Orbital integrals on Lorentzian spaces

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Abstract. We present a rotation symmetric model in the Euclidean space for the Lorentzian of curvature -1 in which the Lorentzian spheres around the points of an apriori fixed spacelike totalgeodesic are straightlines. Investigating the mean value operators in this model yields to various representations of functions by means of their integrals over Lorentzian spheres.

1. Introduction

In this talk, I will speak on the problem of recovering a function from its integrals over the spheres in pseudo Riemannian spaces. As Helgason pointed out in [1,2], formulas representing functions by their mean values plays very important role in the theory of differential operators on higher rank symmetric spaces. Although more generality could be allowed, we shall concentrate mainly on the two dimensional Lorentzian space of curvature -1, because most of the interesting features of our subject appear in this setting and the formulation remains relatively natural.

An easy example to demonstrate the intimate connection between the orbital integrals and the differential operators is the well known Cauchy problem in three dimension

$$Lu = \frac{\partial^2 u}{\partial t^2}, \qquad u(x,0) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = f(x),$$

where $x \in \mathbb{R}^3$ and $f \in S(\mathbb{R}^3)$ is a given function. Using the commonly accepted notation $M^r f(x)$ for the integral of f on the sphere around x of radius $r > 0$, the solution takes the form $u(x,t) = tM^t f(x)$, that is Huygens' principle.

On the Riemannian manifolds it is very easy to determine a function f from its spherical mean values $M^r f$ simply by $f = \lim_{r\to 0} M^r f / M^r 1$. In the Lorentzian

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spaces the situation changes considerable – a Lorentzian sphere does not shrink to its center as its radius approaches zero and it is nor connected, as the light cone separates it into parts, neither compact.

We present a model for $\mathcal{L}^2$, that is obtained as the orthogonal projection of S. Helgason's quadratic surface model [2] along the straightline of the two ideal points. The set $\mathcal{M}^2$ of Lorentzian circles with center on an apriori fixed spacelike totalgeodesic E, the equator, becomes straightlines in this model, that helps a lot in representing the function f with its spherical integrals $M^r f$. First we show a representation similar in spirit to the Riemannian formulation. Then we use only the restricted set $\mathcal{M}^2$ of circles to give an other representation via the corresponding integral transform M, that integrates functions on circles in $\mathcal{M}^2$. This representation is also unique, i.e. M is invertible as an integral operator, despite of the restricted set of circles.

2. The model

Our following model for the Lorentz space of signature $(1,1)$ and of curvature -1, can be obtained from Helgason's quadratic hypersurface model [2]. This is defined in $\mathbb{R}^3$ by the bilinear form $B(x,y) = x_1y_1 - x_2y_2 - x_3y_3$ on the surface $Q^2_{-1} = \{x \in \mathbb{R} : B(x,x) = -1\}$. Q^2_{-1} with its Lorentz structure is axially symmetric around the x_1-axis, and is symmetric with respect to the origin. Helgason proved [2] that the geodesics are the intersections of Q^2_{-1} with the two dimensional subspaces of $\mathbb{R}^3$.

Our model $\mathcal{L}^2$ of the Lorentzian is the projection of Q^2_{-1} with its structure into the plane $x_1 = 0$ along the axis x_1. Then $\mathcal{L}^2$ is rotational symmetric around the origin. Let us take a point P on Q^2_{-1}, and take the plane π of $\mathbb{R}^3$ containing P and the x_1-axis. Clearly, the intersection of π with Q^3_{-1} is a hyperbola. Say, this hyperbola intersects the plane $x_1 = 0$ of $\mathbb{R}^3$ in the point O. Let r denote the Lorentzian distance of P from O. Then r is the Lorentzian distance of P from the equator E, the intersection of Q^2_{-1} and $x_1 = 0$. Let the coordinates of P be (p_1, p_2, p_3) in $\mathbb{R}^3$. Relative to π we may use the coordinates $\rho_1 = p_1$ and $\rho_2 = \sqrt{p_2^2 + p_3^2}$. These coordinates are $p_1 = \rho_1 = \sinh(r)$ and $\rho_2 = \cosh(r)$. Hence the projection $\mu(P)$ of P, where $\mu\colon Q^2_{-1} \to \mathcal{L}^2$ is the above mentioned projection, is a point in the corresponding direction, i.e. in π, and $|\mu(P)| = \cosh r$. We parameterize $\mathcal{L}^2$ according to the Euclidean polar coordinates of $\mathbb{R}^2$ so that (ω, r) means the point $\mu(P) = \omega \cosh r$, where $\omega \in S^1$. Because all the objects considered are symmetric to the plane $x_1 = 0$, we may, without restrictions, identify the points of Q^2_{-1} symmetric to this plane.

