talk at the Universität der Erlangen-Nürnberg, 16$^{\underline{h}}$ November 15., 1993

Shape recognition in convex geometry

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First of all, I would like to thank to Prof. D. Kölzow for his invitation and for his unselfish help. It is my pleasure to make research and especially to give talks on my subject at such a distinguished university as the Universität der Erlangen-Nürnberg.

In this talk I do not want to present serious computations nor any complicated definition. I would like to call the attention for this beautiful part of the mathematics, which has a lot of open problems that do not need too much prerequisites. I hope this lecture will give also a good introduction to the forthcoming lecture of Volcič, as the discussed problems are closely related.

The circle of the problems I am speaking about was born when Hammer [9] asked in 1961 on the AMS Symposium on Convexity that "*How many X-ray pictures of a convex body must be taken to permit its exact reconstruction?*". Although there are some "prehistorical" article about this question, it was first raised at this time and the nature of the problem had a big influence on the mathematicians. The question itself was not a big surprise, because the investigation of the X-ray pictures and its generalizations, the Radon transform for example, was very popular at that time. The really new part in it was its nature, that is, it asked for *shape recognition.*

All I want to speak about happen on the two dimensional Euclidean plane. One could allow bigger generality, but the beauties of the problems are more accessible and more natural on the plane.

The X-ray picture of a convex compact domain $\mathcal{D}$ is defined differently according to the use of parallel beam or divergent beam X-ray.

AMS subject classifications (1980): 0052,0054
Key words and phrases: convex geometry

In the parallel beam case the X-ray picture of the domain $\mathcal{D}$ ordered to a direction, i.e. to a unit vector $\omega \in \mathrm{S}^1$, is a function on the one dimensional subspace orthogonal to ω, defined by

$$X_\omega^{\mathcal{D}}(x) = \int_{-\infty}^{\infty} \chi_{\mathcal{D}}(x\omega^\perp + \lambda\omega)\,\mathrm{d}\lambda \quad (x \in \mathbb{R},\ \omega^\perp \perp \omega), \tag{1}$$

where $\chi_{\mathcal{D}}$ is the indicator function of $\mathcal{D}$.

In the divergent beam case the X-ray picture ordered to a point $P \in \mathbb{R}^n$, called source, is a function on S^1 defined by

$$X_P^{\mathcal{D}}(\omega) = \int_{-\infty}^{\infty} \chi_{\mathcal{D}}(P + \lambda\omega)\,\mathrm{d}\lambda \quad (\omega \in \mathrm{S}^1). \tag{2}$$

The real geometric content of these formal, but useful, definitions is that these give the lengths of the chords the domains $\mathcal{D}$ cut out from the straight lines through P or, in the other case, parallel to ω. That is why these functions (1) and (2) are called chord-functions too.

The uniqueness part of Hammer's question is quite well known already:

- First O. Giering [7] proved in 1963, that for a given convex domain $\mathcal{D}$ three appropriate parallel beam X-ray pictures are enough to distinguish it from any other.(Gardner [6] proved in 1983 that two can not be enough.)
- R.J. Gardner and P. McMullen [3] showed in 1980 that four universal but well chosen parallel beam X-ray pictures make difference between any two convex compact domains.
- K.J. Falconer [1] in 1983 proved that two divergent beam X-ray pictures distinguish any two convex compact domains if they intersect either the segment of the sources or one of the half lines determined by these sources.
- A. Volčič [15] gave the same result as Falconer in 1985 for four X-ray pictures with sources in general position, but without the intersection condition.
- Finally, as I know, R.J. Gardner [4] in 1987 proved the result of Volčič for four X-ray pictures the sources of which are collinear.

Contrary the uniqueness, the existence part of Hammer's problem is almost completely unknown (to my knowledge). To see the exceptionally known results one needs to generalize the X-ray picture using measures. For a measure we take a locally integrable strictly positive, we need this not to loose the geometric meaning, function μ on $\mathbb{R} \times \mathrm{S}^1 \times \mathbb{R}$ and make the generalizations as follows.

