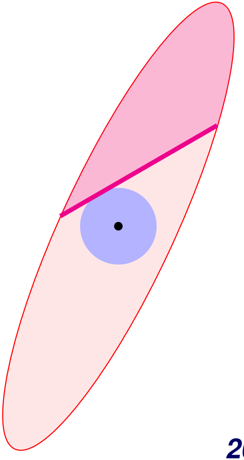
It pays to measure twice!

Lemma of double measuring



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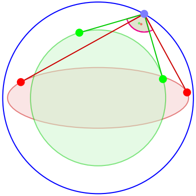
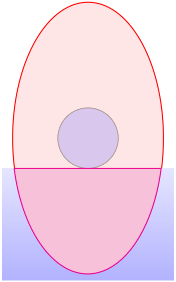


Szeged, 14. november, 2015

Joint results with T. Ódor
(Bolyai Institute, University of Szeged)



2015 Szeged Workshop on Convexity

Lemma of double measurement[1]. (Kurusa & Ódor, 2015).

Let w_i $(i=1,2)$ *be weights,* $\mathcal{K}$ *and* $\mathcal{L}$ *be convex bodies containing the unit ball* $\mathcal{B}$*, and* $c \geq 1$*. If there is a constant* $c_{\mathcal{L}}$ *such that* $w_2 = c_{\mathcal{L}} w_1$ *occurs almost nowhere, then*

$$cV_1(\mathcal{L}) \leq V_1(\mathcal{K}) \text{ and } \left\{\begin{matrix} w_2(X) \leq c_{\mathcal{L}} w_1(X) & \text{for } X \in \mathcal{L}, \\ w_2(X) \geq c_{\mathcal{L}} w_1(X) & \text{for } X \notin \mathcal{L}, \end{matrix}\right\} \text{ imply } cV_2(\mathcal{L}) \leq V_2(\mathcal{K}), \tag{1.1}$$

$$V_1(\mathcal{K}) \leq cV_1(\mathcal{L}) \text{ and } \left\{\begin{matrix} w_2(X) \geq c_{\mathcal{L}} w_1(X) & \text{for } X \in \mathcal{L}, \\ w_2(X) \leq c_{\mathcal{L}} w_1(X) & \text{for } X \notin \mathcal{L}, \end{matrix}\right\} \text{ imply } V_2(\mathcal{K}) \leq cV_2(\mathcal{L}), \tag{1.2}$$

and in both cases equality happens if and only if $\mathcal{K} = \mathcal{L}$ *and* $c = 1$*.*

Proof. In both statements $\mathcal{K} \triangle \mathcal{L} = \emptyset$ implies $V_1(\mathcal{K}) = V_1(\mathcal{L})$, hence $c = 1$ and $V_1(\mathcal{K}) = V_1(\mathcal{L})$. Assume from now on that $\mathcal{K} \triangle \mathcal{L} \neq \emptyset$. Having (1.1) we proceed as

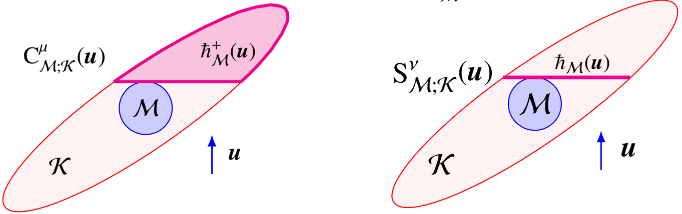
$$\begin{aligned} V_2(\mathcal{K}) - cV_2(\mathcal{L}) &= V_2(\mathcal{K}) - V_2(\mathcal{L}) + (1-c)V_2(\mathcal{L}) = V_2(\mathcal{K} \setminus \mathcal{L}) - V_2(\mathcal{L} \setminus \mathcal{K}) + (1-c)V_2(\mathcal{L}) \\ &= \int_{\mathcal{K}\setminus\mathcal{L}} \frac{w_2(x)}{w_1(x)} w_1(x)dx - \int_{\mathcal{L}\setminus\mathcal{K}} \frac{w_2(x)}{w_1(x)} w_1(x)dx + (1-c)V_2(\mathcal{L}) \\ &> c_{\mathcal{L}}(V_1(\mathcal{K} \setminus \mathcal{L}) - V_1(\mathcal{L} \setminus \mathcal{K})) + (1-c)V_2(\mathcal{L}) = c_{\mathcal{L}}(V_1(\mathcal{K}) - V_1(\mathcal{L})) + (1-c)V_2(\mathcal{L}) \\ &\geq (c-1)(c_{\mathcal{L}}V_1(\mathcal{L}) - V_2(\mathcal{L})) = (c-1)\Big(\int_{\mathcal{L}} \Big(c_{\mathcal{L}} - \frac{w_2(x)}{w_1(x)}\Big) w_1(x)dx\Big) \geq 0 \end{aligned}$$