The Lorentzian inner product on $T_{(\omega,r)}\mathcal{L}^n$ is

$$\langle (dr, d\omega), (dr', d\omega')\rangle_{(\omega,r)} = dr dr' \sinh^{-2} r - \sum_{i=1}^{n-1} d\omega_i d\omega_i',$$

where dr means the radial part of the vector $(dr, d\omega) \in T_{(\omega,r)}\mathcal{L}^n$ and $d\omega$ means the part orthogonal, in Euclidean meaning, to the radius. We shall call $d\omega$ the spherical part.

The Lorentzian circle falls into four parts, determined by the timelike and spacelike vectors. We call these parts timelike and spacelike spheres, respectively. There are two connected spacelike spheres and two connected timelike spheres. In higher dimensions there are also two connected timelike spheres, but 'only' one spacelike sphere.

As usual, we parameterize the set of straightlines in $\mathbb{R}^2$, so that $H(\omega, p)$ denotes the straightline perpendicular to $\omega \in S^1$ and going through $p \cdot \omega \in \mathbb{R}^2$, where $p \in \mathbb{R}_+$. The corresponding Lorentzian set of points in $\mathcal{L}^2$ will be denoted by $\hat{H}(\omega, p)$. This correspondence is not one-to-one for $p < 1$, because the intersection of the corresponding plane with Q^2_{-1} falls into two parts.

Lemma 2.1. *For $p \neq 1$, the set $\hat{H}(\bar{\omega}, p)$ is the Lorentzian sphere around $(\bar{\omega}, 0) \in \mathcal{L}^2$ of radius*

$$r = \begin{cases} \operatorname{arccosh} p & \text{if } p > 1, \text{ timelike} \\ \arccos p & \text{if } p < 1, \text{ spacelike.} \end{cases}$$

$\hat{H}(\bar{\omega}, 1)$ is lightlike geodesic.

With the natural parameter p [3], the geodesics take the form

$$y^2 \frac{p^2 - 1}{p^2} + z^2 = 1.$$

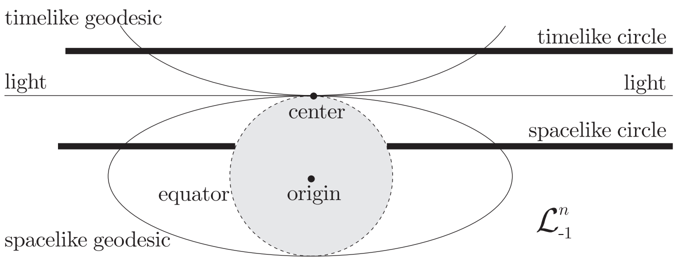

Figure 1. Circles and geodesics in $\mathcal{L}^2$.

Summing up, the Euclidean unit circle in $\mathcal{L}^2$ is the equator $Q^2_{-1} \cap \{x_1 = 0\}$. The Euclidean circle around the origin of radius $\cosh r$ is the 'ring' $Q^2_{-1} \cap \{x_1 = \sinh r\}$. The lightlike geodesics are the straightlines touching the unit sphere. The spacelike resp. timelike geodesics are ellipses resp. hyperbolas. The spacelike resp. timelike circles around the points of the equator are the straightlines intersecting or avoiding the Euclidean unit circle. (Note that we call a part of a sphere timelike or spacelike according to what type of geodesics it meets. We found this more natural than the naming convention [1,2], that gives the name according to the type of the tangent vectors, generally used for submanifolds.)

