = \cdots = \lim_{s\to 0} \frac{\delta^{(k)}(\alpha)}{\delta^{(k)}(\gamma)} \Big(\lim_{s\to 0} \frac{\dot{\alpha}}{\dot{\gamma}}\Big)^k = \Big(\lim_{s\to 0} \frac{\dot{\alpha}}{\dot{\gamma}}\Big)^k.$$

This proves the lemma. □

4. One hyperbola in a Minkowski plane

We start by considering the Minkowski plane $(\mathbb{R}^2, d_{\mathcal{I}})$ with indicatrix $\mathcal{I}$.

By [4, (ii) of Theorem 3] every straight line parallel to the main axis intersects a hyperbola in exactly two points, hence if a hyperbola is a quadric, then it is a hyperbolic quadric.

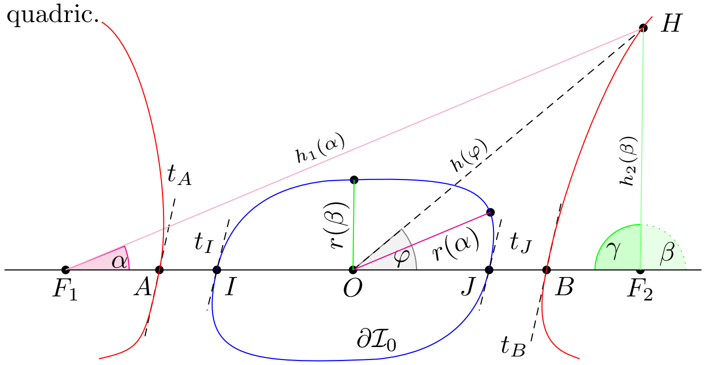

Figure 4.1. A hyperbola in a Minkowski plane

Let A, B be the intersections of line F_1F_2 with $\mathcal{H}^a_{d_{\mathcal{I}};F_1,F_2}$ such that $A \in \overline{F_1B}$ and $B \in \overline{AF_2}$. Let $\mathcal{I}_O$ be the translate of the indicatrix centered at the midpoint O of $\overline{F_1F_2}$, and let I, J be the intersections of line F_1F_2 with $\partial\mathcal{I}_O$, so that $I \in \overline{OF_1}$

and $J \in O\overline{F_2}$. Furthermore, let t_A, t_B and t_I, t_J, respectively, denote the tangents of the appropriate curve $\mathcal{H}^a_{d_\mathcal{I};F_1,F_2}$ or $\partial\mathcal{I}_O$ at A, B and I, J. See Figure 4.1.

Given the Euclidean metric d_e, we let r be the radial function of $\mathcal{I}_O$ with respect to O, $\alpha = \angle(HF_1O)$, $\gamma = \angle(HF_2B)$ ($\beta := \pi - \gamma$) and $\varphi = \angle(HOB)$ for the points H on the B-branch (that contains B) of $\mathcal{H}^a_{d_\mathcal{I};F_1,F_2}$. Finally, we define the lengths $h_1(\alpha) := d_e(F_1, H)$, $h_2(\beta) := d_e(F_2, H)$, and $h(\varphi) := d_e(O, H)$. Then $d_\mathcal{I}(F_1, H) = h_1(\alpha)/r(\alpha)$, and $d_\mathcal{I}(F_2, H) = h_2(\beta)/r(\beta)$, so we have

$$2a = \frac{h_1(\alpha)}{r(\alpha)} - \frac{h_2(\beta)}{r(\beta)}. \qquad (4.1) \quad \langle 6, 8\rangle$$

Lemma 4.1. *If the hyperbola $\mathcal{H}^a_{d_\mathcal{I};F_1,F_2}$ is a quadric, then $t_A \parallel t_B \parallel t_I \parallel t_J$.*

Proof. Since $\mathcal{H}^a_{d_\mathcal{I};F_1,F_2}$ is a quadric, φ and H are bijectively related, hence the functions $\alpha(\varphi)$, $\beta(\varphi)$ are well defined.

The symmetry of $\mathcal{I}$ entails that $t_I \parallel t_J$, and it also follows that the affine center of the quadric $\mathcal{H}^a_{d_\mathcal{I};F_1,F_2}$ coincides with its metric center O, hence $t_A \parallel t_B$ too.

