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TILING A CIRCULAR DISC WITH CONGRUENT PIECES

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Abstract. In this note we prove that any monohedral tiling of the closed circular unit disc with $k \leq 3$ topological discs as tiles has a k-fold rotational symmetry. This result yields the first nontrivial estimate about the minimum number of tiles in a monohedral tiling of the circular disc in which not all tiles contain the center, and the first step towards answering a question of Stein appearing in the problem book of Croft, Falconer and Guy in 1994.

1. Introduction

A *tiling* of a convex body $\mathcal{K}$ in Euclidean d-space $\mathbb{R}^d$ is a finite family of compact sets in $\mathbb{R}^d$ with mutually disjoint interiors, called *tiles*, whose union is $\mathcal{K}$. A tiling is *monohedral*, if all tiles are congruent.

In this paper we deal with the monohedral tilings of the closed circular unit disc $\mathcal{B}^2$ with center O, in which the tiles are Jordan regions; i.e. are homeomorphic to a closed circular disc. The easiest way to generate such tilings, which we call *rotationally generated tilings*, is to rotate around O a simple, continuous curve connecting O to a point on the boundary $\mathcal{S}^1$ of $\mathcal{B}^2$. The following question, based on the observation that any tile of such a monohedral tiling of $\mathcal{B}^2$ contains O, seems to arise regularly in recreational mathematical circles [13]:

Question 1. *Are there monohedral tilings of $\mathcal{B}^2$ in which not all of the tiles contain O?*

The answer to Question 1 is affirmative; the usual examples to show this are the first two configurations in Figure 1. The following harder variant is attributed to Stein by Croft, Falconer and Guy in [2, last paragraph on p. 87].

Question 2 (Stein)**.** *Are there monohedral tilings of $\mathcal{B}^2$ in which O is in the interior of a tile?*

A systematic investigation of monohedral tilings of $\mathcal{B}^2$ was started in [6] by Haddley and Worsley. In their paper they called a monohedral tiling *radially generated*,

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if every tile is *radially generated*, meaning that its boundary is a continuous simple curve consisting of three parts: a circular arc of length α and two other curves one of which is the rotation of the other one about their common point by angle α. The following ambitious conjecture appears in [6, Conjecture 6.1].

Conjecture 1 (Haddley and Worsley)**.** *Every monohedral tiling is a subtiling of a radially generated tiling.*

A similar problem was investigated in [5] by Goncharov, who, for any O-symmetric convex body in $\mathbb{R}^d$, determined the smallest number of congruent copies of a subset of the body that cover the body. In the spirit of this approach we raise the following variant of Question 1:

Question 3. *What is the minimum cardinality $n(\mathcal{B}^2)$ of a monohedral tiling of $\mathcal{B}^2$ in which not all of the tiles contain O?*

As the configurations in Figure 1 show, we have $n(\mathcal{B}^2) \leq 12$. On the other hand, the lower bound $n(\mathcal{B}^2) \geq 3$ is also relatively easy to prove: it was posed as a problem in 2000 on the Russian Mathematical Olympiads [14]. Presently, to the authors' knowledge, the best bounds on $n(\mathcal{B}^2)$ are still the trivial ones: $3 \leq n(\mathcal{B}^2) \leq 12$.

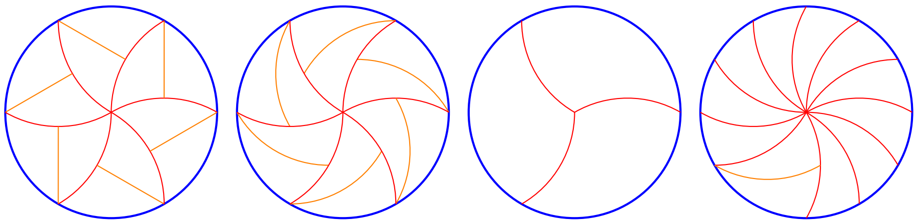

FIGURE 1. A non-radially[1], a radially, and a rotationally generated monohedral tiling of $\mathcal{B}^2$. In contrast to these three, the rightmost, radially generated monohedral tiling is not rotationally invariant.

Our main result is the following.

Theorem 1.1. *Any monohedral tiling of $\mathcal{B}^2$ with at most three topological discs is rotationally generated.*

This result implies Conjecture 1 for tilings with at most 3 tiles, yields the first nontrivial lower bound for $n(\mathcal{B}^2)$, and in particular proves that the answer for Question 2 is refuting for tilings with at most three tiles.

[1] It may be worth noting that this configuration also appears regularly in various places: this was chosen, for example, as the logo of the MASS program at Penn State University, it appears on the front page of five issues of the Hungarian problem-solving mathematical journal Középiskolai Matematikai Lapok [10], and it can be found also in the book [2, Figure C8].

Corollary 1.2. *We have* $n(\mathcal{B}^2) \geq 4$.

Corollary 1.3. *There is no monohedral tiling of* $\mathcal{B}^2$ *with at most three topological discs as tiles such that the center of* $\mathcal{B}^2$ *is contained in exactly one of them.*

In Section 2 we introduce the notions used in the paper, investigate the basic properties of monohedral tilings of $\mathcal{B}^2$, and prove a series of lemmas that we use in the proof of Theorem 1.1. In Section 3 we prove Theorem 1.1.

Finally, in Section 4 we collect our additional remarks and propose some open problems.

2. Notations and preliminaries

Throughout the proof, we denote by $\mathcal{B}^2$ the closed unit circular disc with the origin $O = (0,0)$ as its center, and its boundary by $\mathcal{S}^1 = \partial\,\mathcal{B}^2$. We say that two points $P, Q \in \mathcal{S}^1$ are *antipodal* if $d(P,Q) = 2$, where $d(\cdot,\cdot)$ denotes Euclidean distance. For points $P, Q \in \mathbb{R}^2$, the closed segment with endpoints P, Q is denoted by $\overline{PQ}$.

For any $P, Q \in \mathbb{R}^2$ with $d(P,Q) \leq 2r$, the r*-spindle* $\ominus^r_{P,Q}$ of two points P, Q is by definition (cf. [1] or [3]) the intersection of all Euclidean discs of radius $r > 0$ that contain P and Q. In other words, $\ominus^r_{P,Q}$ is the region bounded by the two circular arcs of radius $r > 0$ that connect P and Q and are not longer than a half-circle.

A set homeomorphic to $\mathcal{B}^2$ is called a *topological disc*. The boundary of a topological disc is a simple, closed, continuous curve, called *Jordan curve*. On the other hand, the Jordan–Schoenflies theorem [17] yields that every Jordan curve is the boundary of a topological disc. We remark that since all topological discs are compact, they are Lebesgue measurable; we denote their measure by $\operatorname{area}(\cdot)$. Nevertheless, there are topological discs (see, e.g. the Koch snowflake, or for more examples [16]) whose boundary is not rectifiable. Our next lemma, which we use in the proof, holds for these topological discs as well.

