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Recovering functions from their integrals on curves or surfaces

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In this talk, all I want to speak about happen on the two dimensional Euclidean space. One could allow more generality, but all the interesting features of our subject appear in the plane, while the formulation remains more natural.

The general framework is the following. Suppose that for each $\omega \in \Omega$, Ω is a set of indices, there is a family Γ_ω, we call this a *spread*, of nonintersecting curves resp. surfaces $g_{\omega,t}$, we call these *support curves* resp. *support surface*, in $\mathbb{R}^2$ resp. $\mathbb{R}^3$ so that $G_\omega = \cup_{t\in T} g_{\omega,t}$ is a connected submanifold of $\mathbb{R}^2$ resp. $\mathbb{R}^3$. Further, on each $g_{\omega,t}$ there is a measure $\mu_{\omega,t}$ so that we can introduce the *generalized Radon transform* R by

$$Rf(\omega,t) = \int_{g_{\omega,t}} f\mu_{\omega,t}ds \tag{1}$$

for functions integrable on each $g_{\omega,t}$, where ds is the arclength measure on $g_{\omega,t}$.

Note that $g_{\omega,t}$ is a foliation of G_ω, and the classical Radon transform defined on $\mathbb{R}^2$ by $\Omega = S^1$, $T = [0,\infty]$ and $g_{\omega,t} = \{x : \langle x,\omega\rangle = t\}$.

When the spreads and measures are known, the most interesting problems are

(a) to invert the transform R, and

(b) to characterize the range and the null space of R.

These are fairly well known by now. Also, a number of results are known in the more general point of view, when we ask

(c) for what type of the spreads is R invertible,

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(d) for what type of the measures is R invertible.

Recently a new direction came into interest. The problem is

(e) to determine the spreads Γ_ω,

(f) and the function f from only the knowledge of Rf.

Obviously the problem (f) depends very much on problem (e), that we call the *identification problem*. This will constitute the subject of this talk.

The identification problem has of course great relevance in applications, as in a number of applications one can measure only integrals. At the same time in the practice one knows very often at least some rough estimates or symmetry of the spreads and measures, and also often the functions, we call these *test function*, or at least the type of the functions, can be chosen to test the transform R. Anyhow, we see from (1), that in the identification problem we have much less data than we are interested in. Therefore a mathematically complete answer should impose very restrictive conditions on the spreads and measures or has to restrict the considerations onto functions in a small set.

To my knowledge the first result is of Mukhometov [20]. Let D be a compact domain with a piecewise smooth boundary ∂D, that is parameterized by arclength with α so that $\partial D = \{(x(\alpha), y(\alpha)) : \alpha \in [0, \delta]\}$ in the plane coordinated by (x, y), where δ is the length of ∂D. Suppose we have a Riemannian structure of the form $g_{(x,y)}(v, w) = r(x, y)\langle v, w\rangle$ on $\dot{D} = D \setminus \partial D$, where $\langle ., .\rangle$ is the ordinary Euclidean inner product. The curve $g_{\alpha,\beta}$ is the unique geodesic joining $(x(\alpha), y(\alpha))$ and $(x(\beta), y(\beta))$ and we suppose also that any two points in $\dot{D}$ can be joined by a unique geodesic. Further necessary assumption is on the geodesic $h_{x,y}(\omega, t)$ starting from $(x, y) \in \dot{D}$ in direction $\omega \in S^1$ with arclength t, the existence of a constant $C > 0$ so that $tC \le \det |\frac{\partial h}{\partial \omega}, \frac{\partial h}{\partial t}|$. As measure we take the Riemannian length, so our definition for the Radon transform gives

$$Rf(\alpha, \beta) = \int_{g_{\alpha,\beta}} f r \, ds \tag{2}$$

for, say, continuous functions on D, where ds is the Euclidean arclength measure on $g_{\alpha,\beta}$. Mukhometov's result is then the following.

Theorem 1. *If $r \in C^4(D)$, and Rf is known for $f \equiv 1$, then r, and so all the spreads are uniquely determined in D.*

Applying this result we get immediately the following: Led D be the unit circle, and $R1(\alpha, \beta) = 2\sin\frac{|\beta-\alpha|}{2}$. Then the spreads are parallel straightlines. In [21] Mukhometov generalized this for higher dimensions. Note however, that this result gives only uniqueness not any reconstruction.