Choose a Euclidean metric d_e so that $t_A \perp F_1F_2 \perp t_B$.

Differentiating (4.1) with respect to φ leads to

$$0 = \frac{\frac{dh_1(\alpha)}{d\alpha} r(\alpha) - h_1(\alpha)\frac{dr(\alpha)}{d\alpha}}{r^2(\alpha)}\frac{d\alpha}{d\varphi} - \frac{\frac{dh_2(\beta)}{d\beta} r(\beta) - h_2(\beta)\frac{dr(\beta)}{d\beta}}{r^2(\beta)}\frac{d\beta}{d\varphi}. \qquad (4.2) \quad \langle 6\rangle$$

As $\varphi = 0$ implies $\alpha = 0 = \pi - \beta$, and $\frac{dh_1}{d\alpha}(0) = \frac{dh_2}{d\beta}(\pi) = 0$ by $t_A \perp F_1F_2 \perp t_B$, (4.2) gives at $\varphi = 0$ that

$$r'(0)\Big[- h_1(0)\frac{d\alpha}{d\varphi}(0) + h_2(\pi)\frac{d\beta}{d\varphi}(0)\Big] = 0.$$

Applying (3.5) for the present configuration, we obtain that the second factor in the left-hand side is negative, hence $r'(0) = 0$. Thus $t_J \perp F_1F_2$, so the lemma follows. □

Lemma 4.2. *If the hyperbola $\mathcal{H}^a_{d_\mathcal{I};F_1,F_2}$ is an analytic curve in a neighborhood of A and B, then the curve $\partial\mathcal{I}_O$ is analytic in a neighborhood of I and J.*

Proof. By Lemma 3.1 and its proof, the functions h_1, h_2, the angles $\alpha(s)$, $\beta(s)$, and the inverses of the angles, where s is the arc length parameter, are clearly analytic, hence we deduce that $\beta(\alpha)$ and $\alpha(\beta)$ are also analytic functions.

As $x \mapsto 1/x$ is analytic in a neighborhood of 1, in order to prove that $r(\alpha)$ is analytic in a neighborhood of 0, it is enough to prove that $\bar{r}(\alpha) := 1/r(\alpha)$ is analytic in some neighborhood of 0. Bearing this in mind, we reformulate (4.1) as

$$\bar{r}(\alpha) = \frac{h_2(\gamma(\alpha))}{h_1(\alpha)}\bar{r}(\gamma(\alpha)) + \frac{2a}{h_1(\alpha)}. \qquad (4.3) \quad \langle 7, 9\rangle$$

Introduce the functions $f(\alpha) := \gamma(\alpha)$, $g(\alpha) := \frac{h_2(\gamma(\alpha))}{h_1(\alpha)}$, and $e(\alpha) := \frac{2a}{h_1(\alpha)}$. Then f, g and e are analytic in a neighborhood of 0, $f(0) = 0$, $\frac{df}{d\alpha}(0) = \frac{h_2(0)}{h_1(0)} < 1$,

$g(0) = \frac{h_2(0)}{h_1(0)} < 1$, and $h(0) = \frac{2a}{h_1(0)} < 1$. Furthermore, by (4.3), the function $\phi(\alpha) := \bar{r}(\alpha)$ solves the functional equation $\phi(\alpha) = g(\alpha)\phi(f(\alpha)) + h(\alpha)$. However, by [3, Theorem 4.6], such a functional equation has a unique solution, which additionally is analytic in a neighborhood of 0. Consequently, $r(\alpha)$ is the reciprocal of that unique analytic solution, so $\partial\mathcal{I}_O$ is analytic around J, and, by its symmetry, around I too. □

Theorem 4.3. *A Minkowski plane is a model of the Euclidean plane if and only if at least one hyperbola is a quadric.*

Proof. As every hyperbola is a quadric in the Euclidean plane, we only have to prove that a Minkowski plane is Euclidean if at least one hyperbola is a quadric.

Assume that $\mathcal{H}^a_{d_\mathcal{I};F_1,F_2}$ is a quadric.