Lemma 2.1. *Let* Γ *be a Jordan curve and* $\mathcal{C}$ *be a simple, continuous curve. Then* Γ *contains finitely many congruent copies of* $\mathcal{C}$ *which are mutually disjoint, apart from possibly their endpoints.*

Proof. Assume for contradiction that Γ contains infinitely many congruent copies $\mathcal{C}_n$ $(n = 1, 2, \ldots)$ of $\mathcal{C}$ which are mutually disjoint, apart from possibly their endpoints. Let P_n and Q_n denote the endpoints of $\mathcal{C}_n$. Since Γ is compact, we may assume that $\lim_{n\to\infty} P_n = P$ and $\lim_{n\to\infty} Q_n = Q$ for some $P, Q \in \Gamma$. By the properties of congruence, $P \neq Q$. On the other hand, since Γ is homeomorphic to $\mathcal{S}^1$, the congruent copies of $\mathcal{C}$ correspond to mutually nonoverlapping circular arcs on $\mathcal{S}^1$. Clearly, this implies that $P = Q$, a contradiction. □

Lemma 2.2. *Let* $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$*, where* $\mathcal{D}, \mathcal{D}_1$ *and* $\mathcal{D}_2$ *are topological discs, and* $\operatorname{int}\mathcal{D}_1 \cap \operatorname{int}\mathcal{D}_2 = \emptyset$. *Then* $\mathcal{S}_1 = \mathcal{D}_1 \cap \partial\,\mathcal{D}$, $\mathcal{S}_2 = \mathcal{D}_2 \cap \partial\,\mathcal{D}$ *and* $\mathcal{S} = \partial\,\mathcal{D}_1 \cap \partial\,\mathcal{D}_2$ *are simple continuous curves.*

Proof. As $\mathcal{D}$ is a topological disc, we have a homeomorphism χ such that $\chi(\mathcal{D}) = \mathcal{B}^2$. Since the statement of the lemma is topologically invariant, it is sufficient to prove it in the case $\mathcal{D} = \mathbf{B}^2$. Thus, we may assume that $\mathcal{S}_i = \mathcal{S}^1 \cap \mathcal{D}_i$ for $i = 1, 2$, where we observe that since $\mathcal{D}_i$ and $\mathcal{S}^1$ are closed, so is $\mathcal{S}_i$.

First, we show that $\mathcal{S}_1$ and $\mathcal{S}_2$ are connected. Assume, for example, that some $X_1, Y_1 \in \mathcal{S}_1$ cannot be connected by an arc in $\mathcal{S}_1$. Then there are some points $X_2, Y_2 \notin \mathcal{S}_1$ that separate X_1 and Y_1 in $\mathcal{S}^1$. Clearly, we have $X_2, Y_2 \in \mathcal{S}_2$. For any $i = 1, 2$, since $\mathcal{D}_i$ is a topological disc, there is a simple, continuous curve γ_i with endpoints X_i, Y_i such that apart from these points γ_i is contained in $\operatorname{int} \mathcal{S}_i$. By continuity, $\gamma_1 \cap \gamma_2 \neq \emptyset$, implying that $\operatorname{int} \mathcal{D}_1 \cap \operatorname{int} \mathcal{D}_2 \neq \emptyset$, a contradiction. Thus, $\mathcal{S}_1$ and $\mathcal{S}_2$ are connected, which yields that they are closed circular arcs in $\mathcal{S}^1$. Let the (common) endpoints of these arcs be P and Q.

The points $P, Q \in \mathcal{S}_1 \cap \mathcal{S}_2$ are also in $\partial \mathcal{D}_1 \cap \partial \mathcal{D}_2$, hence they are connected by a simple continuous curve in $\partial \mathcal{D}_1 \setminus \mathcal{S}_1$ and also in $\partial \mathcal{D}_2 \setminus \mathcal{S}_2$. These curves coincide because $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$, hence it is $\mathcal{S}$, and the proof of Lemma 2.2 is complete. □

Lemma 2.3. *Let $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$ be a tiling of the topological disc $\mathcal{D}$ where for $i = 1, 2, 3$, $\mathcal{D}_i$ is a topological disc such that $\mathcal{S}_i = \mathcal{D}_i \cap \partial \mathcal{D}$ is a nondegenerate simple continuous curve. Then $\mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathcal{D}_3$ is a singleton $\{M\}$, and for any $i \neq j$, $\mathcal{D}_i \cap \mathcal{D}_j$ is a simple continuous curve connecting M and a point in $\partial \mathcal{D}$.*

Proof. Suppose for contradiction that there are two distinct points $M_1, M_2 \in \mathcal{D}_i$ for $i = 1, 2, 3$. For any i, let Γ_i be a simple, continuous curve connecting M_1 and M_2 which is contained in $\operatorname{int} \mathcal{D}_i$, apart from M_1 and M_2. Note that for any $i \neq j$, $\Gamma_i \cup \Gamma_j$ is a simple, closed, continuous curve. Thus, the union of a pair of the curves, say $\Gamma_1 \cup \Gamma_2$ encloses the third one. This implies that $\Gamma_1 \cup \Gamma_2$ encloses $\mathcal{D}_3$. Since $M_1, M_2 \notin \mathcal{S}^1$ by our conditions, it follows that $\mathcal{D}_3$ is disjoint from $\mathcal{S}^1$; a contradiction. Thus, $\mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathcal{D}_3$ contains at most one point. On the other hand, since the closure $\mathcal{X}_1$ of $(\partial \mathcal{D}_1) \setminus \mathcal{S}_1$ is a simple, connected curve and it can be decomposed into the closed sets $\mathcal{X}_1 \cap \mathcal{D}_2$ and $\mathcal{X}_1 \cap \mathcal{D}_3$, it follows that these sets intersect, that is $\mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathcal{D}_3$ is not empty. Thus, $\mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathcal{D}_3 = \{M\}$ for some $M \in \operatorname{int} \mathcal{D}$.

To prove the second part of Lemma 2.3, we may apply an argument like in the proof of Lemma 2.2. □

By the *circumcircle* of a topological disc $\mathcal{D}$ we mean the unique smallest closed Euclidean circle encompassing $\mathcal{D}$. The convex hull of the circumcircle is the *circumdisc* of $\mathcal{D}$, the radius of the circumcircle is the *circumradius* of $\mathcal{D}$. Observe that the center of the circumcircle $\mathcal{C}$ of $\mathcal{D}$ is in $\operatorname{conv}(\mathcal{C} \cap \mathcal{D})$, as otherwise a smaller circle would encompass $\mathcal{D}$.