Lemma 2.2. *The arclength measure on the circle $\hat{H}(\bar{\omega}, p)$ at the point $X = (\omega, r) \in \mathcal{L}^2$ is*

$$dA = \frac{p\sqrt{|p^2-1|}}{\langle \omega, \bar{\omega}\rangle \sqrt{p^2 - \langle \omega, \bar{\omega}\rangle^2}} d\omega,$$

where $\cosh r = p/\langle \omega, \bar{\omega}\rangle$, $0 < \langle \omega, \bar{\omega}\rangle < p$ and $\langle ., .\rangle$ is the standard Euclidean inner product.

Proof. If $d\omega$ is the infinitesimal element at ω on S^1, and $\overline{di}$ is the corresponding Euclidean arclength element on $H(\bar{\omega}, p)$ at X, then $\overline{di} = \frac{p}{\cos^2\alpha} d\omega$, where $\cos\alpha = \langle \omega, \bar{\omega}\rangle$. The radial resp. spherical part of $\overline{di}$ are $dr = \overline{di}\sin\alpha$ resp. $ds = \overline{di}\cos\alpha$. Therefore

$$\begin{aligned} di^2 &= \langle \overline{di}, \overline{di}\rangle_X = (dr^2 \sinh^{-2} r - ds^2) \\ &= (\sin^2\alpha \sinh^{-2} r - \cos^2\alpha) \cdot \frac{p^2 d\omega^2}{\cos^4\alpha}. \end{aligned}$$

Via $\sinh^{-2} r = (\cosh^2 r - 1)^{-1} = \cos^2\alpha/(p^2 - \cos^2\alpha)$ this gives the statement. ∎

For $X \in \mathcal{L}^2$, let $L_X \subset T_X$ be the cone of lightlike vectors, C_X be the set of timelike vectors and D_X be the set of spacelike vectors. The angle γ of $\omega_1, \omega_2 \in T_X$ is defined for timelike and spacelike vectors differently:

$$\begin{aligned} \cosh^2\gamma &= d(\omega_1, \omega_2) \quad &\text{if} \quad &\omega_1, \omega_2 \in C_X \\ \sinh^2\gamma &= -d(\omega_1, \omega_2) \quad &\text{if} \quad &\omega_1, \omega_2 \in D_X, \end{aligned}$$

where

$$d(\omega_1, \omega_2) = \frac{\langle \omega_1, \omega_2\rangle_X^2}{\langle \omega_1, \omega_1\rangle_X \langle \omega_2, \omega_2\rangle_X}.$$

Some simple calculation gives

Lemma 2.3. *Let $X = (\bar{\omega}, r) \in \mathcal{L}^2$ and for every $\omega \in S^1$ let $\hat{\omega} \in \Sigma^1_X$ be tangent to $\hat{H}(\omega, \langle \omega, \bar{\omega} \rangle \cosh r)$. Then*

$$d\hat{\omega} = \frac{|\sinh r|}{\cosh^2 r \langle \omega, \bar{\omega} \rangle^2 - 1} d\omega \quad \text{on} \quad \Sigma^1_X \setminus L_X.$$

Note, that Lemma 2.2 and 2.3 give the same measure in basically different coordinate systems.

3. Spherical means on concentric spheres

We call a function on the Lorentzian space even resp. odd, if its representation f on Q^2_{-1} satisfies $f(x_1, x_2, x_3) = f(-x_1, x_2, x_3)$ resp. $f(x_1, x_2, x_3) = -f(-x_1, x_2, x_3)$. From the symmetry of the spacelike circles centered to points on the equator we deduce, that the odd functions have zero integral on these spheres. The timelike spheres centered to points on the equator never meet with the equator, hence the formulas representing the functions can not depend on the parity of the functions. Therefore, we restrict the considerations onto the even functions, that makes the mapping μ essentially one-to-one, but keep in mind that the timelike formulas are valid for any function.

