Identifying X-ray transforms: the boundary-distance rigidity of projective metrics

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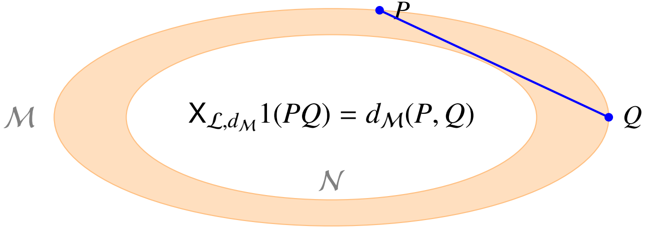

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In the Footsteps of Allan MacLeod Cormack On the Fortieth Anniversary of his Nobel Prize at Tufts University (Medford, Massachusetts, USA)
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Definition of generalized Radon transforms

Let $\mathcal{S}_w$ be a set of hypersurfaces $S_{w,t}$ in $\mathbb{R}^n$ such that $w \in \mathbb{S}^{n-1}$ and $t \in [0, \infty)$.

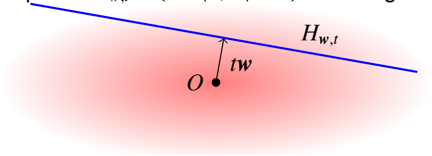
The *Radon transform* $\mathrm{R}_{\mathcal{S},\mu}$ of functions $f : \mathbb{R}^n \to \mathbb{R}$ integrable on each $S_{w,t}$ is defined by

$$\text{(1.1)} \qquad \mathrm{R}_{\mathcal{S},\mu} f(w, t) = \int_{S_{w,t}} f(x)\mu_{w,t}(x)\, dx,$$

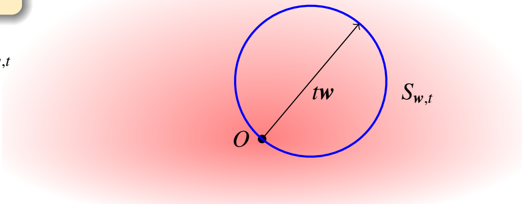
where dx is the natural surface measure on $S_{w,t}$ and $\mu_{w,t}$ is a strictly positive continuous function on $S_{w,t}$ that depends continuously on w and t.

In this definition, the hypersurfaces $S_{w,t}$ are called *petals*, set $\mathcal{S} = \bigcup_{w\in\mathbb{S}^{n-1}} \mathcal{S}_w$ is called *flower* and $\mu_{w,t}$ is called the *weight* on the petal $S_{w,t}$.

The “classic” Radon transform is $\mathrm{R}_{\mathcal{H},1}$, where $\mathcal{H} = \{H_{w,t} : w \in \mathbb{S}^{n-1},\ t \in [0, \infty)\}$ is the flower of the petals $H_{w,t} = \{x : \langle x, w\rangle = t\}$ with weight 1.



The “dual” Radon transform is $\mathrm{R}_{\mathcal{S},1}$, where $\mathcal{S} = \{S_{w,t} : w \in \mathbb{S}^{n-1},\ t \in [0, \infty)\}$ is the flower of the petals $S_{w,t} = \{x : \langle tu_w - x, x\rangle = 0\}$ with weight 1.



Characterizations and identifications

Characterizing by invariances. **Quinto** proved in [22] that *translation invariant* Radon transforms have *exponential weight*. (See also [12] for a slightly different result.) **Quinto** proved in [23] that *rotationally invariant* Radon transforms have rotational weight. (See also [13] for general rotational flower.) **Hertle** characterized the "*classic*" Radon transform in [8] as the *continuous, rotationally, dilationally, translationally invariant operator*. See also [11].

Identifying by image. **Hertle** proved in [9] that for *exponential Radon transform* $\mathrm{R}_{\mathcal{H},\mu}$ the map $(f,\mu) \mapsto \mathrm{R}_{\mathcal{H},\mu} f$ is injective on the set of the compactly supported, not radial distributions f. **Solmon** computed the weight from $\mathrm{R}_{\mathcal{H},\mu} f$ in [24].

