Projective metrics with quadratic metric ellipses

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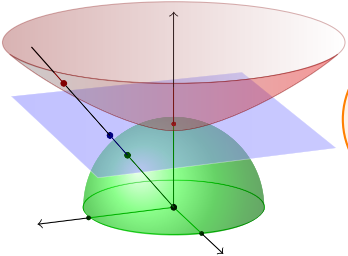
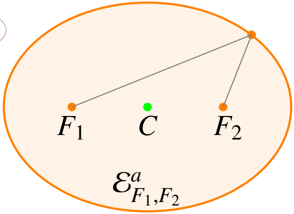
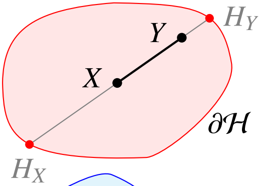
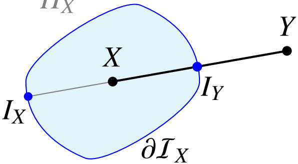








27th July, 2016.

Potenza

Hilbert's IV. problem (Hilbert 1900, [7])

Give all the metrics on every projective space $\mathbb{P}^n$ of dimension n that satisfies the strict triangle inequality.

Classification of projective metrics (Hamel 1903, [8])

Let $\mathcal{D}$ be the support of such a projective metric. Then there are exactly three cases: elliptic *($\mathcal{D} = \mathbb{P}^n$),* parabolic *($D = \mathbb{P}^n \setminus \mathbb{R}^n$) and* hyperbolic *($\mathcal{D} \subsetneq \mathbb{P}^n \setminus \mathbb{R}^n$ convex)*

Construction (Busemann 1961, [3]; based on Blaschke's extension [2] of the Crofton-formula [6])

Let μ be a quasi pozitive[1] "measure" on the set of the hyperplanes.

Let the length of a curve be the measure $\mu/2$ of the set of intersecting hyperplanes.

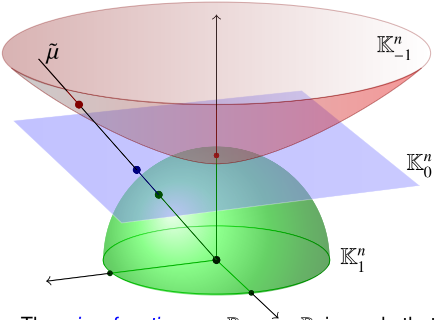
Theorem. (Pogorelov 1973, [13] on the plane; Szabó 1986, [14] any dimension).

Busemann's construction gives all the projective metrics.

Busemann [4]: "... It is clear from Hilbert's comments that he was not aware of the immense number of these metrics, so that the second part of the problem ... has inevitably been replaced by the investigation of special, or special classes of, interesting geometries." .

[1] not necessarily positive everywhere, but for any non-collinear triple ABC of points the "measure" of the hyperplanes intersecting $\overline{AB} \cup \overline{BC}$ is positive and the "measure" of those hyperplanes passing through B is zero.

Classic metrics.



The *size function* $\nu_\kappa\colon \mathbb{R}_+ \to \mathbb{R}$ is such that $\nu_\kappa(r)\mathcal{S}^{n-1}$ is isometric to a metric sphere of radius $r > 0$ in $\mathbb{K}^n_\kappa$. The *projection function* $\mu_\kappa\colon [0, \mathring{\imath}_\kappa) \to \mathbb{R}_+$ makes geodesic correspondence by $\tilde{\mu}_\kappa\colon \operatorname{Exp}_O(r\omega) \mapsto \mu_\kappa(r)\omega$.

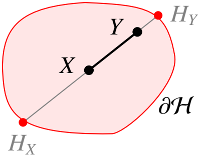
$\mathbb{K}^n_\kappa$	κ	ν_κ	μ_κ	$\mathring{\imath}_\kappa$
$\mathbb{H}^n$	-1	$\sinh r$	$\tanh r$	∞
$\mathbb{R}^n$	0	r	r	∞
$\mathbb{S}^n$ $(\mathbb{P}^n)$	$+1$	$\sin r$	$\operatorname{tg} r$	$\pi/2$

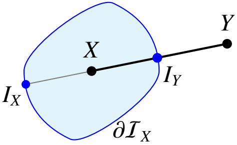
Here we identified the space $\mathcal{T}_O\mathbb{K}^n_\kappa$ with $\mathbb{R}^n$ by the natural way, and used $\omega \in \mathcal{S}^{n-1}$ in both senses.