Lemma 2.4. *Assume that $\mathcal{S}^1$ is the common circumcircle of the non-overlapping congruent topological discs $\mathcal{D}_1$ and $\mathcal{D}_2$. Then there is a diameter $\overline{PQ}$ of $\mathcal{B}^2$ separating $\mathcal{S}_1 = \mathcal{D}_1 \cap \mathcal{S}^1$ and $\mathcal{S}_2 = \mathcal{D}_2 \cap \mathcal{S}^1$. Furthermore, any congruence g with $g(\mathcal{D}_1) = \mathcal{D}_2$ is either the reflection about the line of $\overline{PQ}$, or the reflection about O.*

Proof. Using the idea of the proof of Lemma 2.2, it follows that there are no pairs of points $X_1, Y_1 \in \mathcal{S}_1$ and $X_2, Y_2 \in \mathcal{S}_2$ that strictly separate each other on $\mathcal{S}^1$. In other words, there is a line ℓ separating $\mathcal{S}_1$ and $\mathcal{S}_2$. On the other hand, as $O \in \operatorname{conv} \mathcal{S}_1 \cap \operatorname{conv} \mathcal{S}_2$, ℓ contains O and $\ell \cap \mathcal{S}^1 \subseteq \mathcal{S}_1 \cap \mathcal{S}_2$, proving the first statement with $\{P, Q\} = \ell \cap \mathcal{S}^1$. We note that from this argument it also follows that $\mathcal{S}_1 \cap \mathcal{S}_2 = \{P, Q\}$.

Consider some isometry g with $g(\mathcal{D}_1) = \mathcal{D}_2$. The uniqueness of the circumcircle clearly implies that $g(\mathcal{S}^1) = \mathcal{S}^1$, and thus, $g(\{P, Q\}) = \{P, Q\}$. This implies that g is either the reflection about the line of $\overline{PQ}$, the reflection about the line bisecting $\overline{PQ}$, or the reflection about O. We show that the conditions of the lemma exclude the second case: Consider a simple, continuous curve Γ from P to Q such that $\Gamma \setminus \{P, Q\} \subset \operatorname{int} \mathcal{D}_1$. Then at least one point R of Γ lies on the line $\ell^\perp$ bisecting $\overline{PQ}$. If g is the reflection about $\ell^\perp$, then $g(R) = R$, and hence, $R \in \operatorname{int} \mathcal{D}_1 \cap \operatorname{int} \mathcal{D}_2$; a contradiction. □

In the remaining part of Section 2, we deal only with a monohedral tiling of $\mathcal{B}^2$, where the tiles $\mathcal{D}_i$, $i = 1, 2, \ldots, n$, are congruent copies of a topological disc $\mathcal{D}$. For any $j \neq 1$, we fix an isometry g_{1j} mapping $\mathcal{D}_1$ into $\mathcal{D}_j$, and for any values of i, j, we set $g_{ij} = g_{1i}^{-1} \circ g_{1j}$. Then, by definition, we have $g_{ji} = g_{ij}^{-1}$ for all values of i, j. Finally, we set $\mathcal{S}_i = \mathcal{D}_i \cap \mathcal{S}^1$ for all values of i.

Lemma 2.5. *If $\mathcal{D}$ contains two points at the distance 2, then $n = 1$ or $n = 2$, and the tiling is rotationally generated.*

Proof. If $\mathcal{D}$ contains two points at the distance 2, then each tile contains two antipodal points of $\mathcal{B}^2$. Thus, $\mathcal{B}^2$ is the circumdisc of each tile, which implies that $g_{ij}(\mathcal{B}^2) = \mathcal{B}^2$ for all values of i, j. Since $O \in \mathcal{D}_i$ for some value of i, it also yields that $O \in \mathcal{D}_i$ for all values of i. Then, by Lemma 2.4, there is a diameter $\overline{PQ}$ of $\mathcal{B}^2$ whose endpoints belong to every tile, and the congruence between any two of them is either a reflection about the line through $\overline{PQ}$, or the reflection about the midpoint of $\overline{PQ}$. This implies that there are at most two tiles.

To prove that the tiling is rotationally generated, assume that $n = 2$, and $\mathcal{D}_2$ is a reflected copy of $\mathcal{D}_1$ about the line through $\overline{PQ}$. Since in this case $\mathcal{D}_1$ and $\mathcal{D}_2$ are the two closed half discs of $\mathcal{B}^2$ containing $\overline{PQ}$ in their boundaries, the statement follows. □

Lemma 2.6. *For all values of i, $\mathcal{S}_i$ $(i = 1, \ldots, n)$ is a closed, connected arc in $\mathcal{S}^1$.*

Proof. As $\mathcal{S}_1$ is compact, there are points $P, Q \in \mathcal{S}_1$ farthest from each other in $\mathcal{S}_1$. If P, Q are antipodal points of $\mathcal{S}^1$, then every $\mathcal{D}_i = g_{1i}(\mathcal{D}_1)$ $(i = 1, \ldots)$ contains antipodal points, hence $\mathcal{B}^2$ is the circumdisc of every tile. Then Lemma 2.5 yields that $n = 1$ or $n = 2$. The case $n = 1$ is trivial, and if $n = 2$, then by Lemma 2.4, there is a diameter $\overline{PQ}$ separating $\mathcal{S}_1$ and $\mathcal{S}_2$, which implies that $\mathcal{S}_1$ and $\mathcal{S}_2$ are closed half-circles. Thus, we may assume that P, Q are not antipodal.

Let $\Gamma \subset \mathcal{S}^1$ be the shorter arc connecting P and Q. We show that $\Gamma \subset \mathcal{D}_1$.

For contradiction, suppose that a point $X \in \Gamma$ does not belong to $\mathcal{D}_1$. Then, without loss of generality, we may assume that $X \in \mathcal{D}_2$, and that $X \neq P$, $X \neq Q$.

Let $r > 0$ be the radius of the circumdisc $\mathcal{B}$ of $\mathcal{D}_1$. Since $\mathcal{D}_1$ is compact, and it does not contain antipodal points of $\mathcal{S}^1$, we have $r < 1$, implying that $\ominus^r_{P,Q}$ contains $\Gamma \setminus \{P, Q\}$ in its interior. Thus $\Gamma \subset \mathcal{B}$, and $\Gamma \setminus \{P, Q\} \subset \operatorname{int} \mathcal{B}$. Let Γ' be a continuous curve connecting P and Q such that $\Gamma' \setminus \{P, Q\} \subset \operatorname{int} \mathcal{D}_1$. This yields that $\Gamma \cup \Gamma'$ is a simple, continuous, closed curve in $\mathcal{B}$ enclosing $\mathcal{D}_2$. which, by the congruence of $\mathcal{D}_1$ and $\mathcal{D}_2$, implies that the $\mathcal{B}$ is the circumdisc of $\mathcal{D}_2$ as well. Hence, by Lemma 2.4 it follows that $\operatorname{conv}(\partial \mathcal{B} \cap \mathcal{D}_1 \cap \mathcal{D}_2)$ is a diameter δ of $\mathcal{B}$. As P, Q are the only points of $\Gamma \cup \Gamma'$ that may fall on $\partial \mathcal{B}$, we have $\delta = \overline{PQ}$.