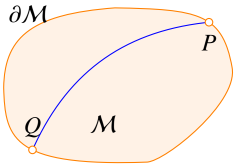
Identifying by Dirac-test. Following **Natterer** [18, 19] and **Boman** [4], it is proved in [13], that a conformal, differentiable and not self-tangent flower $\mathcal{S}$ is determined by $F(\bar{\boldsymbol{w}}, r) = \mathrm{R}_{\mathcal{S},\mu} f(\bar{\boldsymbol{w}}, r)$ if the unknown function f is of the form $f = \sum_{i=1}^{m} \delta_{\boldsymbol{x}_i} + g$, where $g \in L^2(\mathbb{R}^n)$ and $\delta_{\boldsymbol{x}_i}$ is the Dirac measure at point $\boldsymbol{x}_i$. Further, $\mu_{\boldsymbol{w},r}(\boldsymbol{x}_i)$ can be calculated if $\boldsymbol{x}_i \in S_{\boldsymbol{w},r}$ is satisfied.

Identifying by a measurement [13]. A conformal Radon transform $\mathrm{R}_{\mathcal{S},\mu}$ is reconstructible by means of f and $F = \mathrm{R}_{\mathcal{S},\mu} f$, if $f \in L^2(\mathbb{R}^n)$ is radial without vanishing moments and is of compact support, and either the flower or the weight is known and symmetric.

Identification of X-ray transform by a measurement

Let $\mathcal{M} \subseteq \mathbb{R}^n$ be a compact connected domain. Let C be a set of curves in $\mathcal{M}$, such that two curves intersect each other in at most one point, and there is a unique curve $C_{X,Y} \in$ C for any two different points $X, Y \in \mathcal{M}$ that contains both points X, Y.

The *X-ray transform* $\mathsf{X}_{\mathsf{C},\mu}$ maps a function $f\colon \mathcal{M} \to \mathbb{R}$ integrable on each curve of C into function $\mathsf{X}_{\mathsf{C},\mu} f\colon \partial\mathcal{M}^2 \to \mathbb{R}$ such that



$$\mathsf{X}_{\mathsf{C},\mu} f\colon \partial\mathcal{M}^2 \ni (P,Q) \mapsto \mathsf{X}_{\mathsf{C},\mu} f(P,Q) = \int_{C_{P,Q}} f(\boldsymbol{x})\, d\mu_{P,Q}(\boldsymbol{x}), \tag{2.1}$$

where $d\boldsymbol{x}$ is the natural arc-length measure and $\mu_{P,Q}$ is a distribution on $C_{P,Q}$. We call the curves *petals*, C the *flower*, and $\mu_{P,Q}$ the *weight*.

Identification. **Mukhometov** proved in [16, 17] that function $\mathsf{X}_{\mathsf{C},1} 1$ determines the flower if the petals are the geodesics of a Riemannian metric conformal to the Euclidean metric.

Michel, Uhlman et al. generalized this to Riemannian manifolds with boundary [15, 20, 25], etc., but without conformality rigidity is up to diffeomorphism.

Contemplating these results the feeling comes that the system of the petals may play more important role than that the differentiability allows us to see. This motivated the investigation of the *boundary-metric of projective metrics* [14], where the petals, i.e. the geodesics, are known, but there is no differentiability restriction on the metric.

The extension and unicity theorems presented here are the results of joint work with Tibor Ódor.

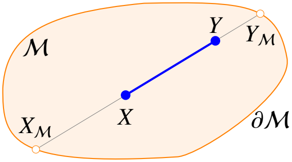
A **projective metric** is a continuous metric defined on a compact[1] convex domain $\mathcal{M}$ of the Euclidean space such that its geodesics are the chords of $\mathcal{M}$.

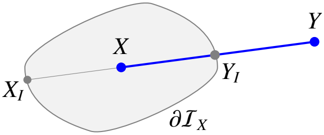
Here are two projective metrics for instance.

Hilbert metric. $d_{\mathcal{M}}\colon \mathcal{M}\times\mathcal{M}\to\mathbb{R}$ is defined by

$$d_{\mathcal{M}}(X,Y)=\begin{cases}0, & \text{if } X=Y,\\ \left|\ln(X,Y;X_{\mathcal{M}},Y_{\mathcal{M}})\right|/2, & \text{if } X\neq Y,\end{cases}$$

where $\mathcal{M}$ is an open, strictly convex, bounded domain in $\mathbb{R}^n$ and $\overline{X_{\mathcal{M}}Y_{\mathcal{M}}}=\mathcal{M}\cap XY$.