Hilbert metric. $d_{\mathcal{H}}\colon \mathcal{H}\times\mathcal{H} \to \mathbb{R}$ is defined by

$$d_{\mathcal{H}}(X,Y) = \begin{cases} 0, & \text{if } X = Y,\\ \frac{1}{2}\left|\ln(X,Y;H_X,H_Y)\right|, & \text{if } X \neq Y, \end{cases}$$

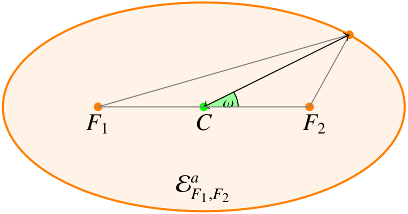
where $\mathcal{H}$, the *infinity*, is an open, strictly convex, bounded domain in $\mathbb{R}^n$ and $\overline{H_XH_Y} = \mathcal{H}\cap XY$.





Minkowski metric. $d_{\mathcal{I}}\colon \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ is defined by $d_{\mathcal{I}}(X,Y) = (Y, I_Y; X)$, where $\mathcal{I}$, the *indicatrix*, is an open, strictly convex, bounded domain centrally symmetric at O in $\mathbb{R}^n$ and $\overline{I_XI_Y} = \mathcal{I}_X\cap XY$.

Elliptic metric. $d\colon \mathbb{P}^n\times\mathbb{P}^n \to \mathbb{R}$ is defined by $d(X,Y) = \arccos|\langle\overrightarrow{OX},\overrightarrow{OY}\rangle|$, where the points of $\mathbb{P}^n$ are the diagonal point pairs of $\mathcal{S}^n$.



A set $\boxed{\mathcal{E}^a_{d;F_1,F_2} := \{P : 2a = d(F_1,P) + d(P,F_2)\}}$, where F_1, F_2, the *focuses*, are fixed points and $f := d(F_1,F_2)/2 < a$, is called *metric ellipse*. The metric ellipse is called *metric circle* if $f = 0$. The metric midpoint of the segment $\overline{F_1F_2}$ is called the *metric center* of $\mathcal{E}^a_{d;F_1,F_2}$.

A projective metric d is called *strongly quadratic* if for every $a > 0$ and pair of points F_1, F_2, such that $a > d(F_1,F_2)$, the metric ellipse $\mathcal{E}^a_{d;F_1,F_2}$ is quadratic.

Folkloric result. *The classic metrics are strongly quadratic.*

Proof for the hyperbolic plane only[2]**.** In the quadratic model $\mathbb{H}^2 = \{(x,y,z) : x^2 + y^2 - z^2 = -1,\ z \geq 1\}$ the metric is $d(\boldsymbol{p},\boldsymbol{q}) = \cosh^{-1}\langle \boldsymbol{p},\boldsymbol{q}\rangle$ and $\kappa = -1$. Fix $f > 0$ and $a > f$, and let $F_1 = (\sinh(-f), 0, \cosh f)$ and $F_2 = (\sinh f, 0, \cosh f)$. Then $C = (0,0,1)$. For any point $P = (x,y,z) \in \mathcal{E}^a_{d;F_1,F_2} \subset \mathbb{H}^2$ there is a $t \in [0,f]$ such that

$$a + t = d(F_1, X) = \cosh^{-1}(-x\sinh f + z\cosh f), \quad \text{and} \quad a - t = d(X, F_2) = \cosh^{-1}(x\sinh f + z\cosh f).$$

Taking cosh gives $\cosh a \cosh t \pm \sinh a \sinh t = \cosh(a \pm t) = z\cosh f \mp x\sinh f$, hence $\cosh a\cosh t = z\cosh f$ and $\sinh a \sinh t = -x\sinh f$. This implies

$$\frac{z^2}{\cosh^2 a/\cosh^2 f} - \frac{x^2}{\sinh^2 a/\sinh^2 f} = \cosh^2 t - \sinh^2 t = 1, \quad \text{hence } z^2 - x^2\frac{\tanh^2 f}{\tanh^2 a} = \frac{\cosh^2 a}{\cosh^2 f} = \frac{1-\tanh^2 f}{1-\tanh^2 a}$$