The generalized parallel beam X-ray picture of the domain $\mathcal{D}$ for a direction ω by

$$X_\omega^{\mathcal{D}}(x) = \int_{-\infty}^{\infty} \chi_{\mathcal{D}}(x\omega^\perp + \lambda\omega)\mu(\lambda, \omega^\perp, x)\,\mathrm{d}\lambda. \tag{3}$$

The generalized divergent beam X-ray picture for a point P is defined by

$$X_P^{\mathcal{D}}(\omega) = \int_{-\infty}^{\infty} \chi_{\mathcal{D}}(P + \lambda\omega)\mu(\lambda, \omega^\perp, \langle P, \omega^\perp\rangle)\,\mathrm{d}\lambda, \tag{4}$$

where $\langle .,. \rangle$ is the usual inner product of $\mathbb{R}^2$. Here $\mu(., \omega, r)$ is the "measure" on the straight line going through $r\omega$ and perpendicular to ω, and therefore we need $\mu(\lambda, \omega, r) = \mu(\lambda, -\omega, -r)$. We call μ the weight function. It is somewhat surprising, that almost the same uniqueness results can be proved for these generalized X-ray pictures as for the original ones. This is partially done by Gardner [4].

In this setting a lot of convex geometric problems become a part of our original question:

- *The equichordal problem*; Is there a convex compact domain with two equichordal points? An inner point is said to be equichordal if the X-ray picture function is constant at that point. [1]
- *The equireciprocal problem*; Is the ellipse the only compact convex domain with two equireciprocal points? An inner point is said to be equireciprocal, if the generalized X-ray picture function for the "measure" $(1 - \delta(\lambda))/\lambda^2$ is constant at that point [2]. (From the polar coordinatization centered to a focus of the ellipse we get that the ellipse has two such points.)
- *The equiproduct problem*; Is the circle the only convex compact domain with two equiproduct points? An inner point is said to be equiproduct if the generalized X-ray picture function for the "measure" $(1-\delta(\lambda))/|\lambda|$ is constant at that point [16]. (The butterfly theorem shows that all the points of a disc is equiproduct.)

Although the latter two problems seem more complicated than the first one, the existence problem is solved only for these two [2,16]! In general meaning, the existence problem is equivalent to the characterization of the range of the map which corresponds some X-ray pictures to a convex domain, but even in this direction nothing is known.

One can regard the uniqueness results for the X-ray pictures as a proof that the X-ray pictures contain really much information about the domains. This gives the idea to try to obtain similar results for poorer pictures that contain less information. J. Kincses, one of my colleagues, proposed to use the shadow picture [10]. This can be interpreted in the way that the domain is impenetrable for the X-rays.

Exactly, we have two types of the shadow pictures again. The shadow picture of a domain from a direction is defined as the distance of the domain's two tangents parallel to the given direction [10]. This , actually, is called the width of the domain, and obviously does not determine a convex domain even if we know it for all the directions, as the Roleaux triangle and the circle show, for example.

For a point, the shadow picture is defined as the angle of the domain's two tangents through the given point. This is called the visual angle of the domain at the given point [10]. Of course, a finite set of such shadow pictures does not determine a convex domain, but it turned out recently, that uniqueness can be proved for shadow pictures taken at all the points of some curves [10,11,12].

Concerning this problem, J. Green [8] and J.C.C. Nietsche [13] must be mentioned. Green proved that a set subtending a constant angle on a circle must be a circle except some types of the angle ($\frac{2m}{n}$, where n is odd) in which case he gave counterexamples (ellipses) [8]. Then Nietsche [13] showed that a set subtending constant angles on two concentric circles must be a circle. We must note that the tricky calculations they used do not make any sense for the general problem.

From our investigations with J. Kincses [10] it became clear that one curve will not be enough to get uniqueness for general convex domains. However, we obtained the following result for convex polygons.