that implies $V_2(\mathcal{K}) - cV_2(\mathcal{L}) > 0$. The lemma is proved. ∎

[1] A preliminary version of *Lemma of double measurement* first appeared in [13].

Cap and section functions

If the convex body $\mathcal{M}$ contains the origin O, let $\hbar_{\mathcal{M}}(\boldsymbol{u})$ be the supporting hyperplane of $\mathcal{M}$ that is perpendicular to the unit vector $\boldsymbol{u} \in \mathbb{S}^{n-1}$ and has positive perpendicular projection onto $\boldsymbol{u}$. The half space of $\hbar_{\mathcal{M}}(\boldsymbol{u})$ that contains $\mathcal{M}$ is denoted by $\hbar^-_{\mathcal{M}}(\boldsymbol{u})$, the other half space is $\hbar^+_{\mathcal{M}}(\boldsymbol{u})$.



If the convex body $\mathcal{K}$ contains a convex body $\mathcal{M}$, the *kernel*, we define the functions

$$C^{\mu}_{\mathcal{M};\mathcal{K}}(\boldsymbol{u}) = \int_{\langle \boldsymbol{x},\boldsymbol{u}\rangle \geq h_{\mathcal{M}}(\boldsymbol{u})} \chi_{\mathcal{K}}(\boldsymbol{x}) \mu_{\hbar_{\mathcal{M}}(\boldsymbol{u})}(\boldsymbol{x})\, d\boldsymbol{x} \qquad (\textit{cap function})$$

$$S^{\nu}_{\mathcal{M};\mathcal{K}}(\boldsymbol{u}) = \int_{\langle \boldsymbol{x},\boldsymbol{u}\rangle = h_{\mathcal{M}}(\boldsymbol{u})} \chi(\boldsymbol{x} \in \mathcal{K}) \nu_{\hbar_{\mathcal{M}}(\boldsymbol{u})}(\boldsymbol{x})\, d\boldsymbol{x}_{\hbar_{\mathcal{M}}(\boldsymbol{u})} \qquad (\textit{section function}^2)$$

where $h_{\mathcal{M}}$ is the support function of $\mathcal{M}$, μ and ν are strictly positive weights, and $d\boldsymbol{x}_{\hbar_{\mathcal{M}}(\boldsymbol{u})}$ is the appropriate Lebesgue measure on the hyperplane $\hbar_{\mathcal{M}}(\boldsymbol{u})$.

[2] This is usually called *chord function* in the plane.

Weights

From now on $\mathcal{M} = r\mathcal{B}$, the ball of radius r centered at $\mathbf{0}$, and we deal only with rotationally invariant weights, hence

$\nu_{\hbar(\boldsymbol{u},r)}(\boldsymbol{x}) = \bar{\nu}(r, \langle \boldsymbol{x}, \boldsymbol{u}\rangle, |\boldsymbol{x}|)$

and

$\mu_{\hbar(\boldsymbol{u},r)}(\boldsymbol{x}) = \bar{\mu}(r, \langle \boldsymbol{x}, \boldsymbol{u}\rangle, |\boldsymbol{x}|)$

for some $\bar{\nu}, \bar{\mu}$.