First we introduce notations for the orbital integrals, that fits to our situation. For a function f integrable on all the circles $Mf(\omega, p)$ denotes its integral over the circle $\hat{H}(\omega, p)$ with the Lorentzian arclength measure determined in Lemma 2.2. We point our attention to the set $C^\infty_c(\mathcal{L}^2)$ of infinitely differentiable functions of compact support. Then by Lemma 2.2 we have

$$Mf(\bar{\omega}, p) = \int_{S^1_{\bar{\omega},p}} f\Big(\omega, \operatorname{arccosh}\Big(\frac{p}{\langle \omega, \bar{\omega} \rangle}\Big)\Big) \frac{p\sqrt{|p^2 - 1|}}{\langle \omega, \bar{\omega} \rangle (p^2 - \langle \omega, \bar{\omega} \rangle^2)^{1/2}} d\omega, \tag{3.1}$$

where $S^1_{\bar{\omega},p} = \{\omega \in S^1 : 0 < \langle \omega, \bar{\omega} \rangle < p\}$, and we used the coordinates of $\mathcal{L}^2$ for parameterizing f. Note that both arguments in the notation of $Mf(\omega, p)$ are Euclidean objects.

We shall frequently use the radially acting differential operators $\mathcal{D}_t$ resp. $\mathcal{D}_s$ that is defined on the functions $f \in C^\infty_c(\mathcal{L}^2)$ as

$$\mathcal{D}_t f = \frac{d}{dr}\Big(\frac{f(\omega, r)}{\sinh r \cosh r}\Big) \quad \text{and} \quad \mathcal{D}_s f = \frac{d}{dr}\Big(\frac{f(\omega, r)}{\sin r \cos r}\Big)$$

Theorem 3.1. *For $f \in C_c^\infty(\mathcal{L}^2)$*

$$f(\omega,0)=\lim_{r\to 0}\frac{-\sinh r}{2}\mathcal{D}_t\Big(\cosh r Mf(\omega,\cosh r)\Big) \qquad \textit{(timelike)}$$

and

$$f(\omega,0)=\lim_{r\to 0}-\sin r\mathcal{D}_s\Big(\cos r Mf(\omega,\cos r)\Big) \qquad \textit{(spacelike).}$$

Note that $\sinh r$ as well as $\sin r$ can be replaced with r.

Proof. We deal here only with the timelike case. Formula (3.1) takes the form

$$Mf(\alpha,p)=\int_{\alpha-\pi/2}^{\alpha+\pi/2} f\Big(\beta,\operatorname{arccosh}\Big(\frac{p}{\cos\beta}\Big)\Big)\frac{p\sqrt{|p^2-1|}}{\cos\beta(p^2-\cos^2\beta)^{1/2}}d\beta,$$

where we used angles instead of vectors.

The timelike circle $\hat{H}(\alpha,p)$ is symmetric to its point closest to the equator, therefore it is enough to prove for symmetric functions. With this in mind, substituting $\cos\beta=\frac{\cosh r}{\cosh z}$ yields to

$$Mf(\alpha,\cosh r)=2\int_r^\infty f(z)\frac{\cosh z\sinh r}{\sqrt{\cosh^2 z-\cosh^2 r}}dz.$$

Since $\cosh^2 z-\cosh^2 r=\sinh^2 z-\sinh^2 r$, partial integration with $f(z)/\sinh z$ gives

$$Mf(\alpha,\cosh r)=-2\sinh r\int_r^\infty\Big(\frac{f(z)}{\sinh z}\Big)'\sqrt{\sinh^2 z-\sinh^2 r}dz.$$

Now we can differentiate with respect to r and get

$$\begin{aligned}&\frac{-\sinh r}{2}\mathcal{D}_t\Big(\cosh r Mf(\omega,\cosh r)\Big)\\ &\qquad=\sinh^2 r\cosh r\int_r^\infty\frac{f(z)\cosh z-f'(z)\sinh z}{\sinh^2 z\sqrt{\sinh^2 z-\sinh^2 r}}dz.\end{aligned} \tag{3.4}$$