Minkowski metric. $d_{\mathcal{I}}\colon \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is defined by $d_{\mathcal{I}}(X,Y)=(Y,Y_I;X)$, where $\mathcal{I}$, the indicatrix, is an open, strictly convex, bounded, centrally symmetric domain in $\mathbb{R}^n$, $\mathcal{I}_X$ is its translate symmetric in X, and $\overline{X_IY_I}=\mathcal{I}_X\cap XY$.

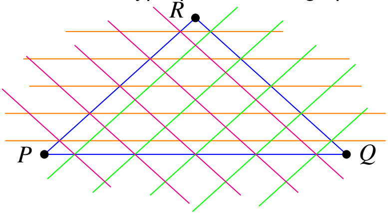
By **Beltrami's theorem** [3], if a projective metric is Riemannian than it has constant curvature (Euclidean, hyperbolic or elliptic metric). (Non hyperbolic Hilbert metrics are Finslerian.)

According to the solution of Hilbert's fourth problem, the class of the projective metrics is really huge [21, 1, 2, 26], and the projective metrics can be generated by the BB-construction.

[1] *Elliptic case is intentionally left out from the definition.*

BB-construction of projective metrics I

$\mathcal{M}$ is a convex non-empty domain in $\mathbb{R}^n$. P^* denotes the set of hyperplanes through point P.
A measure $\mu\colon 2^{\mathcal{M}^*} \to \mathbb{R}_+$ is *p-admissible* if



1. $\mu(P^*) = 0$ (μ is *definit*),
2. $\mu(\bigcup_{X\in\overline{PQ}} X^*) > 0$ (μ is *positive*), and
3. $\mu(\bigcup_{X\in\overline{PQ}} X^* \cap \bigcup_{Y\in\overline{QR}} Y^*) > 0$ (μ is *strict*)

for every non-collinear points $P, Q, R \in \mathcal{M}$.

Blaschke–Busemann construction [6]. *If $\mu\colon 2^{\mathcal{M}^*} \to \mathbb{R}_+$ is a p-admissible measure, then the function $d\colon \mathcal{M}\times\mathcal{M} \to \mathbb{R}_+$ defined by $d(P,Q) = \mu(\bigcup_{X\in\overline{PQ}} X^*)/2$ is a projective metric.*

Proof. If $\mu\colon 2^{\mathcal{M}^*} \to \mathbb{R}_+$ is a p-admissible measure, then $d(P,Q) = \mu(\bigcup_{X\in\overline{PQ}} X^*)/2$ is positive and vanishes if and only if $P \equiv Q$. Moreover,

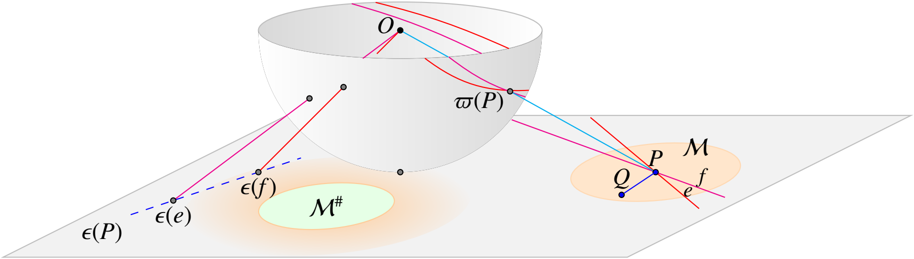
$$2d(P,Q) + 2d(Q,R) = \mu\Big(\bigcup_{X\in\overline{PQ}} X^* \cup \bigcup_{Y\in\overline{QR}} Y^*\Big) = \mu\Big(\bigcup_{Z\in\overline{PR}} Z^*\Big) + \mu(\{XY : X\in\overline{PQ} \wedge Y\in\overline{QR}\})$$

$$= 2d(P,R) + \mu\Big(\bigcup_{X\in\overline{PQ}} X^* \cap \bigcup_{Y\in\overline{QR}} Y^*\Big) > 2d(P,R)$$

is the triangle inequality. ∎

BB-construction of projective metrics II

Lines in $\mathcal{M}^*$ correspond to the complement of $\mathcal{M}^\#$, the poles of straight lines avoiding $\mathcal{M}$, through a polarity ϵ, because the *pole* (point) $\epsilon(e)$ of every straight line e through a point P lays on the *polar* (straight line) $\epsilon(P)$ of P. $\mathcal{M}^\#$ is a bounded open non-empty convex domain.