We have $t_A \parallel t_I \parallel t_J \parallel t_B$ by Lemma 4.1, and, as every (planar) quadric is an analytical curve, the border $\partial\mathcal{I}_O$ is analytic in a neighborhood of I and J by Lemma 4.2, where O is the midpoint of $\overline{F_1F_2}$. Furthermore, by the central symmetry of $\mathcal{I}_O$ and the definition of $\mathcal{H}^a_{d_\mathcal{I};F_1,F_2}$, we have $c = d_\mathcal{I}(F_1, O)$, $\overrightarrow{AF_1} = \overrightarrow{F_2B}$ and $\overrightarrow{IA} = \overrightarrow{BJ}$, so O is the (affine) midpoint of both $\overline{IJ}$ and $\overline{AB}$. Additionally, we have $a \cdot d_\mathcal{I}(O, J) = d_\mathcal{I}(O, B)$, because the definition of $\mathcal{H}^a_{d_\mathcal{I};F_1,F_2}$ implies

$$\begin{aligned} 2d_\mathcal{I}(O, B) &= 2d_\mathcal{I}(O, F_2) - 2d_\mathcal{I}(F_2, B) \\ &= d_\mathcal{I}(F_1, O) + d_\mathcal{I}(O, F_2) - d_\mathcal{I}(F_2, B) + 2a - d_\mathcal{I}(F_1, B) = 2a. \end{aligned}$$

Being a hyperbolic quadric, $\mathcal{H}^a_{d_\mathcal{I};F_1,F_2}$ has two asymptotes ℓ_+ and ℓ_- through O. Let C_1 and C_2 be the points where they intersect the straight line t_A.

Fix the affine coordinate system such as $O = (0, 0)$, $J = (1, 0)$, and $C_1 = (c, \sqrt{c^2 - a^2})$, and choose the Euclidean metric d_e so that $\{(1, 0), (0, 1)\}$ is an orthonormal basis.

Let $\mathcal{C}$ denote the unit circle of d_e. See Figure 4.2.

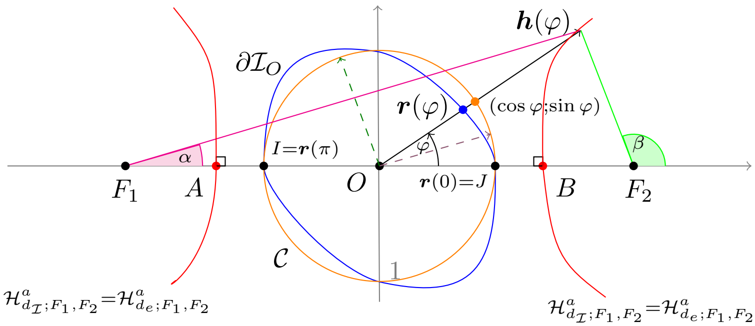

Figure 4.2. Coinciding hyperbolas $\mathcal{H}^a_{d_\mathcal{I};F_1,F_2} \equiv \mathcal{H}^a_{d_e;F_1,F_2}$

Then both $\mathcal{H}^a_{d_e;F_1,F_2}$ and $\mathcal{H}^a_{d_\mathcal{I};F_1,F_2}$ are hyperbolic quadrics, and have two common tangents t_A and t_B, two common asymptotes, and two common points A and B, hence they coincide.

By the definition of $\mathcal{H}^a_{d_e;F_1,F_2}$ we have $h_1(\alpha)-h_2(\beta)=2a$, which together with (4.1) implies

$$\delta(\alpha)=\delta(\beta)\frac{h_2(\beta)}{h_1(\alpha)+2a\delta(\beta)}, \tag{4.4}$$ ⟨8⟩

where $\delta(\alpha)=1-r(\alpha)$ is the radial difference of $\mathcal{C}$ and $\partial\mathcal{I}_O$.

If in every neighborhood of I curves $\mathcal{C}$ and $\partial\mathcal{I}_O$ differ, then (4.4) implies

$$\lim_{\varphi\to 0}\frac{\delta(\alpha)}{\delta(\beta)}=\frac{c-a}{c+a}=(F_2,F_1;B),$$

which, by (3.4), implies $(F_2,F_1;B)=1$. This contradicts $a>0$, so the curves $\mathcal{C}$ and $\partial\mathcal{I}_O$ coincide in a neighborhood of I.