For $r<1$ we break up the integral as $\int_r^\infty=\int_r^{\sqrt{r}}+\int_{\sqrt{r}}^\infty$. For any $g\in C_c^\infty(\mathcal{L}^2)$

$$\left|\sinh^2 r\int_{\sqrt{r}}^\infty\frac{g(z)}{\sinh^2 z\sqrt{\sinh^2 z-\sinh^2 r}}dz\right|\le\int_0^\infty|g(z)|dz\frac{(\sinh r/\sinh\sqrt{r})^2}{\sqrt{\sinh^2\sqrt{r}-\sinh^2 r}},$$

where the right hand side goes to zero as $r \to 0$, therefore we are only interested in the integration $\int_r^{\sqrt{r}}$. Changing the variable $t = \sinh r / \sinh z$ we see

$$\left| \sinh^2 r \int_r^{\sqrt{r}} \frac{g(z)}{\sinh^2 z \sqrt{\sinh^2 z - \sinh^2 r}} dz \right| \leq \int_0^1 |g(\operatorname{arcsinh}(\frac{\sinh r}{t})) \frac{t}{\sqrt{1-t^2}} dt.$$

In case of $g(0) = 0$ the right hand side goes to zero along $r \to 0$, therefore the only not vanishing part in (3.4) is

$$\sinh^2 r \cosh r \int_r^{\infty} \frac{f(0) \cosh z}{\sinh^2 z \sqrt{\sinh^2 z - \sinh^2 r}} dz.$$

The substitution $\sinh z = \sinh r / t$ gives immediately the desired result.

∎

In a similar way we can show the following generalization of Helgason's similar formula [2], but with some differences. Our formula works in any dimensions and also works for spacelike spheres in even dimensions. Further, our differential operator is not invariant, but has half of the rank than Helgason's invariant operator.

Theorem 3.2. *Let $f \in C_c^\infty(\mathcal{L}^n)$. For even dimensions*

$$f(\omega, 0) = \lim_{r \to 0} \frac{(-1)^{[n/2]}(n-3)!!}{|S^{n-2}|} \mathcal{D}_t^{[n/2]} \Big(\cosh r M f(\omega, \cosh r) \Big)$$

and for odd dimensions

$$f(\omega, 0) = \lim_{r \to 0} \frac{(n-3)!! \sinh r}{|S^{n-2}|(-1)^{[n/2]}} \mathcal{D}_t^{[n/2]} \Big(\cosh r M f(\omega, \cosh r) \Big).$$

Proof. Observe, that the timelike spheres are rotation symmetric and the operator M commutes with the rotations around ω. This allows working only with functions rotation symmetric around ω.

∎

Considering the spacelike spheres in odd dimensions, it turns out, that there can not exist formulas representing general functions with formulas of this Riemannian style, because only some moments of f appears in Mf. However in even dimensions formulas can also be obtained.

Theorem 3.3. *Let $f \in C_c^\infty(\mathcal{L}^n)$. For even dimensions*
$$f(\omega, 0) = \lim_{r\to 0} \frac{(n-3)!!}{|S^{n-2}|} \mathcal{D}_s^{[n/2]} \Big(\cos r M f(\omega, \cos r) \Big).$$

The missing result for odd dimension raises the idea to look for other representations. We only sketch our results.

4. Spherical means on a restricted set of spheres

We consider the representation of functions by their spherical integrals over spheres centered to points of the equator. (Note that any spacelike totalgeodesic can be chosen as equator by the homogeneity of the space $\mathcal{L}^2$.) These spheres in our higher dimensional model are the hyperplanes. We restrict our considerations onto the even functions, to make the mapping μ essentially one-to-one, but keep in mind that the timelike formulas remains valid for any function.

We use all the spherical integrals $Mf(\omega, p)$, and from now on we think of $Mf(\omega, p)$ as a transform of the function f into a function on the set of spheres $\hat{H}(\omega, p)$.