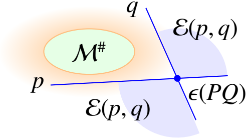
A segment $\overline{PQ}$ corresponds to the *two-edge*

$$\epsilon(\overline{PQ}^*) = \bigcup_{X\in\overline{PQ}} \epsilon(X^*) = \bigcup_{X\in\overline{PQ}} \epsilon(X) =: \mathcal{E}(\epsilon(P), \epsilon(Q)).$$

It is bounded by the union of the polar lines $p = \epsilon(P)$ and $q = \epsilon(Q)$.

So, we have

(3.1) $$\mu \circ \epsilon(\mathcal{E}(\epsilon(P), \epsilon(Q))) = \mu(\overline{PQ}^*).$$



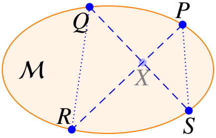
Notice that in the Blaschke–Busemann construction this is just $d(P,Q)$, so we are looking for the proper measure on $\mathcal{M}^\#$.

Extending boundary-metrics to projective metrics I

Due to the triangle inequality, and to that the diagonal point $X = \overline{PR} \cap \overline{QS}$ of any convex quadrangle $\square(PQRS)$ in $\mathcal{M}$ falls in $\mathcal{M}$, every projective metric $d\colon \mathcal{M}\times\mathcal{M} \to \mathbb{R}_+$ satisfies the *quadrangle inequality*

(3.2) $$d(P,R) + d(Q,S) - d(P,S) - d(Q,R) \geq 0,$$

where equality happens only if $\square(PQRS)$ degenerates to a segment.



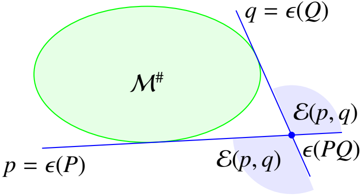
Extension theorem [14]. *If a continuous bounded metric $\delta\colon \partial\mathcal{M}\times\partial\mathcal{M} \to \mathbb{R}_+$ satisfies the quadrangle inequality* (3.2) *for any convex non-degenerate quadrangle $\square(PQRS)$ inscribed in $\partial\mathcal{M}$, then it is the restriction of a projective metric $d\colon \mathcal{M}\times\mathcal{M} \to \mathbb{R}_+$.*

Sketch of proof. A p-admissible measure $\mu\colon 2^{\mathcal{M}^*} \to \mathbb{R}_+$ needs to be constructed such that $\delta(P,Q) = \mu(\bigcup_{X\in\overline{PQ}} X^*)/2$ for every points $P,Q \in \partial\mathcal{M}$.
We first construct a set function ν on $\mathbb{P}^2$ that will generate $\mu\circ\epsilon$.

For a chord $\overline{PQ}$ inscribed in $\partial\mathcal{M}$, we set

(3.3) $$\nu(\mathcal{E}(\epsilon(P),\epsilon(Q))) := \delta(P,Q).$$

This is necessary by (3.1).
Two-edges outside $\mathcal{M}^\#$ are called *support two-edges* if their bounding edges are tangent to $\mathcal{M}^\#$.

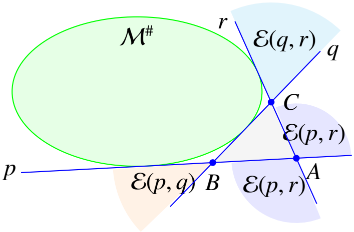


Extending boundary-metrics to projective metrics II

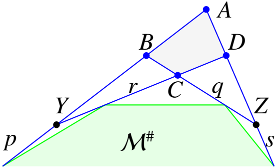
Let $\triangle(ABC)$ be a triangle with side-lines $p = AB = \epsilon(P)$, $q = BC = \epsilon(Q)$, and $r = CA = \epsilon(R)$, such that $\mathcal{E}(p,q)$, $\mathcal{E}(q,r)$, and $\mathcal{E}(r,p)$ support $\mathcal{M}^\#$. Observe that outside lines p,q,r we have $2\chi_{\triangle(ABC)} = \chi_{\mathcal{E}(p,q)} + \chi_{\mathcal{E}(q,r)} - \chi_{\mathcal{E}(r,p)}$. Therefore, using (3.3), we define

$$\nu(\triangle(ABC)) := (\delta(P,Q) + \delta(Q,R) - \delta(P,R))/2, \tag{3.4}$$

which is non-negative by the triangle inequality. Triangles considered here are called *support triangles*.