Theorem [10]. *Let $\mathcal{C}$ be the nonempty intersection of finitely many convex domains with analytic boundaries. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be convex polygons in the interior of $\mathcal{C}$. If $\mathcal{P}_1$ and $\mathcal{P}_2$ subtend the same visual angle at all the points of $\partial\mathcal{C}$ then they coincide.*

This result is really unexpected, because the convex domains and the polygons usually behave very similarly. (Similar result for the parallel beam X-ray picture is recently proved by Gardner and Gritzmann [5]. For the divergent beam X-ray pictures the question is very easy in the case of the polygons — any two of them are enough.)

Much better result can hardly be richen for only one curve, because two equal chords of a circle can not be distinguished locally from an arc of the circle. Therefore the question raised naturally if the boundaries of two convex domains may be enough for the distinction (we drew up this question only for concentric circles in [10]).

In [11] I proved the following result for general domains:

Theorem [11]. *Let $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{D}_1$ and $\mathcal{D}_2$ be closed convex domains having* C^2 *boundaries. $\mathcal{D}_1 \cup \mathcal{D}_2$ is compact and is in the interior of $\mathcal{C}_1 \cap \mathcal{C}_2$. The boundaries of $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect each other in nonzero angles. If $\mathcal{D}_1$ and $\mathcal{D}_2$ subtend the same visual angle at each of the points of $\partial\mathcal{C}_1 \cup \partial\mathcal{C}_2$ then they coincide.*

The intersection condition in this theorem seems very unnatural, and therefore, the next question was to find similar statements for nonintersecting curves. This is done in [12].

Theorem [12]. *Let $\mathcal{D}_1$, $\mathcal{D}_2$ be compact convex domains. $\mathcal{C}$ is a compact domain so that $\mathcal{D}_1 \cup \mathcal{D}_2 \subset \operatorname{Int}\mathcal{C}$. Further, g_1 and g_2 are arbitrary straight lines not intersecting $\mathcal{D}_1 \cup \mathcal{D}_2$. If the visual angles of $\mathcal{D}_1$ and $\mathcal{D}_2$ are equal at each point of g_1, g_2 and $\partial\mathcal{C}$ then $\mathcal{D}_1 \equiv \mathcal{D}_2$.*

Using the same idea, but a more complicated calculation I arrived to the following result. Note, that a hyperbola separates the plane into tree parts two of which are convex. The union of these convex parts for the hyperbola h is denoted by $Conv(h)$.

Theorem [12]. *Let $\mathcal{D}_1$, $\mathcal{D}_2$ be compact convex domains. h_1 and h_2 are hyperbolas so that their respective asymptotes are parallel and $Conv(h_1) \cup Conv(h_2)$ contains $\mathcal{D}_1$ and $\mathcal{D}_2$. If the visual angles of $\mathcal{D}_1$ and $\mathcal{D}_2$ are equal at each point of g_1 and g_2 then $\mathcal{D}_1 \equiv \mathcal{D}_2$.*

Because a mathematical talk without any proof is like a soup without salt, we prove her the following much easier theorem. The others need very extensive calculations, although their spirit is not far from this one.

Theorem [12].Let $\mathcal{C}_i$ $(i \in \mathbb{N})$, $\mathcal{D}_1$ and $\mathcal{D}_2$ be compact convex domains so that $\mathcal{D}_1 \cup \mathcal{D}_2 \subset \operatorname{Int} \mathcal{C}_0$ and $\mathcal{C}_i \subset \operatorname{Int} \mathcal{C}_{i+1}$ for $(i \in \mathbb{N})$. *If the S-pictures of $\mathcal{D}_1$ and $\mathcal{D}_2$ are equal at each point of each $\partial\mathcal{C}_i$, then $\mathcal{D}_1 \equiv \mathcal{D}_2$.*

Proof. Let t_0 be a common tangent of $\mathcal{D}_1$ and $\mathcal{D}_2$. It intersects each $\partial\mathcal{C}_i$ in two points, P_i and Q_i. Let $T_1 = t_0 \cap \mathcal{D}_1$ and $T_2 = t_0 \cap \mathcal{D}_2$. T_1 and T_2 divide the straight line t_0 into tree parts. On one of the two infinite parts are the intersection points P_i and on the other one are the intersection points Q_i $(i = 0, 1, 2, \ldots)$.