Examples of rotationally invariant weights:

1. Volume of the caps uses weight $\mu_{\hbar(\boldsymbol{u},r)}(\boldsymbol{x}) = 1$.
2. Area of the shifted sections uses weight $\nu_{\hbar(\boldsymbol{u},r)}(\boldsymbol{x}) = 1$.
3. The condition of floating in equilibrium in the floating body problem of Ulam leads to weight $\mu_{\hbar(\boldsymbol{u},r)}(\boldsymbol{x}) = \langle \boldsymbol{x}, \boldsymbol{u}\rangle$ [10].
4. Dual of intersecting lines leads to weight $\nu_{\hbar(\boldsymbol{u},r)}(\boldsymbol{x}) = r|\boldsymbol{x}|^{-n}$ [7].

Lemma of weights. *Let the convex body $\mathcal{K}$ contain the ball $\varrho\mathcal{B}$. If μ, ν rotationally invariant weights, then*

$$\int_{\mathbb{S}^{n-1}} \mathrm{C}^{\mu}_{\varrho;\mathcal{K}}(\boldsymbol{u})d\boldsymbol{u} = |\mathbb{S}^{n-2}| \int_{\mathcal{K}\setminus\varrho\mathcal{B}} \int_{\varrho/|\boldsymbol{x}|}^{1} \bar{\mu}(\varrho, \lambda|\boldsymbol{x}|, |\boldsymbol{x}|)(1-\lambda^2)^{\frac{n-3}{2}}\, d\lambda\, d\boldsymbol{x},$$

$$\int_{\mathbb{S}^{n-1}} \mathrm{S}^{\nu}_{\varrho;\mathcal{K}}(\boldsymbol{u})d\boldsymbol{u} = |\mathbb{S}^{n-2}| \int_{\mathcal{K}\setminus\varrho\mathcal{B}} \bar{\nu}(\varrho, \varrho, |\boldsymbol{x}|)\frac{(|\boldsymbol{x}|^2-\varrho^2)^{\frac{n-3}{2}}}{|\boldsymbol{x}|}\, d\boldsymbol{x}.$$

Notice the shorthands $\mathrm{C}^{\cdot}_{\varrho;\cdot}$ or $\mathrm{S}^{\cdot}_{\varrho;\cdot}$ for $\mathrm{C}^{\cdot}_{\varrho\mathcal{B};\mathcal{K}}$ or $\mathrm{S}^{\cdot}_{\varrho\mathcal{B};\cdot}$, respectively.

Thus, cap and section functions lead to weighted volumes of the body by. Moreover, equality of $\mathrm{C}^{\mu}_{\varrho;\cdot}$ (or $\mathrm{S}^{\nu}_{\varrho;\cdot}$) for different bodies gives equality of the appropriate weighted volumes of those bodies. This gives the chance to use Lemma of double measurement.

Bodies with constant cap or section functions

Theorem 3.1 (Kurusa & Ódor, 2015; [9, Theorem 5.1,Theorem 5.2])

Let $0 < \varrho_1 < \varrho_2 < r$*. If* $\mathcal{K} \subset \mathbb{R}^n$ *is a convex body containing* $\varrho_2\mathcal{B}$*, and either* $\boxed{\mathrm{C}_{\varrho_1;\mathcal{K}} = \mathrm{C}_{\varrho_1;r\mathcal{B}},\ \mathrm{C}_{\varrho_2;\mathcal{K}} = \mathrm{C}_{\varrho_2;r\mathcal{B}}}$*, or* $\boxed{\mathrm{S}_{\varrho_1;\mathcal{K}} \equiv \mathrm{S}_{\varrho_1;r\mathcal{B}},\ \mathrm{S}_{\varrho_2;\mathcal{K}} \equiv \mathrm{S}_{\varrho_2;r\mathcal{B}}\ (n \neq 3)}$*, then* $\mathcal{K} \equiv r\mathcal{B}$*.*