Let $\square(ABCD)$ be a convex quadrangle with side-lines $p = AB = \epsilon(P)$, $q = BC = \epsilon(Q)$, $r = CD = \epsilon(R)$, and $s = DA = \epsilon(S)$, such that $\mathcal{E}(p,q)$, $\mathcal{E}(q,r)$, $\mathcal{E}(r,s)$, and $\mathcal{E}(s,p)$ support $\partial\mathcal{M}^\#$. Let the diagonal points of $\square(ABCD)$ be $X = AC \cap BD$, $Y = p \cap r$, and $Z = q \cap s$. Since the side-lines of triangles $\triangle(YBC)$, $\triangle(ZDC)$, $\triangle(YAD)$, and $\triangle(ZAB)$ support $\mathcal{M}^\#$, using (3.4), we define



$$\begin{aligned} 2\nu(\square(ABCD)) &:= \nu(\triangle(YAD)) + \nu(\triangle(ZAB)) - \nu(\triangle(YBC)) - \nu(\triangle(ZCD)) \\ &= \delta(P,Q) + \delta(R,S) - \delta(R,Q) - \delta(P,S) \end{aligned} \tag{3.5}$$

which is non-negative by the quadrangle inequality. Quadrangles considered here are called *support quadrangles*.

Extending boundary-metrics to projective metrics III

Let Q be the set of the quadrangles all of whose side-lines support $\mathcal{M}^\#$. Let R be the smallest semiring containing Q. Then the sets in R are the union of mutually disjoint closed polygons all of whose side lines support $\mathcal{M}^\#$.

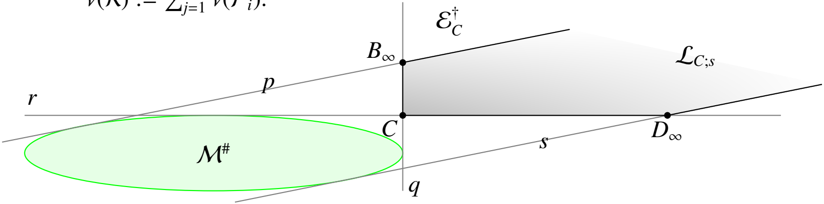
As all side-lines of a closed polygon $\mathcal{P} \in$ R support $\mathcal{M}^\#$, the side-lines cut $\mathcal{P}$ into finitely many mutually disjoint quadrangles $\mathcal{Q}_i \in$ Q $(i = 1, \ldots, n)$. So we define

(3.6) $$\nu(\mathcal{P}) := \sum_{i=1}^{n} \nu(\mathcal{Q}_i).$$

As every set $\mathcal{R} \in$ R is the union of such mutually disjoint closed polygons $\mathcal{P}_j \in$ R $(j = 1, \ldots, \ell)$, we can finish defining ν by

(3.7) $$\nu(\mathcal{R}) := \sum_{j=1}^{\ell} \nu(\mathcal{P}_j).$$

Let r and q be the supporting lines of $\mathcal{M}^\#$ through $C \notin \mathcal{M}^\#$. One of the two-edges, say $\mathcal{E}_C$, with vertex C contains $\mathcal{M}^\#$. Let $\mathcal{E}_C^\dagger$ be the quadrant in $\mathcal{E}_C$ that does not contain $\mathcal{M}^\#$. Let s be a tangent of $\mathcal{M}^\#$ that contains C and $\mathcal{M}^\#$ on the same side. Let $p \neq s$ be the tangent of $\mathcal{M}^\#$ parallel to s. Let the *pointed lane* $\mathcal{L}_{C;s}$ be the intersection of $\mathcal{E}_C^\dagger$ and the strip between s and p. Let $D_\infty = r \cap s$ and $B_\infty = q \cap p$.