Let P be a limit point of $\{P_i\}$, which may be the infinity. At each point P_i there must be an other common tangent of $\mathcal{D}_1$ and $\mathcal{D}_2$, say t_i. Since P is a limit point, the sequence t_i must have a limit straight line t^P through P which is a common tangent. Since t^P is a limit of common tangents, it must touch the domains in a point of $\partial\mathcal{D}_1 \cap \partial\mathcal{D}_2$.

The point P must also be a limit point of the intersection points $t^P \cap \partial\mathcal{C}_i$. Through these intersection points there must be other common tangents t_i^P. Obviously t_0 is a limit straight line of these common tangents t_i^P, hence T_1 should coincide with T_2, i.e. every common tangent touches the two domains in a point of $\partial\mathcal{D}_1 \cap \partial\mathcal{D}_2$.

Also every point of $\partial\mathcal{D}_1 \cap \partial\mathcal{D}_2$ is a touching point of a common tangent, because otherwise a common tangent bridging the point of $\partial\mathcal{D}_1 \cap \partial\mathcal{D}_2$ would exist, that touches $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_2$ in different points.

Let $\tilde{P}_i \in \tilde{\mathcal{C}}_i$ be a monotone subsequence of P_i tending to P. Then the sequence $\tilde{\mathcal{C}}_i$ is also monotone with respect to the inclusion relation, i.e. either $\tilde{\mathcal{C}}_i \subset \tilde{\mathcal{C}}_{i+1}$ for each i or $\tilde{\mathcal{C}}_{i+1} \subset \tilde{\mathcal{C}}_i$ for each i. Let $\tilde{Q}_i \in (t_0 \cap \partial\tilde{\mathcal{C}}_i)$ be different from $\tilde{P}_i$. The sequence $\tilde{Q}_i$ is monotone, because $\tilde{\mathcal{C}}_i$ is monotone. Moreover, if R denotes the common touching point of t_0, then $|\tilde{P}_i R|$ is decreasing or increasing together with $|\tilde{Q}_i R|$. Let Q be the limit of $\tilde{Q}_i$, which may be the infinity. As in the case of P we obtain a new common tangent through Q, say t^Q. Q is a limit point of the intersection points $t^Q \cap \partial\tilde{\mathcal{C}}_i$, which implies a sequence of common tangents t_i^Q through these points tending to t_0. Obviously, the common touching points of t_i^Q and t_j^P, $i, j \in \mathbb{N}$, are on the same side of t_0 and they tend to R.

Therefore a touching point of a common tangent is a limit point of $\partial\mathcal{D}_1 \cap \partial\mathcal{D}_2$ so that it has convergent sequences from both directions, clockwise and anti-clockwise. Hence, each point of $\partial\mathcal{D}_1 \cap \partial\mathcal{D}_2$ is a limit point of $\partial\mathcal{D}_1 \cap \partial\mathcal{D}_2$ from both directions.

Assume now an arc on $\partial\mathcal{D}_1$ not intersecting $\partial\mathcal{D}_2$. Then there must exist two points of $\partial\mathcal{D}_1 \cap \partial\mathcal{D}_2$, because this is closed of course, that are the closest ones on $\partial\mathcal{D}_1$ to that arc from clockwise and anti-clockwise directions on $\partial\mathcal{D}_1$. But these points must have convergent point sequences in $\partial\mathcal{D}_1 \cap \partial\mathcal{D}_2$ from both direction, i.e. they can not be the first in any directions.