From Lemma 2.4 it also follows that g_{12} is the reflection about the line of $\overline{PQ}$, or the reflection about the midpoint of $\overline{PQ}$, and in particular $g_{21} = g_{12}$. On the other hand, observe that $g_{12}(\ominus^1_{P,Q}) = \ominus^1_{P,Q} = \mathcal{B}^2 \cap g_{12}(\mathcal{B}^2)$. Since $\mathcal{D}_1 \subset \mathcal{B}^2$ and $\mathcal{D}_1 = g_{12}(\mathcal{D}_2) \subset g_{12}(\mathcal{B}^2)$, this implies that $\mathcal{D}_1, \mathcal{D}_2 \subset \ominus^1_{P,Q}$. Now, if there is a point $R \in \mathcal{D}_2 \cap \partial \ominus^1_{P,Q} \setminus \Gamma$ then R and X can be connected with a continuous curve in $\operatorname{int} \mathcal{D}_2$, while $P, Q \in \operatorname{int} \mathcal{D}_1$, a contradiction. Hence $\mathcal{D}_2 \cap \partial \ominus^1_{P,Q} \subset \Gamma$, and accordingly $\mathcal{D}_1 \cap \mathcal{S}^1 = \{P, Q\}$, and, in particular, $\mathcal{D}_1 \cap \Gamma = \{P, Q\}$.

Assume that there is an interior point Y of Γ that belongs to, say, $\mathcal{D}_3$. Since $P, Q \in \mathcal{D}_2$, we may repeat the argument in the previous paragraph, replacing $\mathcal{D}_1$ and $\mathcal{D}_2$ by $\mathcal{D}_2$ and $\mathcal{D}_3$, respectively, and obtain that $\mathcal{D}_2 \cap \Gamma = \{P, Q\}$ contradicting our assumption that there is an interior point $X \in \mathcal{D}_2$ of Γ. Thus, $\Gamma \subset \mathcal{D}_2$, which yields by Lemma 2.4 that $\mathcal{D}_2 \cap \mathcal{S}^1 = \Gamma$ and $\ominus^1_{P,Q} = \mathcal{D}_1 \cup \mathcal{D}_2$. From this, in particular, it follows that $\operatorname{area}(\mathcal{D}_1) = \operatorname{area}(\mathcal{D}_2) = \operatorname{area}(\ominus^1_{P,Q})/2$.

Since for all values of i, $g_{2i}(\{P, Q\}) \subset \mathcal{B}^2$, the definition of 1-spindle implies that $\mathcal{D}_i \subset \ominus^1_{g_{2i}(P), g_{2i}(Q)} \subset \mathcal{B}^2$, and $g_{2i}(\mathcal{D}_2) \setminus \Gamma \subset \operatorname{int} \ominus^1_{g_{2i}(P), g_{2i}(Q)}$ is disjoint from $\mathcal{S}^1$. In other words, the sets $g_{2i}(\Gamma)$ cover $\mathcal{S}^1$. Note that these arcs may intersect each other only at their endpoints, and if $|\mathcal{S}^1 \cap g_{2i}(\Gamma)| \geq 3$, then $g_{2i}(\Gamma) \subset \mathcal{S}^1$. Thus, $\mathcal{S}^1$ can be decomposed into finitely many, say $k < n$ circular arcs, each of which is congruent to Γ.

Let $s = 2\pi/k$ denote the arclength of Γ. Then $ks = 2\pi$ on one hand, and

$$\frac{\pi}{n} = \frac{\operatorname{area}(B^2)}{n} = \operatorname{area}(\mathcal{D}_2) = \frac{\operatorname{area}(\ominus^1_{P,Q})}{2} = \frac{s - \sin s}{4} = \frac{\frac{2\pi}{k} - \sin\frac{2\pi}{k}}{4}$$

on the other hand. Thus, we have $\sin\frac{2\pi}{k} = \pi(\frac{2}{k} - \frac{4}{n})$. The left-hand side is an algebraic number (see, e.g. [18, Theorem 2.1]), from which $\frac{2}{k} = \frac{4}{n}$ follows, hence

$\sin \frac{2\pi}{k} = 0$, implying that k is a divisor of 2, contradicting our assumption that Γ is shorter than a half-circle. □

Remark 2.7. Since $\mathcal{S}_i \subset \partial \operatorname{conv} \mathcal{D}_i$, it follows that for all values of i, j, we have $g_{ij}(\mathcal{S}_i) \subset \partial \operatorname{conv} \mathcal{D}_j$.

Remark 2.8. For any values of i, j, k, the arcs $g_{ik}(\mathcal{S}_i)$ and $g_{jk}(\mathcal{S}_j)$ share at most some of their endpoints, or they coincide.

Proof. Observe that since for any i, $\mathcal{S}_i$ is contained in the convex hull of $\mathcal{D}_i$, if the arcs $g_{ik}(\mathcal{S}_i)$ and $g_{jk}(\mathcal{S}_j)$ are not disjoint apart from (possibly) their endpoints, then $g_{ik}(\mathcal{S}_i) \cap g_{jk}(\mathcal{S}_j)$ is a nondegenerate unit circular arc, and thus, $g_{ik}(\mathcal{S}_i) \cup g_{jk}(\mathcal{S}_j)$ lies on a unit circle $\mathcal{S}$.