The key is to connect the orbital integral transform M with the exterior Radon transform on the Euclidean space. We define $\mathbb{E}^n = \mathbb{R}^n \setminus B^n$, where B^n is the unit open ball.

Theorem 4.2. *Let $n > 2$ and $g \in C_c^\infty(\mathbb{E}^n)$. If $f(\omega, r) = g(\omega, \cosh r)|\sinh r|$ then*
$$\bar{R}g(\bar{\omega}, p) = Mf(\bar{\omega}, p) \cdot \frac{1/2}{\sqrt{|p^2-1|}} \qquad (p \neq 1),$$
where $\bar{R}$ denotes the Euclidean Radon transform on $\mathbb{R}^n$.

Proof. First, f is well defined as an even function on $\mathcal{L}^n$. It is well known [2] that
$$2 \cdot \bar{R}g(\bar{\omega}, p) = \int_{S^1} g\Big(\omega \cdot \frac{p}{\langle \omega, \bar{\omega}\rangle}\Big) \frac{p}{|\langle \omega, \bar{\omega}\rangle|^2} d\omega.$$
On the other hand, by the higher dimensional analog of formula (3.1) we have
$$\begin{aligned} \frac{Mf(\bar{\omega}, p)}{2\sqrt{|p^2-1|}} &= \frac{1}{2} \int_{S^{n-1}_{\bar{\omega},p}} f\Big(\omega, \operatorname{arccosh}\Big(\frac{p}{\langle \omega, \bar{\omega}\rangle}\Big)\Big) \frac{p^{n-1}}{\langle \omega, \bar{\omega}\rangle^{n-1} (p^2 - \langle \omega, \bar{\omega}\rangle^2)^{1/2}} d\omega \\ &= \frac{1}{2} \int_{S^{n-1}_{\bar{\omega},p}} g\Big(\omega \frac{p}{\langle \omega, \bar{\omega}\rangle}\Big) \Big(\frac{p^2}{\langle \omega, \bar{\omega}\rangle^2} - 1\Big)^{1/2} \frac{p^{n-1}\langle \omega, \bar{\omega}\rangle^{1-n}}{(p^2 - \langle \omega, \bar{\omega}\rangle^2)^{1/2}} d\omega \end{aligned}$$
that gives the statement immediately. ∎

Worth to note that $g \in C_c^\infty(\mathbb{E}^n)$ implies $f \in C_c^\infty(\mathcal{L}^n)$. Furthermore, if $g \in S(\mathbb{E}^n)$ then $f \in S(\mathcal{L}^n)$, that is, f and all of its derivatives are fast decreasing at the infinity.

Theorem 4.2 can be used to transfer most of the results known about the Radon transform on the Euclidean space to the spherical integral transform on the Lorentzian space. We make the transfer only for the support theorem and the inversion formula.

Theorem 4.3. *If $f \in C(\mathcal{L}^n)$ (not necessarily even!), $f(\omega, r)\cosh^k r$ is bounded for all $k \geq 0$ and for an $A > 0$ the spherical integrals $Mf(\omega, r)$ are zero for $r \geq A$ then f is zero for $r \geq A$.*

Proof. Let $g\colon \mathbb{E}^n \to \mathbb{R}$ be $g(\omega, \cosh r) = f(\omega, r)/\sinh r$. Then $\bar{R}g(\bar{\omega}, p) = Mf(\bar{\omega}, p)\frac{1/2}{\sqrt{|p^2-1|}}$. Our condition says $Mf(\bar{\omega}, p) = 0$ if $p > A$. Therefore we only have to show that g satisfies the conditions of Helgason's support theorem [2]. ∎

To formulate the transferred inversion formula, we need the operator

$$\Lambda\Phi(\omega, p) = \begin{cases} \dfrac{\partial^{n-1}}{\partial p^{n-1}}\Phi(\omega, p) & n \text{ odd} \\ \dfrac{i}{\pi}\displaystyle\int_{-\infty}^{\infty} \dfrac{\partial^{n-1}}{\partial t^{n-1}}\Phi(\omega, p)\dfrac{dt}{t-p} & n \text{ even} \end{cases}$$

for Φ in the Schwartz space of the set of hyperplanes in $\mathbb{R}^n$.