Substituting the C-based polar coordinates (ω, r) of P, we have $P = (\sinh r\cos\omega, \sinh r\sin\omega, \cosh r)$, which results in

$$\mathcal{E}^a_{d_\kappa;F_1,F_2}: \qquad \frac{1}{\nu_\kappa^2(r(\omega))} = \frac{\cos^2\omega}{\nu_\kappa^2(a)} + \frac{\sin^2\omega}{(\mu_\kappa^2(a) - \mu_\kappa^2(f))(1 - \kappa\nu_\kappa^2(a))}.$$

■

[2] It is very much the same for the Euclidean plane and the sphere. See [11].

Problem. (Kurusa, 2015).

Which projective metrics are such that *one, many or all* of its metric *ellipses* are *quadratic*?

A projective metric d is called *strongly ε-quadratic* for an $\varepsilon \in [0,1)$ if for every $a > 0$ and pair of points F_1, F_2, where $\varepsilon a = d(F_1, F_2)$, the metric ellipse $\mathcal{E}^a_{d;F_1,F_2}$ is quadratic.

Beltrami's theorem. (1865, [1]).

If a projective metric is Riemannian, then it is elliptic, Euclidean or Bolyai's hyperbolic.

Condition "Riemmannian" can be thought of as every infinitesimal sphere is quadratic.

Tétel. (Busemann 1953, [5, 25.4]).

A Minkowski metric is Euclidean if and only if its unit metric spheres *are quadratic.*

Tétel. (Kay 1967, [9, 296. old.]).

If the unit metric sphere *of every associated Minkowski metrics of a Hilbert metric is quadratic, then it is Bolyai's hyperbolic metric.*

A projective metric is called *weakly quadratic* if it has a quadratic metric ellipse.

Question (Kurusa, 2015)

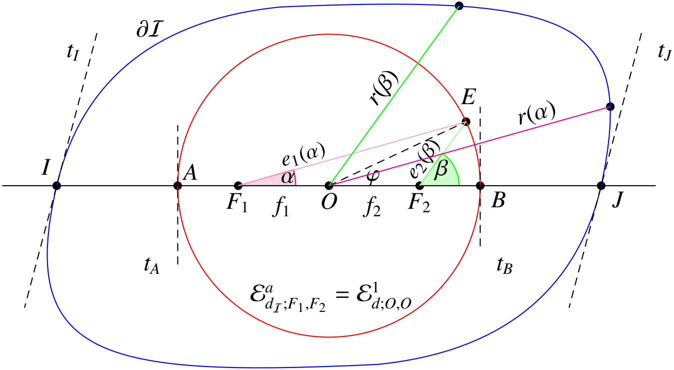
Is a weakly quadratic projective metric classic?

There is no known counterexample (yet?) but we have positive results.

Theorem. (Kurusa 2016, [11]).

Weakly quadratic Minkowski metric is Euclidean.

Detailed sketch of proof. Suppose that $\mathcal{E}^a_{d_{\mathcal{I}};F_1,F_2}$ is quadratic. Then it is the unit circle $\mathcal{E}^1_{d;O}$ of a Euclidean metric d, where O is the midpoint of segment $\overline{F_1F2}$. Place $\mathcal{I}$ so that its center is O. Since $t_A \perp F_1F_2 \perp t_B$, obviously $\dot{e}_1(0) = \dot{e}_2(0) = 0$.



As $d_{\mathcal{I}}(F_1,E) = \frac{e_1(\alpha)}{r(\alpha)}$ and $d_{\mathcal{I}}(E,F_2) = \frac{e_2(\beta)}{r(\beta)}$, we have

$$2a = \frac{e_1(\alpha)}{r(\alpha)} + \frac{e_2(\beta)}{r(\beta)}.$$

The derivative of this with respect to φ at 0 and the consequence $t_I \parallel t_J$ of the symmetry of $\mathcal{I}$ proves $t_I \perp F_1F_2 \perp t_J$.