Since $\mathcal{S}_i = g_{ki}(g_{ik}(\mathcal{S}_i)) \subset \mathcal{S}^1$, we have $g_{ki}(\mathcal{S}) = \mathcal{S}^1$. Thus, $\mathcal{S}_i \cup g_{ki}(g_{jk}(\mathcal{S}_j)) \subset \mathcal{S}^1 \cap \mathcal{D}_i = \mathcal{S}_i$. This implies that $g_{jk}(\mathcal{S}_j) \subseteq g_{ik}(\mathcal{S}_i)$. The containment relation $g_{ik}(\mathcal{S}_i) \subseteq g_{jk}(\mathcal{S}_j)$ can be obtained using a similar argument, which yields the desired equality. □

Lemma 2.9. *Let $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n$ be a monohedral tiling of $\mathcal{B}^2$, where $n > 1$. Then at least two of the arcs $\mathcal{S}_1$, $g_{21}(\mathcal{S}_2)$, $\ldots$, $g_{n1}(\mathcal{S}_n)$ coincide.*

Proof. Suppose for contradiction that the arcs $\mathcal{S}_1, g_{21}(\mathcal{S}_2), \ldots, g_{n1}(\mathcal{S}_n)$ are disjoint apart from possibly their endpoints. By our earlier observation these arcs are in $\partial \operatorname{conv} \mathcal{D}_1$. As the total turning angle of these n arcs is 2π, and the total turning angle along the boundary of a convex body is also 2π, $\partial \operatorname{conv} \mathcal{D}_1$ may only consist in excess of these arcs some segments that connect the endpoints of these arcs in a smooth way. In other words, $\operatorname{conv} \mathcal{D}_1 = \mathcal{P} + \mathcal{B}^2$ for some convex n-gon $\mathcal{P}$. This implies that the circumradius of $\mathcal{D}_1$ is at least 1, with equality if and only if $\mathcal{D}_1 = \mathcal{B}^2$, a contradiction. □

Definition 2.10. A *multicurve* (cf. also [11]) is a finite family of continuous simple curves, called the *members of the multicurve*, which are parameterized on non-degenerate closed finite intervals, and any point of the plane belongs to at most one member, or it is the endpoint of exactly two members. If $\mathcal{F}$ and $\mathcal{G}$ are multicurves, $\bigcup \mathcal{F} = \bigcup \mathcal{G}$, and every member of $\mathcal{F}$ is the union of some members of $\mathcal{G}$, we say that $\mathcal{G}$ is a *partition* of $\mathcal{F}$.

Definition 2.11. Let $\mathcal{F}$ and $\mathcal{G}$ be multicurves. If there are partitions $\mathcal{F}'$ and $\mathcal{G}'$ of $\mathcal{F}$ and $\mathcal{G}$, respectively, and a bijection $f \colon \mathcal{F}' \to \mathcal{G}'$ such that $f(\mathcal{C})$ is congruent to $\mathcal{C}$ for all $\mathcal{C} \in \mathcal{F}'$, we say that $\mathcal{F}$ and $\mathcal{G}$ are *equidecomposable.*

The following lemma can be proved very similarly to the analogous statement for equidecomposability of polygons [4], thus we omit the proof.

Lemma 2.12. *Equidecomposability is an equivalence relation on the family of multicurves in $\mathbb{R}^2$.*

Corollary 2.13. *If $\mathcal{F}$ and $\mathcal{G}$ are multicurves with $\bigcup \mathcal{F} = \bigcup \mathcal{G}$, then $\mathcal{F}$ and $\mathcal{G}$ are equidecomposable.*

Proof. Clearly, it is sufficient to prove the statement for the connected components of $\bigcup \mathcal{F}$, and by Lemma 2.12 we may assume that one of the multicurves, say $\mathcal{G}$, is a simple continuous curve. But then $\mathcal{F}$ is a partition of $\mathcal{G}$, in which case the statement is obvious. □

Corollary 2.14. *If $\mathcal{F}$ and $\mathcal{G}$ are equidecomposable, and their subfamilies $\mathcal{F}' \subseteq \mathcal{F}$ and $\mathcal{G}' \subseteq \mathcal{G}$ are equidecomposable, then $\mathcal{F} \setminus \mathcal{F}'$ and $\mathcal{G} \setminus \mathcal{G}'$ are equidecomposable.*

Proof. By Lemma 2.12, we may assume that $\bigcup \mathcal{F} = \bigcup \mathcal{G}$. Without loss of generality, we may also assume that $\bigcup \mathcal{F}$ is connected, which yields that we may regard both $\mathcal{F}$ and $\mathcal{G}$ as different partitions of the same simple, continuous curve. More specifically, after reparametrizing if necessary, we may assume that there is some curve $\mathcal{C} \colon [a,b] \to \mathbb{R}^2$, and partitions P_F and P_G of $[a,b]$ such that the elements of $\mathcal{F}$ and $\mathcal{G}$ are the restrictions of $\mathcal{C}$ to the subintervals of P_F and P_G, respectively. By Corollary 2.13, a multicurve is equidecomposable with any of its partitions, and hence, we may assume that $P_F = P_G$, and there is a bijection between the elements of $\mathcal{F}'$ and $\mathcal{G}'$ such that the corresponding elements are congruent. Since congruence is an equivalence relation, it is clear that any such bijection can be extended to all subintervals of P_F, which proves the assertion. □

3. Proof of Theorem 1.1

First, consider a monohedral tiling of $\mathcal{B}^2$ with the topological discs $\mathcal{D}_1$ and $\mathcal{D}_2$. The containment $O \in \mathcal{D}_1 \cap \mathcal{D}_2$ can be proved in a number of elementary ways (see, e.g. [7]); here we also show that the tiling is rotationally generated.

By Lemma 2.6, for $i = 1, 2$, $\mathcal{S}_i = \mathcal{D}_i \cap \mathcal{S}^1$ is a connected arc and hence, $\mathcal{S}_1$ or $\mathcal{S}_2$ is an arc of length at least π. Thus, $\mathcal{D}_1$ or $\mathcal{D}_2$ contains a pair of antipodal points of $\mathcal{B}^2$, which, by Lemma 2.5, implies that the tiling is rotationally generated.

From now on, we consider the case that $\mathcal{B}^2$ is decomposed into three congruent topological discs $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$, and for $i = 1, 2, 3$, we set $\mathcal{S}_i = \mathcal{D}_i \cap \mathcal{S}^1$. By Lemmas 2.6 and 2.5, we may assume that each tile intersects $\mathcal{S}^1$ in a nondegenerate cicle arc, which is smaller than a half-circle.

By Lemma 2.3, we have that $\mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathcal{D}_3$ consists of a single point $M \in \operatorname{int} \mathcal{B}^2$, and that for any $i \neq j$, $\mathcal{D}_i \cap \mathcal{D}_j$ is a simple, continuous curve connecting M and a point of $\mathcal{S}^1$.

To prove the assertion, we distinguish some cases. Before we do it, we observe that by Remark 2.8, any pair of the curves $\mathcal{S}_1$, $g_{21}(\mathcal{S}_2)$ and $g_{31}(\mathcal{S}_3)$ intersect in at most a common endpoint, or they coincide.

Case 1: *No pair of the arcs $\mathcal{S}_1$, $g_{21}(\mathcal{S}_2)$ and $g_{31}(\mathcal{S}_3)$ coincide.*
In this case we immediately have a contradiction by Lemma 2.9.