Sketch of proof. According to *Lemma of weights* integration of $\mathrm{C}_{\varrho_i;\cdot}$ and $\mathrm{S}_{\varrho_i;\cdot}$ measures the bodies $\mathcal{K}$ and $r\mathcal{B}$ with the pairs of weights $\boxed{\omega_1(\boldsymbol{x}) = I_{1-\varrho_1^2/|\boldsymbol{x}|^2}(\frac{n-1}{2}, \frac{1}{2}),\ \omega_2(\boldsymbol{x}) = I_{1-\varrho_2^2/|\boldsymbol{x}|^2}(\frac{n-1}{2}, \frac{1}{2})}$, and $\boxed{\bar\omega_1(\boldsymbol{x}) = (|\boldsymbol{x}|^2 - \varrho_1^2)^{\frac{n-3}{2}}|\boldsymbol{x}|^{-1},\ \bar\omega_2(\boldsymbol{x}) = (|\boldsymbol{x}|^2 - \varrho_2^2)^{\frac{n-3}{2}}|\boldsymbol{x}|^{-1}}$, respectively, where I is the regularized incomplete beta function. As ω_1/ω_2 has $r\mathbb{S}^{n-1}$ as a level surface, *Lemma of double measurement* proves $\mathcal{K} \equiv r\mathcal{B}$. Except for $n = 3$ the weight $\bar\omega_1/\bar\omega_2$ also has $r\mathbb{S}^{n-1}$ as a level surface, and again *Lemma of double measurement* proves $\mathcal{K} \equiv r\mathcal{B}$. ∎

The problem for section functions is still open in dimension $n = 3$*.* The same way leads to:

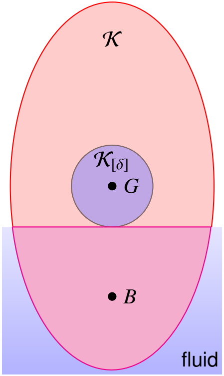
Theorem 3.2 (Kurusa & Ódor, 2015; [9, Theorem 5.3])

Let $0 < \varrho_1 \le \varrho_2 < r$*. If* $\mathcal{K} \subset \mathbb{R}^n$ *is a convex body containing* $\varrho_2\mathcal{B}$ *and* $\boxed{\mathrm{S}_{\varrho_1;\mathcal{K}} \equiv \mathrm{S}_{\varrho_1;r\mathcal{B}},\ \mathrm{C}_{\varrho_2;\mathcal{K}} \equiv \mathrm{C}_{\varrho_2;r\mathcal{B}}}$*, then* $\mathcal{K} \equiv r\mathcal{B}$*.*

Bodies with spherical floating body

Theorem 3.3 (Kurusa & Ódor, 2015; [10, Theorem 6.1])

Assume that the convex body $\mathcal{K}$ floats indifferently stable in every position. If $\mathcal{K}$ and $r\mathcal{B}$ have equal volume, equal density $\delta \in (0, 1/2)$ and common floating body $\mathcal{K}_{[\delta]} = (r\mathcal{B})_\delta$, then $\mathcal{K} \equiv r\mathcal{B}$.



G is the center of gravity,
B is the center of buoyancy.