The next result belongs to Natterer [22]. Now the spreads are known to be straightlines, that we parameterize as $g_{\omega,t} = \{(x,y) : x\omega_1 + y\omega_2 = t\}$, where $\omega = (\omega_1, \omega_2) \in S^1$ and $t \in \mathbb{R}_+$. We are interested in the measure and suppose that it has the form

$$d\mu_{\omega,t} = \exp\big(- \int_{g^+_{\omega,t}} \nu(x,y) ds \big),$$

where ν is a smooth compactly supported function on $\mathbb{R}^2$, $g^+_{\omega,t}$ is the half line $t\omega + s\omega^\perp$, where $s > 0$ and $\omega^\perp = (-\omega_2, \omega_1)$ and ds is the arclength measure. The result is the following.

Theorem 2. *If Rf is known for the unknown function f known to be a finite sum of Dirac measures, then $\mu_{\omega,t}(x,y)$ can be computed up to a multiplicative constant for all (ω,t) if (x,y) is in the support of f.*

Although the class of measures that are covered by this theorem seems to be quite small there is still not better general result that would cover *all* these measures in this reconstructive way. Although this reconstruction is valid only in the support of f that is a discrete point set of $\mathbb{R}^2$, assuming ν to be rotation symmetric, this determines μ completely and then by the known results on inverting the attenuated Radon transform will give also the function f. (This observation seems to have missed the attention of Natterer.) However, the theorem suffers the serious disadvantage, that no noise in f is allowed. A noise would be an L^2 function, but this is out of the range of the proof of this theorem.

J. Boman [5] tried to avoid the special conditions imposed on the measure in Natterer's result, but unfortunately he lost the reconstruction in exchange. To formulate his result we need to introduce two generalized Radon transform R and $\bar{R}$ in the usual meaning, like in Natterer's result, but without specializing the corresponding measures, μ and $\bar{\mu}$.

Theorem 3. *Assume f, g are finite sums of Dirac measures and functions in $L^2_c(\mathbb{R}^2)$ and $Rf = \bar{R}g$. Then the Dirac distributions of f and g have to be concentrated to the same points and there is a function a on $\mathbb{R}^2$ so that $\mu_{\omega,t}(x,y) = a(x,y)\bar{\mu}_{\omega,t}(x,y)$ for all (ω,t) if (x,y) is in the support of the Dirac deltas of f.*

To get back the lost reconstructibility one should find a description of the range of R. By now this seems to be almost impossible. Even for constant ν in Natterer's special class of measures no characterization of the range is known; only some necessary conditions are known. On the other hand one should pay attention, that here noise can occur in the test functions.

All these results show that to have more usable results more additional conditions are necessary. The set of test functions can not in fact further restricted, so one comes to find conditions on the spreads and measures.

So, I was looking for conditions on the spreads and measures that can guarantee the reconstructibility of the transform R, i.e. the spreads and measures in whole (!), meanwhile allows to use test functions with noise. These are found in [17] and now I would like to present the main results of my work [17].

We work with the spreads Γ_α, so that Γ_α is Γ_0 rotated around the origin by angle α. Γ_0 consists of support curves $g_{0,r}$, where $r \in (0,\infty)$, that are closed, passing through the origin and have exactly two intersections with each of the circle of radius $\varrho \in (0,r)$ centered to the origin. We assume further, that $|P_r| = r$ for the point $P_r \in g_{0,r}$ farthest from the origin and that $g_{0,r}$ is symmetric with respect to the straight line, that is, say, the first axis, through P_r and the origin. The curve $g_{\alpha,r}$ is $g_{0,r}$ rotated around the origin by the angle α.

The conditions imply that in polar coordinates $g_{0,r}$ can be parameterized as $g_{0,r}(\varrho) = (\varrho, \varphi_r(\varrho))$ in the positive (upper) half plane and as $g_{0,r}(\varrho) = (\varrho, -\varphi_r(\varrho))$ in the negative (downward) half plane so that $|g_{0,r}(\varrho)| = \varrho \in [0,r]$ and $\varphi_r(\varrho)$ is the angle of $g_{0,r}(\varrho)$ to the first axis. We assume further that $\frac{\varphi_r(\varrho)}{\sqrt{r-\varrho}} \in \mathrm{C}^2(\{(r,\varrho) \in \mathbb{R}^2 : 0 \le \varrho \le r\})$.

The arclength measure on $g_{0,r}$ is $\sqrt{1+\varrho^2\dot{\varphi}_r^2(\varrho)}d\varrho$. Therefore, using polar coordinates, the Radon transform takes the form

$$R_\Gamma f(\alpha,r) = \int_0^r \left(f(\alpha+\varphi_r(\varrho),\varrho) + f(\alpha-\varphi_r(\varrho),\varrho)\right)\sqrt{1+\varrho^2\dot{\varphi}_r^2(\varrho)}\,\mathrm{d}\varrho.$$

In [18] I showed that an invertible Radon transform should have, in a certain sense, support curves like the above defined ones, without the symmetry, therefore the conditions are not as restrictive.