However, if $\delta(\beta_0)\neq 0$ for a β_0, then no value of the 0-convergent sequence β_{2i} constructed in Lemma 3.2 can vanish by (4.4), therefore no β_0 can exist for which $\delta(\beta_0)\neq 0$. Similarly follows that no α exists for which $\delta(\alpha)\neq 0$, hence $\mathcal{C}$ and $\partial\mathcal{I}_O$ coincide. □

This kind of implication extends over to analyticity too.

Theorem 4.4. *The indicatrix of a Minkowski plane is analytic if and only if one of the hyperbolas of the Minkowski plane is analytic.*

Proof. First, assume that the Minkowski plane $(\mathbb{R}^2,d_{\mathcal{I}})$ is analytic.

We use the notations introduced in the previous sections, and consider the hyperbola $\mathcal{H}^a_{d_{\mathcal{I}};F_1,F_2}$.

Fix an arbitrary point $H_0\in\mathcal{H}^a_{d_{\mathcal{I}};F_1,F_2}$, and let the point $R_i\in\mathcal{I}$ $(i=1,2)$ be such that $O\overline{R_i}\parallel F_i\overline{H_0}$. Let the straight line t_i $(i=1,2)$ be tangent to $\mathcal{I}$ at R_i. Let d_e be the Euclidean metric which satisfies $t_2\perp OR_2$, $d_e(O,R_1)=d_e(O,R_2)$, and $d_e(O,J)=1$. Then we have

$$h_2^2(\beta)=h_1^2(\alpha)+4c^2-4h_1(\alpha)c\cos\alpha,\quad\text{and}\quad \beta=\arcsin\frac{h_1(\alpha)\sin\alpha}{h_2(\beta)}.$$

Substituting this into (4.1) results in the analytic equation

$$F(\alpha,h_1(\alpha)):=\Big(2a-\frac{h_1(\alpha)}{r(\alpha)}\Big)^2-\frac{h_1^2(\alpha)+4c^2-4h_1(\alpha)c\cos\alpha}{r^2\Big(\arcsin\frac{h_1(\alpha)\sin\alpha}{\sqrt{h_1^2(\alpha)+4c^2-4h_1(\alpha)c\cos\alpha}}\Big)}=0.$$

Since

$$\begin{aligned}\partial_2F(\alpha,h_1(\alpha))=&2\frac{-h_2(\beta)}{r(\beta)}\frac{-1}{r(\alpha)}-\frac{2h_1(\alpha)-4c\cos\alpha}{r^2(\beta)}+\\&+2\frac{h_2^2(\beta)}{r^3(\beta)}\frac{\dot r(\beta)}{\cos\beta}\Big(\frac{\sin\alpha}{h_2(\beta)}-\frac{1}{2}\frac{h_1(\alpha)\sin\alpha(2h_1(\alpha)-4c\cos\alpha)}{h_2^3(\beta)}\Big),\end{aligned}$$

$\partial_2 F(\alpha, h_1(\alpha))$ vanishes if and only if

$$-\frac{h_2(\beta)}{r(\alpha)} + \frac{h_1(\alpha) - 2c\cos\alpha}{r(\beta)} = \frac{h_2(\beta)\sin\alpha}{r(\beta)\cos\beta}\frac{\dot r(\beta)}{r(\beta)}\Big(1 - \frac{h_1(\alpha)(h_1(\alpha) - 2c\cos\alpha)}{h_2^2(\beta)}\Big).$$

By (3.2), we have $\frac{\dot r(\beta)}{r(\beta)} = \cot\theta$, where θ is the angle between $F_1\overline{H}$ and the tangent vector at H of $\mathcal{H}^a_{d_{\mathcal{I}};F_1,F_2}$. Furthermore, it can be easily seen that $h_1(\alpha) - 2c\cos\alpha = h_2(\beta)\cos(\beta-\alpha)$. Thus, the above equation is equivalent to

$$-\frac{h_2(\beta)}{r(\alpha)} + \frac{h_2(\beta)\cos(\beta-\alpha)}{r(\beta)} = \frac{\cot\theta}{r(\beta)\cos\beta}(h_2(\beta)\sin\alpha - h_1(\alpha)\cos(\beta-\alpha)\sin\alpha).$$