Case 2: *Two of the arcs $\mathcal{S}_1$, $g_{21}(\mathcal{S}_2)$ and $g_{31}(\mathcal{S}_3)$ coincide, the third one is different.*
Using a suitable relabeling of the tiles, we may assume that $\mathcal{S}_1 = g_{21}(\mathcal{S}_2)$. Let the

arclength of this arc be $0 < \alpha < \pi$, and the arclength of $\mathcal{S}_3$ be β. The equality $\mathcal{S}_1 = g_{21}(\mathcal{S}_2)$ implies, in particular, that g_{21} is an isometry of $\mathcal{S}^1$; or more generally that it is either the reflection about the symmetry axis ℓ of $\mathcal{S}_1 \cup \mathcal{S}_2$ or a rotation around O with angle α. We may assume without loss of generality that ℓ is the y-axis, the common point of $\mathcal{S}_1$ and $\mathcal{S}_2$ is $(0,1)$, and $\mathcal{S}_1 \subset \{(x,y) : x \leq 0\}$. Furthermore, in the proof we set $\mathcal{C}_1 = \mathcal{D}_1 \cap \mathcal{D}_3$, and $\mathcal{C}_2 = \mathcal{D}_2 \cap \mathcal{D}_3$.

Subcase 2.a*: g_{21} is the reflection about ℓ.*
If there is a point $P \in \operatorname{int} \mathcal{D}_1 \cap \{(x,y) : x > 0\}$, then a continuous curve Γ in $\operatorname{int} \mathcal{D}_1$ connects P and the midpoint of $\mathcal{S}_1$, so $g_{12}(\Gamma)$ connects the midpoint of $\mathcal{S}_2$ to $g_{12}(P)$ in $\operatorname{int} \mathcal{D}_2$. This implies that $\Gamma \cap g_{12}(\Gamma)$ is in $\operatorname{int} \mathcal{D}_1 \cap \operatorname{int} \mathcal{D}_2 = \emptyset$, which is a contradiction. Thus we have $\mathcal{D}_1 \subset \{(x,y) : x \leq 0\}$ and also $\mathcal{D}_2 \subset \{(x,y) : x \geq 0\}$.

Observe that $g_{13}(\mathcal{S}_1) = g_{23}(g_{12}(\mathcal{S}_1)) = g_{23}(\mathcal{S}_2)$, and $\mathcal{D}_3 = \operatorname{cl}(\mathcal{B}^2 \setminus (\mathcal{D}_1 \cup \mathcal{D}_2))$ is symmetric in ℓ. We denote this arc of length α by $\mathcal{S} = g_{13}(\mathcal{S}_1)$. Note that by the conditions of Case 2 $\mathcal{S} \neq \mathcal{S}_3$, and $\mathcal{S} \subset \partial \operatorname{conv} \mathcal{D}_3$ by Remark 2.7. Furthermore, $\partial \operatorname{conv} \mathcal{D}_3$ does not contain any arc of length α apart from $\mathcal{S}$ and possibly $\mathcal{S}_3$, as otherwise the idea of the proof of Lemma 2.9 yields a contradiction. Thus, $\mathcal{S}$ is symmetric in the y-axis.

Since $\mathcal{D}_3$ is connected, and every point of ℓ belongs either to $\mathcal{D}_3$, or to both $\mathcal{D}_1$ and $\mathcal{D}_2$, the segment connecting the midpoint X of $\mathcal{S}$ and the midpoint Y of $\mathcal{S}_3$ belongs to $\mathcal{D}_3$. Let the length of $\overline{XY}$ be $\delta > 0$, and note that the fact $X, Y \in \partial \operatorname{conv} \mathcal{D}_3$ yields that the line through $\overline{XY}$ intersects $\mathcal{D}_3$ exactly in $\overline{XY}$ and $\overline{XY} \setminus \{X, Y\} \subset \operatorname{int} \mathcal{D}_3$. For $i = 1, 2$, $g_{3i}(\overline{XY})$ is the segment of length δ in $\mathcal{B}^2$, starting at the midpoint of $\mathcal{S}_i$, and perpendicular to it. Thus, if $\delta < 1$, then $O \notin \mathcal{D}_i$ for any value of i, if $\delta > 1$, then $O \in \operatorname{int} \mathcal{D}_i$ for all values of i, and if $\delta = 1$, then O is the midpoint of a unit circle arc in the boundary of each of the $\mathcal{D}_i$s, which is a contradiction.

Subcase 2.b*: g_{21} is the rotation around O by angle α in counterclockwise direction.*
As O is a fixed point of g_{21}, it follows that either $O \in \mathcal{D}_1 \cap \mathcal{D}_2$, or $O \notin \mathcal{D}_1 \cup \mathcal{D}_2$. By the definition of tiling and our assumptions, in the first case $O \in \partial \mathcal{D}_1 \cap \partial \mathcal{D}_2$, and in the second case $O \in \operatorname{int} \mathcal{D}_3$.

First, consider the case that $O \in \partial \mathcal{D}_1 \cap \partial \mathcal{D}_2$.

Recall that by Lemma 2.3, $D_1 \cap D_2 \cap D_3$ is a single point M, and for any $i \neq j$, $\mathcal{D}_i \cap \mathcal{D}_j$ is a simple continuous curve connecting M to a point of $\mathcal{S}^1$. Thus, if $O = M$, then $g_{21}(\mathcal{D}_1 \cap \mathcal{D}_2) = \mathcal{D}_1 \cap \mathcal{D}_3$, and $g_{12}(\mathcal{D}_1 \cap \mathcal{D}_2) = \mathcal{D}_2 \cap \mathcal{D}_3$. Since $\partial \mathcal{D}_1$ and $\partial \mathcal{D}_3$ are equidecomposable, this implies that $\mathcal{S}_1$ and $\mathcal{S}_3$ are congruent, and hence $\alpha = 2\pi/3$. In other words, if $O = M$, then the tiling is rotationally generated. Thus, we assume that $O \notin \mathcal{D}_3$, which by the compactness of $\mathcal{D}_3$ yields the existence of a small closed circular disc $\mathcal{B}$ centered at O such that $\mathcal{B} \cap \mathcal{D}_3 = \emptyset$. Let $t \mapsto \mathcal{C}(t)$ be a continuous parameterization of the curve $\mathcal{D}_1 \cap \mathcal{D}_2$ satisfying $O = \mathcal{C}(0)$, and let $t_+ = \sup\{t : \mathcal{C}([0,t])) \subset \mathcal{B}\}$ and $t_- = \inf\{t : \mathcal{C}([t,0]) \subset \mathcal{B}\}$. Then $g_{12}(\mathcal{C}(t_\pm)) = \mathcal{C}(t_\mp)$, which implies that g_{12} is the reflection about O. Thus $\alpha = \pi$ and $\beta = 0$, which contradicts our assumptions.

In the remaining part of Subcase 2.b, we assume that $O \in \operatorname{int} \mathcal{D}_3$.