Theorem 4. *If all the elements of Γ_0 are similar to each other, and $R_\Gamma f(\omega,r) = F(\omega,r)$ is known for an unknown function f known to be a sum of finitely many Dirac measures and an L^2 function, then Γ can be computed.*

Although this is not as general result than the previous one of Boman, but it covers a large set of problems in practice and is *reconstructive*. The only disadvantage we can mention is that the reconstruction needs infinitely many trying in the computation (not in the measuring!). Similar result is proved in [17] for measures on the straightlines:

Theorem 5. *If $\mu_{\omega,r}(x,y) = \nu(\sqrt{x^2+y^2}/r)$ for a smooth function ν, and $Rf(\omega,r) = F(\omega,r)$ is known for an unknown function f known to be a sum of finitely many Dirac measures and an L^2 function, then μ can be computed.*

The idea behind these results is that for the function $f = \sum_{i=1}^m \delta_{P_i} + \ell$, where $\ell \in \mathrm{L}^2(\mathbb{R}^2)$ and δ_{P_i} is the Dirac measure at $P_i \in \mathbb{R}^2$, we have that

$$\langle F,h\rangle = \langle R_\mu f,h\rangle = \langle f,R_\mu^* h\rangle = \langle f,R_{\mu^*}h\rangle = \sum_{i=1}^m R_{\mu^*}h(P_i) + \langle R_\mu \ell,h\rangle$$

for any function $h \in \mathrm{L}^2(\mathbb{R}^2)$, where μ^* is the measure dual to μ. Observing the functions

$$h_{(x,y),j}(x',y') = \begin{cases} 0 & \text{if } (x'-x)^2+(y'-y)^2 > 1/j^2, \\ j & \text{if } (x'-x)^2+(y'-y)^2 < 1/j^2, \end{cases}$$

one sees that $\lim\langle R_\mu \ell, h_{(x,y),j}\rangle = 0$ for all $(x,y) \in \mathbb{R}^2$ as $j \to \infty$. The limit $\lim R_{\mu^*}h_{(x,y),j}(P_i)$ is not zero if and only if (x,y) is on the sphere with diameter OP_i. For these, we find

$$\lim_{j\to\infty}\langle F, h_{(x,y),j}\rangle = \nu(|P_i|/\sqrt{x^2+y^2})$$

that determines μ.

Our following result is of interest when test function can not be chosen, but from some other considerations the existing test function can be determined.

Theorem 6. *If all the elements of* Γ *are similar to each other, then it is determined by knowing* $F(\bar{\omega}, r) = R_\Gamma f(\bar{\omega}, r)$ *together with the function* $f \in \mathrm{C}^2(\mathbb{R}^2) \cap \mathrm{L}^2(\mathbb{R}^2)$ *that is zero in a neighborhood of the origin and has no vanishing negative moment.*

Proof. Let $If(X) = |X| \int_{\mathrm{S}^1} f(|X|\omega)\,\mathrm{d}\omega$. Then $R_\Gamma If(\bar{\omega}, r) = IR_\Gamma f(\bar{\omega}, r) = IF(\bar{\omega}, r)$, hence we can suppose f to be radial.

We have

$$R_\Gamma f(r\bar{\omega}) = 2\int_0^r f(\varrho\bar{\omega})\sqrt{1 + \frac{\varrho^2}{r^2}\dot{\varphi}^2\left(\frac{\varrho}{r}\right)}\,\mathrm{d}\varrho,$$

where $\bar{\omega}$ is a unit vector and $\varphi \equiv \varphi_1$. Substituting $\varrho = rt$ we obtain

$$r^{-1}R_\Gamma f(\bar{\omega}, r) = 2\int_0^1 f(rt\bar{\omega})\sqrt{1 + t^2\dot{\varphi}^2(t)}\,\mathrm{d}t.$$

Integration by r^{-k}, where $k \geq 1$, over $[0, \infty]$ results in

$$\int_0^\infty r^{-1-k}R_\Gamma f(\bar{\omega}, r)\,\mathrm{d}r = 2\int_0^\infty \varrho^{-k} f(\varrho\bar{\omega})\,\mathrm{d}\varrho \times \\ \times \int_0^1 t^{k-1}\sqrt{1 + t^2\dot{\varphi}^2(t)}\,\mathrm{d}t.$$

Since the negative moments of f are not zero, this gives the moments of

$$v(\varrho) = \sqrt{1 + \varrho^2\dot{\varphi}(\varrho)}$$

and this finishes the proof. $\square$

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