Since $h_2(\beta)\sin\beta = h_1(\alpha)\sin\alpha$, this equation simplifies to

$$\frac{-1}{r(\alpha)} + \frac{\cos(\beta-\alpha)}{r(\beta)} = \frac{\cot\theta}{r(\beta)}\frac{\sin\alpha - \sin\beta\cos(\beta-\alpha)}{\cos\beta} = -\sin(\beta-\alpha)\frac{\cot\theta}{r(\beta)}.$$

In sum, $\partial_2 F(\alpha, h_1(\alpha))$ vanishes if and only if

$$\frac{-r(\beta)}{r(\alpha)} + \sin(\beta-\alpha)(\cot\theta + \cot(\beta-\alpha)) = 0. \tag{4.5}$$ ⟨9⟩

At H_0 we have $\theta = \pi/2$, and $r(\beta) = r(\alpha)$, therefore (4.5) reduces to $\cos(\beta-\alpha) = 1$, resulting $\beta = \alpha$, a contradiction. Thus $\partial_2 F(\alpha, h_1(\alpha)) \neq 0$ at H_0, hence the analytic implicit function theorem [3, Theorem 4.1] implies the analyticity of h_1 in a neighborhood of α. As the point H_0 was chosen arbitrarily on $\mathcal{H}^a_{d_{\mathcal{I}};F_1,F_2}$, this proves that $\mathcal{H}^a_{d_{\mathcal{I}};F_1,F_2}$ is analytic.

Assuming now that the hyperbola is analytic, Lemma 4.2 proves the analyticity of the boundary of the indicatrix, where F_1F_2 intersects it. By (4.3) we have

$$\bar r(\beta(\alpha)) = \frac{h_1(\alpha)}{-h_2(\beta(\alpha))}\bar r(\alpha) + \frac{2a}{h_2(\beta(\alpha))}.$$

This shows that if $\bar r$ is analytic in an interval $(-\varepsilon, \varepsilon)$, then it is also analytic in the interval $(-\beta(\varepsilon), \beta(\varepsilon))$. According to Lemma 3.2, this means that the boundary of the indicatrix is analytic at all the directions. □

5. Quadrics in a Minkowski geometry

Before presenting the proof of Theorem 5.1, let us rephrase its statement for the planar case: if one hyperbola is a quadric, then the Minkowski plane is a model of the Euclidean geometry.

In a Minkowski geometry $(\mathbb{R}^n, d)$

(D_2) a set $\mathcal{E}^a_{d;F_1,F_2} := \{E : 2a = d(F_1,E) + d(E,F_2)\}$, where $a > d(F_1,F_2)/2$, is called an *ellipse* if $n = 2$, and an *ellipsoid* in higher dimensions,

where $F_1, F_2 \in \mathcal{M}$ are called the *focuses*, and $a > 0$ is called the *radius*.

Theorem 5.1. *A Minkowski geometry is a model of the Euclidean geometry if and only if through a point every planar section of at least one quadric is either a hyperbola or an ellipse.*

Proof. As every planar section of each hyperbolic quadric is either a hyperbola or an ellipse in the Euclidean geometry, it is enough to prove that a Minkowski geometry is Euclidean if every planar section of at least one quadric is either a hyperbola or an ellipse.

Let the quadric $\mathcal{Q}$ be such that its every planar section is either a hyperbola or an ellipse. If the planar section is a hyperbola, then Theorem 4.3 implies, that the parallel central planar section of the indicatrix is an ellipse. If the planar section is an ellipse, then [7, Theorem 4.3] implies, that the parallel central planar section of the indicatrix is an ellipse. Thus, the statement of the theorem follows immediately from [2, II.16.12], which states for any integers $1 < k < n$ that the border $\partial\mathcal{K}$ of a convex body $\mathcal{K} \subset \mathbb{R}^n$ is an ellipsoid if and only if every k-plane through an inner point of $\mathcal{K}$ intersects $\partial\mathcal{K}$ in a k-dimensional ellipsoid. □

We omit the easy proof of the following result that closes this paper.

Theorem 5.2. *A Minkowski geometry is a model of the Euclidean geometry if and only if there is a hyperplane and a point in it such that every line in the hyperplane through the point is parallel to main axis of some ellipsoid that is a quadric.*

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