Let $M_1 = g_{21}(M)$ and $M_2 = g_{12}(M)$. Since $\alpha > 0$, we have $M_2 \neq M$. On the other hand, we clearly have $M_2 \in \partial\mathcal{D}_2$ and $M_2 \notin \mathcal{S}^1$.

Let $\mathcal{B}$ be the circular disc in $\mathcal{D}_3$ that is centered at O and is of maximum radius $r > 0$. Then $\mathcal{B}$ is tangent to at least one of the curves $\mathcal{C}_1$ and $\mathcal{C}_2$, say $\mathcal{C}_2$ touches $\mathcal{B}$ in $X_2 \in \partial\mathcal{B} \cap \mathcal{C}_2$. Let $X_1 = g_{21}(X_2)$. Then $X_1 \in \mathcal{B} \cap \mathcal{D}_1 = \mathcal{B} \cap \mathcal{C}_1$ clearly, hence $X_2 \in g_{12}(\mathcal{C}_1) \cap \mathcal{C}_2 \neq \emptyset$. Since $g_{12}(\mathcal{C}_1)$ is a continuous curve in $\partial\mathcal{D}_2$, connecting the intersection point of $\mathcal{S}_1$ and $\mathcal{S}_2$ to M_2 in $\operatorname{int}\mathcal{B}^2$, it follows that $M \in g_{12}(\mathcal{C}_1)$, that is, $M_1 \in \mathcal{C}_1$, implying also $M_2 \in \mathcal{C}_2$.

Thus, M_1 divides the curve $\mathcal{C}_1$ into two parts: one from M to M_1, which we denote by $\mathcal{C}_1^M$, and the other one from M_1 to a point of S_1, which we denote by $\mathcal{C}_1^S$. We define the parts $\mathcal{C}_2^M$ and $\mathcal{C}_2^S$ of $\mathcal{C}_2$ similarly, using M_2 in place of M_1. Furthermore, we set $\mathcal{C}_3^S = \mathcal{D}_1 \cap \mathcal{D}_2$.

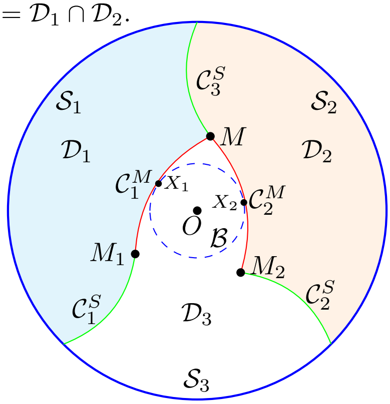

FIGURE 2. $\mathcal{B}^2$ is dissected into three topological discs.

We clearly have $g_{21}(\mathcal{C}_2^M) = \mathcal{C}_1^M$, $g_{21}(\mathcal{C}_2^S) = \mathcal{C}_3^S$ and $g_{21}(\mathcal{C}_3^S) = \mathcal{C}_1^S$. Observe that since $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$ are congruent, their boundaries are equidecomposable. Furthermore, as $\mathcal{C}_1^S$, $\mathcal{C}_2^S$, and $\mathcal{C}_3^S$, and also $\mathcal{C}_1^M$ and $\mathcal{C}_2^M$ are congruent, we obtain by Corollary 2.14 that $\mathcal{S}_1$ and $\mathcal{C}_1^M \cup \mathcal{S}_3$ are equidecomposable. Thus we deduce that $\mathcal{C}_1^M$ (and also $\mathcal{C}_2^M$) is a multicurve such that its every member curve is a unit circular arc, and their total length is $\alpha - \beta \geq 0$.

If a unit circular arc $\mathcal{C}$ is contained in the boundary of a tile $\mathcal{D}_i$, it may happen that the convex side of $\mathcal{C}$ belongs to $\operatorname{int}\mathcal{D}_i$, and the concave side of $\mathcal{C}$ does not belong to $\mathcal{D}_i$. In this case we say that $\mathcal{C}$ is a *convex circular arc* of $\mathcal{D}_i$, and in the opposite case that it is a *concave circular arc* of $\mathcal{D}_i$. Clearly, if $\mathcal{C}$ is a unit circular arc in $\mathcal{D}_i \cap \mathcal{D}_j$ for some $i \neq j$, then it is a convex circular arc of exactly one of $\mathcal{D}_i$ and $\mathcal{D}_j$. Let x and y denote the total length of the convex and concave unit circular arcs of $\mathcal{D}_1$ in $\mathcal{C}_1^M$. Since $\mathcal{C}_1^M$ and $\mathcal{C}_2^M$ are congruent, the total length of the convex and concave unit circular arcs of $\mathcal{D}_2$ in $\mathcal{C}_2^M$ is also x and y, respectively. Thus, the total length of the convex and concave unit circular arcs of $\mathcal{D}_3$ in $\mathcal{C}_1^M \cup \mathcal{C}_2^M$ is $2y$ and $2x$, respectively.

The congruence of the tiles $\mathcal{D}_i$ and the curves $\mathcal{C}_i^S$ for $i = 1, 2, 3$ yields that the total lengths of the convex and concave unit circular arcs of $\mathcal{D}_1$ in $\mathcal{S}_1 \cup \mathcal{C}_1^M$ is equal to the total lengths of the convex and the concave unit circular arcs of $\mathcal{D}_3$ in $\mathcal{S}_3 \cup \mathcal{C}_1^M \cup \mathcal{C}_2^M$, respectively. This equality for convex circular arcs implies that $\alpha + x = \beta + 2y$, and the equality for concave arcs implies $y = 2x$. From these equations it follows that $x = (\alpha - \beta)/3$ and $y = 2(\alpha - \beta)/3$. Thus, in particular, it follows that if $\beta = \alpha$, then $x = y = 0$ and $M = M_1 = M_2$, which yields that $\alpha = 0$, a contradiction. This means that $\beta < \alpha$.

We show that M is not an interior point of a unit circular arc in $\partial\mathcal{D}_3$ longer than $\alpha - \beta$. Suppose for contradiction that M is an interior point of such a circular arc $\mathcal{C}$. If one of M_1 or M_2, say, $M_1 \in \mathcal{C}$, then $\mathcal{C}_1^M \subset \mathcal{C}$, which yields that $\mathcal{C}_2^M = g_{21}(\mathcal{C}_1^M)$ is also a unit circular arc, implying that $\mathcal{C}_1^M \cup \mathcal{C}_2^M$ belongs to the same unit circle $\hat{\mathcal{S}}$. Since this circle is invariant under a rotation around O, we have $\hat{\mathcal{S}} = \mathcal{S}^1$, which contradicts our assumption that $M, M_1, M_2 \in \operatorname{int}\mathcal{B}^2$. Assume that $M_1, M_2 \notin \mathcal{C}$, and let $\mathcal{C}^1$ and $\mathcal{C}^2$ denote $\mathcal{C} \cap \mathcal{C}_1^M$ and $\mathcal{C} \cap \mathcal{C}_2^M$, respectively. Then $g_{21}(\mathcal{C}^2)$ is a unit circular arc in $\mathcal{C}_1^M$ whose length is equal to that of $\mathcal{C}^2$. Thus, $g_{21}(\mathcal{C}^2)$ and $\mathcal{C}^1$ intersect in a unit circular arc, which yields that $g_{21}(\mathcal{C}^2) \cup \mathcal{C}^1 = \mathcal{C}_1^M$ is a unit circular arc, which leads to a contradiction in a similar way.