Sketch of proof. Let ϱ be the radius of $\mathcal{K}_{[\delta]} = (\bar{r}\mathcal{B})_\delta$, $\bar{\mu}_{\hbar(\boldsymbol{u},r)}(\boldsymbol{x}) := 1$ and $\mu_{\hbar(\boldsymbol{u},r)}(\boldsymbol{x}) := \langle \boldsymbol{x}, \boldsymbol{u} \rangle$. Then

$$\mathrm{C}^{\bar{\mu}}_{\varrho;\mathcal{K}}(\boldsymbol{u}) = \int_{\mathcal{K}\cap\hbar^+(\boldsymbol{u},\varrho)} 1\, d\boldsymbol{x} \quad \text{and} \quad \mathrm{C}^{\mu}_{\varrho;\mathcal{K}}(\boldsymbol{u}) = \int_{\mathcal{K}\cap\hbar^+(\boldsymbol{u},\varrho)} \langle \boldsymbol{x}, \boldsymbol{u} \rangle\, d\boldsymbol{x}$$

are constant by the flotation and stability conditions. According to *Lemma of weights* these give the weights

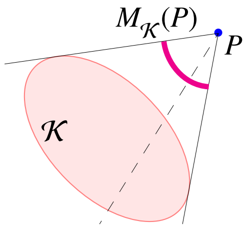
$$\omega_1(\boldsymbol{x}) = \int_{\varrho/|\boldsymbol{x}|}^1 (1-y^2)^{\frac{n-3}{2}}\, dy \quad \text{and} \quad \omega_2(\boldsymbol{x}) = |\boldsymbol{x}| \int_{\varrho/|\boldsymbol{x}|}^1 y(1-y^2)^{\frac{n-3}{2}}\, dy.$$

Now, *Lemma of double measurement* implies $\mathcal{K} \equiv r\mathcal{B}$, because

$$\frac{\omega_1(\boldsymbol{x})}{\omega_2(\boldsymbol{x})} = \frac{n-1}{2} \int_0^1 \frac{z^{\frac{n-3}{2}}}{((1-z)|\boldsymbol{x}|^2 + z\varrho^2)^{1/2}}\, dz$$

has $r\mathbb{S}^{n-1}$ as a level surface. ∎

Isomaskers of convex bodies



Point projection and its measure $M_{\mathcal{K}}(P)$ of the convex body $\mathcal{K}$ at point P.

The *masking number* $M_{\mathcal{K}}(P)$ of the convex body $\mathcal{K}$ at $P \notin \mathcal{K}$ is

$$M_{\mathcal{K}}(P) = \frac{1}{2}\int_{\mathbb{S}^{n-1}} \#(\partial\mathcal{K} \cap \ell(P, \boldsymbol{u}_\xi))d\boldsymbol{\xi}, \qquad ([8, (7.1)])$$

where # is the counting measure, $\partial\mathcal{K}$ is the boundary of $\mathcal{K}$, $\boldsymbol{\xi}$ is the spherical coordinate of the unit vector $\boldsymbol{u}_\xi \in \mathbb{S}^{n-1}$, and $\ell(P, \boldsymbol{u}_\xi)$ is the straight line through P with direction $\boldsymbol{u}_\xi$.

The *α-isomasker*[3] of the convex body $\mathcal{K} \subset \mathbb{R}^n$ is the set $\{P : M_{\mathcal{K}}(P) = \alpha\}$, where $\alpha \in (0, |\mathbb{S}^{n-1}|/2)$.

The α-isomasker ($\alpha \in [0, \pi]$) is called *α-isoptic* in the plane.

Problems. Does an isomasker determine a convex body?
[6, 7] How many isomasker determine a convex body?

Theorem 3.4 (Kurusa, 2013; [6])

Different convex bodies in the plane may have common α-isoptic if and only if $1 - \alpha/\pi \in \mathbb{Q}$ and its numerator is odd in its lowest terms.

If that numerator is odd, then there exist non-circular convex bodies with circular α-isoptic (Green [2]).

Only disks have two different circular isoptics (Nitsche [12]).

[3] The word isoptic refers to the case where the shape of the projection is constant. A result toward this direction can be found in [11].

Bodies with two spherical isomaskers

Theorem 3.5 (Kurusa & Ódor, 2015; [7])

If a ball and a convex body in $\mathbb{R}^n$ $(n \neq 3)$ have two different common isomaskers then they coincide.