Theorem 4.4. *For $f \in S(\mathcal{L}^n)$*

$$cf = \cosh^{1-n} r M^t\Big(\frac{1}{|1-p^2|}\Lambda\Big(\frac{Mf(\omega, p)}{2\sqrt{|1-p^2|}}\Big)\Big),$$

where $c = (-4\pi)^{(n-1)/2}\Gamma(n/2)/\Gamma(1/2)$.

Proof. Observe $|\sinh r|R^t F(\omega, r) = \bar{R}^* G(\omega, \cosh r$ for $G(\omega, p) = F(\omega, p)/|1-p^2|$, where $\bar{R}^*$ is the Euclidean dual Radon transform. Substitute this and Theorem 4.2 into Helgason's Theorem 3.4 in [2]. ∎

Therefore the inversion is local in odd dimensions contrary the even dimensions, where the reconstruction needs Mf on all the spheres. On the other hand, this inversion formula hides the important aspect of the spherical integral transform, namely, that the integrals over the timelike spheres *alone* make the reconstruction possible. To prove this, one needs an extensive calculations with the spherical harmonic expansion of our integral transform, therefore we omit the details and give only the result, that that represents the functions by their integrals over timelike spheres via their spherical harmonic expansion. The functions are expanded into spherical harmonic series like

$$g(\varphi, p) = \sum_{m=-\infty}^{\infty} g_m(p) \exp(im\varphi) \quad \text{and} \quad g(\omega, p) = \sum_{\ell,m}^{\infty} g_{\ell,m}(p) Y_{\ell,m}(\omega)$$

in dimension two and in higher dimensions, respectively. The spherical harmonics $Y_{\ell,m}$, of rank m are known to constitute a complete polynomial orthonormal system in the Hilbert space $L^2(S^{n-1})$.

Theorem 4.6. *For $f \in S(\mathcal{L}^n)$ we have*
(i) *for $n \geq 3$*

$$f_{\ell,m}(s) = (-1)^{n-1} \frac{\Gamma(m+1)\Gamma(\lambda)}{\pi^{n/2}\Gamma(m+n-2)} \delta^{n-1} F_{\ell,m}(s),$$

where $\delta = \frac{d}{ds}\left(\frac{\cdot}{\sinh s}\right)$, $\lambda = (n-2)/2$ *and*

$$F_{\ell,m}(s) = -\int_s^{\infty} (Mf)_{\ell,m}(\cosh r) C_m^{\lambda}\left(\frac{\cosh r}{\cosh s}\right)\left(\frac{\cosh^2 r}{\cosh^2 s} - 1\right)^{\frac{n-3}{2}} \frac{\sinh s \cosh^{n-2} s}{\cosh^{n-1} r} dr.$$

(ii) *for $n = 2$*

$$f_m(s) = -\pi \frac{d}{ds} \int_s^{\infty} (Mf)_m(\cosh r) \frac{\cosh(m \operatorname{arccosh}(\cosh r/\cosh s))}{\cosh r \sqrt{\cosh^2 r/\cosh^2 s - 1}} dp.$$

A singular value decomposition is also possible via the spherical harmonic expansion according to Quinto's result [5].

In sum, the functions can be recovered from their timelike spherical integral transform that raises the similar question for the spacelike spherical integral transform. But the spacelike spherical integral transform $(Mf)_{\ell,m}$ in higher odd dimensions determines only some moments of $f_{\ell,m}$ and therefore the spacelike spherical integral transform does not allow general exact reconstruction. (One can determine its null space and range.)

To end, we give the uniqueness result for even dimensions and call attention that the uniqueness fails in two dimension.

Theorem 4.7. *If $f \in S(\mathcal{L}^n)$ and $n \geq 4$ is even then $f \equiv 0$ if the spacelike spherical integral transform of f is zero.*

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