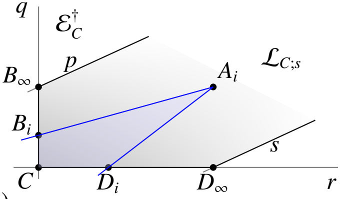


We claim that $\nu_{C;s} = \nu\restriction_{\mathrm{R}_{C;s}}$, *where* $\mathrm{R}_{C;s} = \{\mathcal{R} : \mathrm{R} \ni \mathcal{R} \subset \mathcal{L}_{C;s}\}$ *is extendible to a measure on* $\mathcal{L}_{C;s}$.

Extending boundary-metrics to projective metrics IV

Set-function $\nu_{C;s}$ is clearly additive and σ-subadditive. Its σ-finiteness is needed.

Let points $D_i \in \overline{CD_\infty}$ and $B_i \in \overline{CB_\infty}$ $(i = 1, \dots, \infty)$ be such that $\overrightarrow{CB_\infty} = i\overrightarrow{B_iB_\infty}$ and $\overrightarrow{CD_\infty} = i\overrightarrow{D_iD_\infty}$, respectively. Let $s_i = \epsilon(S_i)$ and $p_i = \epsilon(P_i)$ be the tangent lines of $\mathcal{M}^\#$ such that $D_i \in s_i \neq r$ and $B_i \in p_i \neq q$, respectively. Let $A_i = p_i \cap s_i$. As $\bigcup_{i=1}^k \square(A_iB_iCD_i) = \square(A_kB_kCD_k)$, and $\bigcup_{i=1}^\infty \square(A_iB_iCD_i) = \mathcal{L}_{C;s}$, we obtain



$$2\nu_{C;s}(\mathcal{L}_{C;s}) = 2\lim_{k\to\infty}\nu_{C;s}(\square(A_kB_kCD_k)) = 2\lim_{k\to\infty}\nu(\square(A_kB_kCD_k))$$

$$= \lim_{k\to\infty}(\delta(P_i,R)+\delta(Q,S_i)-\delta(P_i,S_i)-\delta(Q,R)) = \delta(P,R)+\delta(Q,S)-\delta(P,S)-\delta(Q,R) < \infty,$$

where $P = \lim_{k\to\infty} P_k$, and $S = \lim_{k\to\infty} S_k$, by the continuity of metric δ, and $q = \epsilon(Q)$, $r = \epsilon(R)$, $p = \epsilon(P)$, $s = \epsilon(S)$.

Thus, by Charatodory's [10, Theorem 1.53], set function $\nu_{C;s}$ extends to a σ-finite measure $\mu_{C;s}$ on $\sigma(\mathrm{R}_{C;s})$, the set of the Borel sets in $\mathcal{L}_{C;s}$.

Observe that *$\mu_{C;s}(\square(ABCD)) = \nu(\square(ABCD))$ for every quadrangle $\square(ABCD)$ in ring* $\mathrm{R}_{C;s}$, hence every measure $\mu_{C;s}$ takes the same value on every quadrangle $\square(ABCD)$ in the common domain, so *all such measures are equal on every Borel set in the common domain*.

Extending boundary-metrics to projective metrics V

Now we can define the measure μ requested in the theorem as follows:
Given a Borel set, divide it to disjoint parts so that every part falls in a set $\mathcal{L}_{C;s}$, then measure every such part by the appropriate measure $\mu_{C;s}$, and sum up the values.

To finish the proof we only have to check that δ is a restriction of the projective metric d defined from μ by the Blaschke–Busemann-construction.
Let $\square(ABCD)$ be a convex quadrangle with side-lines $p = AB = \epsilon(P)$, $q = BC = \epsilon(Q)$, $r = CD = \epsilon(R)$, and $s = DA = \epsilon(S)$, such that $\mathcal{E}(p,q)$, $\mathcal{E}(q,r)$, $\mathcal{E}(r,s)$, and $\mathcal{E}(s,p)$ support $\partial\mathcal{M}^{\#}$. Observe that

$$\begin{aligned}
&\delta(P,Q)+\delta(R,S)-\delta(R,Q)-\delta(P,S)\\
&\quad=2\nu(\square(ABCD))=2\nu_{C;s}(\square(ABCD))=2\mu_{C;s}(\square(ABCD))=2\mu(\square(ABCD))\\
&\quad=d(P,Q)+d(R,S)-d(R,Q)-d(P,S).
\end{aligned}$$

where the first and last equation is proved by the derivation of (3.5). Letting $Q \to R$ and $S \to P$ in this equality, the continuity of δ and d implies $2\delta(P,R) = 2d(P,R)$ that completes the proof.