Let us say that a unit circular arc in $\partial\mathcal{D}_i$ is *maximal*, if it is not a proper subset of another unit circular arc in $\partial\mathcal{D}_i$. By Lemma 2.1, $\partial\mathcal{D}_1$ contains finitely many, say $m \geq 1$ maximal unit circular arcs of length α, one of which is $\mathcal{S}_1$. Thus, $\partial\mathcal{D}_3$ also contains m maximal unit circular arcs of length α. By the previous paragraph, any of these arcs is contained in $\mathcal{C}_1^S \cup \mathcal{C}_1^M$ or in $\mathcal{C}_2^S \cup \mathcal{C}_2^M$. Assume that all these arcs are contained in $\mathcal{C}_1^S$ or in $\mathcal{C}_2^S$. Since $\mathcal{C}_1^S$, $\mathcal{C}_2^S$ and $\mathcal{C}_3^S$ are congruent, we have that the total number of unit circular arcs of length α in $\mathcal{C}_i^S$ is equal to $m/2$. Thus, $\partial\mathcal{D}_1$ contains $m + 1$ arcs, which is a contradiction.

Finally, consider the case that some maximal unit circular arc $\mathcal{S}_\alpha$ of length α in $\partial\mathcal{D}_3$ is not contained in $\mathcal{C}_1^S \cup \mathcal{C}_2^S$. Since $\alpha > \alpha - \beta$, M is not an interior point of $\mathcal{S}_\alpha$, but M_1 or M_2 is. Without loss of generality, we may assume that M_1 is in the interior of $\mathcal{S}_\alpha$. This implies that M is in the interior of $g_{12}(\mathcal{S}_\alpha) \subseteq \mathcal{C}_3^S \cup \mathcal{C}_2^M$ (similarly as Figure 2 shows). Hence, M is not an interior point of a unit circular arc in $\partial\mathcal{D}_1$, which implies that M_2 is not an interior point of any unit circular arc in $\partial\mathcal{D}_2$. On the other hand, again by Lemma 2.1, $\partial\mathcal{D}_3$ contains k maximal unit circular arcs of length β for some $k \geq 1$, one of which is $\mathcal{S}_3$. By our previous argument, any of these arcs is contained in one of $\mathcal{C}_i^M$ or $\mathcal{C}_i^S$ for some $i \in \{1, 2\}$. Let $k_M \geq 0$ and $k_S \geq 0$ denote the number of these arcs in $\mathcal{C}_1^M$ and $\mathcal{C}_1^S$, respectively. Then $\mathcal{C}_1^M$ and $\mathcal{C}_1^S$ contain exactly k_M and k_S of these arcs, respectively. From this it readily follows that $k = 2k_M + 2k_S + 1$. Furthermore, since $\partial\mathcal{D}_1$ also contains k maximal unit circular arcs of length β, we have $k = k_M + 2k_S$. This yields that $k_M = -1$, which is a contradiction.

***Case 3**: all of the arcs $\mathcal{S}_1$, $g_{21}(\mathcal{S}_2)$ and $g_{31}(\mathcal{S}_3)$ coincide.*

In this case g_{21} and g_{31} are either reflections about a line through O, or rotations around O. In particular, O is a fixed point of both of them and thus it is the unique common point M of all tiles. For any $i \neq j$, let $\mathcal{C}_{ij} = \mathcal{D}_i \cap \mathcal{D}_j$. If both g_{12} and g_{13} are rotations around O, then the tiling is clearly rotationally generated. Hence, assume that one of g_{12} and g_{13}, say g_{12} is a reflection about a line ℓ through O. Then $g_{12}(\mathcal{C}_{13} \cup \mathcal{C}_{12}) = \mathcal{C}_{12} \cup \mathcal{C}_{23}$ yields that $\mathcal{C}_{12}$ is a straight line segment in ℓ, which, by the congruence of the tiles implies also that C_{ij} is a segment for all $i \neq j$. Thus, also in this case the tiling is rotationally generated, and the assertion follows.

4. Remarks and open problems

First, we observe that the quantity $n(\mathcal{K})$ can be similarly defined for any O-symmetric convex body $\mathcal{K}$ in $\mathbb{R}^d$ playing the role of $\mathcal{B}^2$. On the other hand, Theorem 1.1 cannot be generalized for any O-symmetric convex body even in the case $d = 2$. Indeed, taking a parallelogram and dissecting it into three congruent parallelograms with two lines parallel to a pair of sides of the parallelogram shows that there are O-symmetric plane convex bodies $\mathcal{K}$ with $n(\mathcal{K}) = 3$. However, it is easy to see that the following generalization of Theorem 1.1 holds.

Theorem 4.1. *If there is a monohedral tiling of an O-symmetric, strictly convex, smooth body $\mathcal{K}$ in $\mathbb{R}^2$ with $k \leq 3$ topological discs, then both K and its tiling has a k-fold symmetry. In particular, for any O-symmetric, strictly convex plane body $\mathcal{K}$ of smooth boundary we have $n(\mathcal{K}) \geq 4$.*

This raises the question what happens if smoothness or the strictness of the convexity is dropped from the conditions.

Following [5], we generalize Question 1 for balls in arbitrary dimensions.

Question 4. *Are there monohedral tilings of the closed unit ball $\mathcal{B}^d$ such that the center of the ball is not contained in all of the tiles? More specifically, what are the values of d for which it is possible?*

We also raise the following, related problem:

Question 5. *If $\mathcal{B}^2$ has a tiling with* similar *copies of some topological disc $\mathcal{D}$, does it follow that the tiles are congruent? Does it follow that $\mathcal{B}^2$ has a tiling with* congruent *copies of $\mathcal{D}$? Do these properties hold under some additional assumption on the tiles, e.g. if they have piecewise analytic boundaries?*

We should finally mention the *divisibility problem*, in which the topological conditions on the tiles are dropped: A subset of $\mathbb{R}^d$ is *m-divisible* if it can be decomposed into $m \in \mathbb{N}$ mutually *disjoint* congruent subsets. It is proved that typical convex bodies are not divisible [15], but balls are not typical in this sense, and they are m-divisible for large values of m if d is divisible by three [8] or d is even [9].

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