As parallelism does not depend on metrics this is proves $t_I \parallel t_A \parallel t_B \parallel t_J$. Let ℓ be the straight line through O which is parallel to t_A and let C_1 and C_2 be the intersection of ℓ with $\mathcal{E}^a_{d_{\mathcal{I}};F_1,F_2}$.

We choose now a different Euclidean metric d_e such that $d_e^2(O,C_1) = a^2 - f^2$ and $d_e(O,I) = 1$. Then $\mathcal{E}^a_{d_\mathcal{I};F_1,F_2} \equiv \mathcal{E}^a_{d_e;F_1,F_2}$ as they have 4 points and two tangents in common. By the definition of these ellipses we have

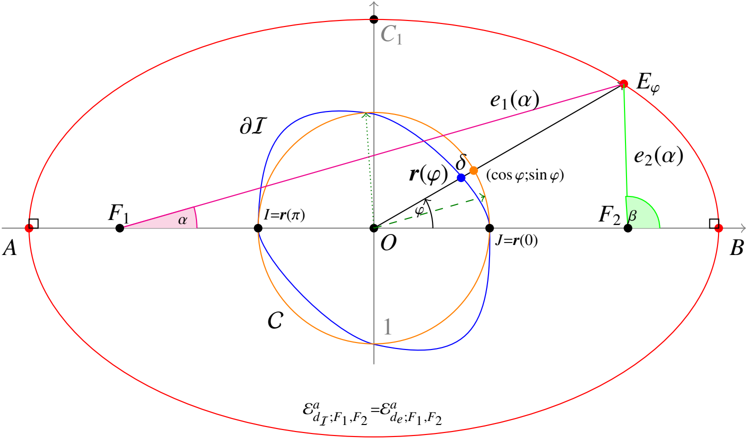
$$e_1(\alpha) + e_2(\beta) = 2a = \frac{e_1(\alpha)}{r(\alpha)} + \frac{e_2(\beta)}{r(\beta)}, \quad \text{i.e.} \quad \delta(\alpha) = -\delta(\beta)\frac{e_2(\alpha)}{e_1(\beta) + 2a\delta(\beta)},$$

where $\delta\colon \varphi \mapsto r(\varphi) - 1$. Taking the limit as $\varphi \to 0$, we obtain $\lim_{\varphi\to 0} \frac{\delta(\alpha)}{\delta(\beta)} = -\frac{a-f}{a+f} = -(F_2, F_1; B)$.

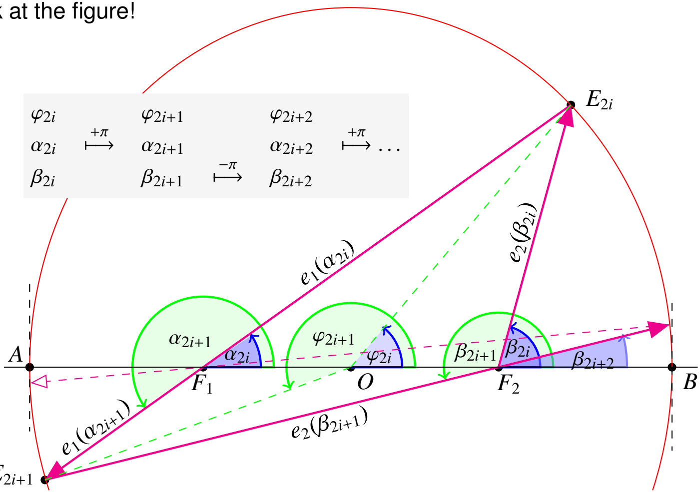
If $\delta(\beta) \neq 0$, then differential geometric reasoning (see [11, Lemma 3.4]), implies $(F_2, F_1; B) = -1$ which is a contradiction.

Thus $\delta(\beta) = 0$ for small enough β.

Observe that $\delta(\beta) \neq 0$ if and only if $\delta(\alpha) \neq 0$.



Take a look at the figure!