Sketch of proof. The conditions imply $M_{\mathcal{K}}(\varrho_1 \boldsymbol{u}) = \alpha = M_{r\mathcal{B}^n}(\varrho_1 \boldsymbol{u})$ and $M_{\mathcal{K}}(\varrho_2 \boldsymbol{u}) = \beta = M_{r\mathcal{B}^n}(\varrho_2 \boldsymbol{u})$ for some $\alpha, \beta, \varrho_1 > \varrho_2 > r > 0$ and every $\boldsymbol{u} \in \mathbb{S}^{n-1}$, where $\mathcal{K} \subset \varrho_2 \mathcal{B}^n$ is the unknown convex body. Dualization gives for any convex body $\mathcal{L}$ that

(3.1) $$\int_{\mathbb{S}^{n-1}} M_{\mathcal{L}}(\varrho \boldsymbol{u})\, d\boldsymbol{u} = \frac{|\mathbb{S}^{n-1}|^2}{|\mathbb{S}^{n-2}|} - \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \mathrm{S}^{\nu}_{\frac{1}{\varrho};\mathcal{L}^\star}(\boldsymbol{u})\, d\boldsymbol{u},$$

where $\nu_{\hbar(\boldsymbol{u},\varrho)}(\boldsymbol{x}) = \varrho |\boldsymbol{x}|^{-n}$. Applying this to $\mathcal{K}$ and $r\mathcal{B}^n$ we obtain

$$\int_{\mathbb{S}^{n-1}} \mathrm{S}^{\nu}_{\frac{1}{\varrho_1};\mathcal{K}^\star}(\boldsymbol{u}_\xi)\, d\boldsymbol{\xi} = \int_{\mathbb{S}^{n-1}} \mathrm{S}^{\nu}_{\frac{1}{\varrho_1};\frac{1}{r}\mathcal{B}^n}(\boldsymbol{u}_\xi)\, d\boldsymbol{\xi} \quad \text{and} \quad \int_{\mathbb{S}^{n-1}} \mathrm{S}^{\nu}_{\frac{1}{\varrho_2};\mathcal{K}^\star}(\boldsymbol{u}_\xi)\, d\boldsymbol{\xi} = \int_{\mathbb{S}^{n-1}} \mathrm{S}^{\nu}_{\frac{1}{\varrho_2};\frac{1}{r}\mathcal{B}^n}(\boldsymbol{u}_\xi)\, d\boldsymbol{\xi}.$$

According to *Lemma of weights* and *Lemma of double measurement* one needs to consider the pairs of weights $\omega_1(\boldsymbol{x}) = \varrho_1 |\boldsymbol{x}|^{-n-1}(|\boldsymbol{x}|^2 - \varrho_1^{-2})^{\frac{n-3}{2}}$, $\omega_2(\boldsymbol{x}) = \varrho_2 |\boldsymbol{x}|^{-n-1}(|\boldsymbol{x}|^2 - \varrho_2^{-2})^{\frac{n-3}{2}}$. As ω_1/ω_2 is strictly monotone and constant on $r\mathbb{S}^{n-1}$, Lemma of Double Measurement proves $\mathcal{K} \equiv r\mathcal{B}$. ■

The problem is still open in dimension $n = 3$.

Thank you for your attention!

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Abstract of the talk

There are several situations that are similar to the following pair of statements about isoptics (the sets of points where a convex set subtends constant angle).

There are non-spherical convex bodies that have spherical isoptics.
If a convex body has two different spherical isoptics, then it is spherical.

These pairs demonstrate that *it pays to measure twice*.
In this talk we show several results of this kind. We prove an easy *Lemma of double measuring* that can be used for the proof of every such result. *Lemma of double measuring* is useful to establish coincidence of convex bodies by considering inequality of their volumes with respect to two different weights.
The talk is based on results of joint work with Tibor Ódor (University of Szeged) published in articles [9, 10, 7].