This contradiction proves that $\partial\mathcal{D}_1 \cap \partial\mathcal{D}_2$ is dense in $\partial\mathcal{D}_1$, i.e. $\partial\mathcal{D}_1 \equiv \partial\mathcal{D}_2$, which was to be proved. □

A number of questions are open in the shadow picture problem.

- The existence problem is totally open. The only known result is that of Green and Nietsche.
- One may feel that the straight lines and the hyperbolas are not really non-intersecting curves as they can be considered to intersect each other at the infinity. Thus in this direction the best result that can be imagined would be the uniqueness proved for any two convex bounded curves. I would be satisfied with two concentric circles although this can happen to be the most difficult to prove.
- T. Ódor called my attention to the fact that there is some duality between the X-ray picture and the shadow picture. He has new results from this relation to both of the problems.

If we allow ourselves to a wider point of view, a number of similar problems can be discovered.

- It is not hard to prove that if $\mathcal{D} \subset \mathcal{C}$ convex compact domains and the two tangents of $\mathcal{D}$ are equal at any point $P \in \mathrm{Out}\mathcal{C}$, then $\mathcal{D}$ is circle. What can we say about $\mathcal{D}$, if we have the difference of the two tangents at any $P \in \mathrm{Out}\mathcal{C}$?
- As a generalization of the X-ray problem we may ask on what kind of a straight line set is necessary to know the chord-function to be able to distinguish any two compact convex domains? For example: Is the tangent bundles of some circles enough?
- Let P be a point in $\mathcal{D}$, a convex compact domain. Any straight line intersects $\partial\mathcal{D}$ in two points, where there are tangents. Thus we can have a function at any inner point, which says for any direction ω the angle of the tangents taken at the intersections of $\partial\mathcal{D}$ and the straight line through the point with the direction ω. How many points are necessary to determine $\mathcal{D}$?

References

[1] K.J. Falconer, *X-ray problems for point sources*, Proc. London Math. Soc. **46**(1983), 241-262.

[2] K.J. Falconer, *On the equireciprocal problem*, Geom. Dedicata **14**(1983), 113-126.

[3] R.J. Gardner and P. McMullen, *On Hammer's X-ray problem*, J. London Math. Soc. **21**(1980), 171-175.

[4] R.J. Gardner, *Chord functions of convex bodies*, J. London Math. Soc. **36**(1987), 314-326.

[5] R.J. Gardner, *Symmetrals and X-rays of planar convex bodies*, Arch. Math. (Basel) **41**(1983), 183-189.

[6] R.J. Gardner and P. Gritzmann, *Successive determination and verification of polytops by their X-rays*, preprint .

[8] J.W. Green, *Sets subtending a constant angle on a circle*, Duke Math. J. **17**(1950), 263-267.

[9] P.C. Hammer, *'Problem 2', Proc. of Symp. in Pure Math. – Vol. VII: Convexity*, Amer. Math. Soc., Providence, R.I., 1963.

[10] J. Kincses and Á. Kurusa, *Can you recognize the shape of a figure from its shadows?*, submitted to Amer. Math. Monthly .

[11] Á. Kurusa, *You can recognize the shape of a figure from its shadows!*, submitted to Geom. Dedicata .

[12] Á. Kurusa, *The shadow picture problem for nonintersecting curves*, submitted to Geom. Dedicata .

[13] J.C.C. Nietsche, *Isoptic characterization of a circle, (Proof of a conjecture of M.S. Klamkin)*, Amer. Math. Monthly **97**(1990), 45-47.

[14] C.A. Rogers, *An equichordal problem*, Geom. Dedicata **10**(1981), 73-78.

[15] A. Volčič, *A three-point solution to Hammer's X-ray problem*, J. London Math. Soc. **34**(1986), 349-359.

[16] K. Yanagihara, *On a characteristic property of the circle and the sphere*, Tôhoku Math. J. **10**(1916), 142-143.

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