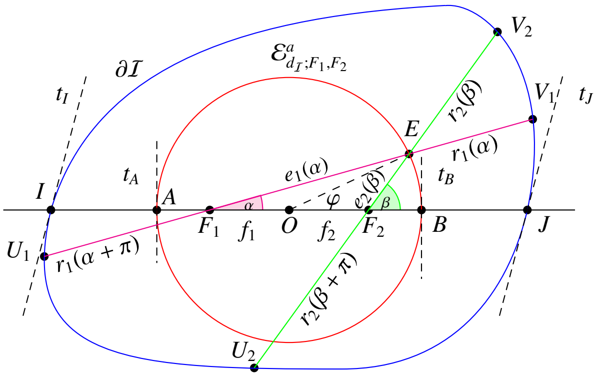


We clearly have $\beta_{2i+2} < \alpha_{2i} < \beta_{2i}$ and $\beta_{2i+1} < \alpha_{2i+1} < \beta_{2i-1}$ and it is easy to prove that $\beta_{2i} \to 0$. Thus, for large enough $i \in \mathbb{N}$ we obtain $\delta(\beta_2 i) = 0$, hence $\delta \equiv 0$. ∎

Theorem. (Kurusa 2016, [11]).

Weakly quadratic Hilbert metric is Bolyai's hyperbolic metric.

Sketch of proof without details. The idea is similar as for the Minkowski case, but needs much more complicated formulas and new tricks. Let $\mathcal{E}^a_{d_{\mathcal{I}};F_1,F_2}$ be a quadric.



Let ℓ be the line F_1F_2 if $F_1 \neq F_2$ and OF_1 if $F_1 = F_2$, where O is the affine center of $\mathcal{E}^a_{d_{\mathcal{I}};F_1,F_2}$. To reach

$$t_I \parallel t_A \parallel t_B \parallel t_J,$$

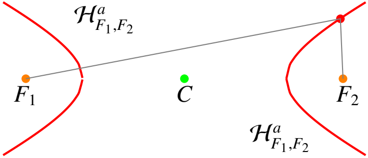
where t_I, t_J and t_A, t_B are the tangents of $\mathcal{I}$ and $\mathcal{E}^a_{d_{\mathcal{I}};F_1,F_2}$, respectively, is the first step and usually needs projectivity. Then a Bolyai plane $\mathcal{H}$ is constructed such that $\mathcal{E}^a_{d_{\mathcal{I}};F_1,F_2} = \mathcal{E}^a_{d_{\mathcal{H}};F_1,F_2}$.

Then a complicated estimate of the difference of the radial functions of $\mathcal{I}$ and $\mathcal{H}$, using differential geometric methods, proves $\mathcal{I} = \mathcal{H}$ in a neighborhood of I. Considering sequences of angles finishes the proof. ■

A projective metric d is called *ε-quadratic* if for every $a > 0$ and point F_1 there is a point F_2, such that $\varepsilon a = d(F_1, F_2)$ and the metric ellipse $\mathcal{E}^a_{d;F_1,F_2}$ is quadratic.

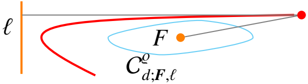
Conjecture. An ε-quadratic projective metric is classic.

A set $\mathcal{H}^a_{d;F_1,F_2} := \{P : 2a = |d(F_1, P) - d(P, F_2)|\}$, where F_1 and F_2, the *focuses*, are fixed points and $0 < a < d(F_1, F_2)/2 = f$, is called *metric hyperbola*. The metric midpoint of the segment $\overline{F_1F_2}$ is called the *center* of $\mathcal{H}^a_{d;F_1,F_2}$.



Theorem. (Kurusa & Kozma, 2016, [12]).

If a metric hyperbola of a Minkowski metric or Hilbert metric in the plane is quadratic, then the metric is Euclidean.



A set $C^\varrho_{d;\ell,F} := \{P : \varrho d(\ell, P) = d(\ell, F)d(F, P)|\}$, where ℓ, the *directrix*, is a fixed straight line, $F \notin \ell$, the *focus*, is a fixed point, and $\varrho > 0$, is called *metric conic*.

Theorem. (Kurusa, 2015, [10]).

If a metric conic of a Minkowski metric in the plane is quadratic, then the metric is Euclidean.


Thank you for your attention!

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Projective metrics with quadratic metric ellipses

1. **Short history and concise theory**
2. **Examples: classic, Minkowski and Hilbert geometries**
3. **Metric ellipses and classic metrics**
4. **Classifications by metric spheres**
5. **Classifications by metric ellipses**
   - Weakly quadratic Minkowski metric
   - Weakly quadratic Hilbert metric
6. **Further considerations**