∎

Notice: The unicity of the projective metric does not follow directly from the unicity part of Caratéodory's theorem, as the unicity of the definition of the basic values of ν is not proven.

Boundary-rigidity of projective metrics I

As a projective metric is projective metric on any plane section of its domain, the following rigidity results are valid in any dimension.

Boundary-rigidity [14].
A projective-metric $d:\mathcal{M}\times\mathcal{M}\to\mathbb{R}_+$ is determined by its boundary-metric $\delta = d\restriction_{\partial\mathcal{M}}$.

Sketch of proof. By the Extension Theorem, we have a p-admissible measure $\mu\colon 2^{\mathcal{M}^*}\to\mathbb{R}_+$, such that $d(P,Q) = \mu(\mathcal{E}(\epsilon(P),\epsilon(Q)))$. So we only need to determine that μ.

Following (3.3), (3.4) and (3.5) we can calculate μ for two-edges, triangles and quadrangles with side-lines supporting $\mathcal{M}^\#$:

$$2\mu(\mathcal{E}(\epsilon(P),\epsilon(Q))) = \delta(P,Q)$$

$$2\mu(\triangle(ABC)) = \delta(P,Q)+\delta(Q,R)-\delta(P,R),$$

$$2\mu(\square(ABCD)) = 2(\delta(P,Q)+\delta(R,S)-\delta(R,Q)-\delta(P,S)),$$

where $P,Q,R,S\in\partial\mathcal{M}$.

Let Q be the set of the convex quadrangles all of whose side-lines support $\mathcal{M}^\#$. Let R be the smallest semiring containing Q. Then the sets in R are the union of mutually disjoint closed polygons all of whose side lines support $\mathcal{M}^\#$. As all side-lines of a closed polygon $\mathcal{P}\in$ R support $\mathcal{M}^\#$, the side-lines cut $\mathcal{P}$ into finitely many mutually disjoint quadrangles $Q_i\in$ Q $(i=1,\dots,n)$.

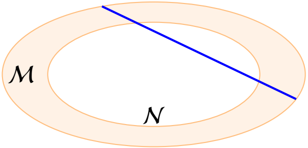
So we have $\mu(\mathcal{P}) := \sum_{i=1}^n \mu(Q_i)$. As every set $\mathcal{R}\in$ R is the union of such mutually disjoint closed polygons $\mathcal{P}_j\in$ R $(j=1,\dots,\ell)$, we also have $\mu(\mathcal{R}) := \sum_{j=1}^{\ell}\mu(\mathcal{P}_i)$. Since μ is a measure, and it is σ-finite in every pointed lane $\mathcal{L}_{C;s}$, it is determined uniquely by its values on R due to the unicity part of Charatodory's [10, Theorem 1.53]. ■

Boundary-rigidity of projective metrics II

Restricting the boundary metric to smaller set of points gives sharper geometric views. Let $\mathcal{N}$ be a compact convex domain in the interior of $\mathcal{M}$.

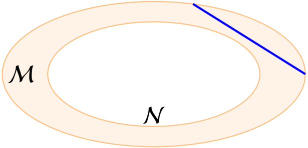
We can localize the Boundary-rigidity Theorem...

Theorem [14]. *A projective-metric $d\colon \mathcal{M}\times\mathcal{M}\to\mathbb{R}_+$ is determined on $\mathcal{N}$ by its restriction on pairs $(P,Q)\in\partial\mathcal{M}\times\partial\mathcal{M}$ such that $\overline{PQ}\cap\mathcal{N}\neq\emptyset$.*



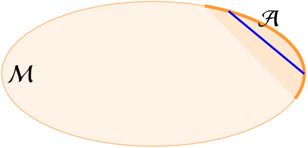
The "peeling argument" of [25] in a way.

Theorem [14]. *A projective metric $d\colon \mathcal{M}\times\mathcal{M}\to\mathbb{R}_+$ is determined on $\mathcal{M}\setminus\mathcal{N}$ by its restriction on pairs $(P,Q)\in\partial\mathcal{M}\times\partial\mathcal{M}$ such that $\overline{PQ}\cap\mathcal{N}=\emptyset$.*



Let $\mathcal{A}$ be a connected open arc in $\partial\mathcal{M}$.

Theorem [14]. *A projective metric $d\colon \mathcal{M}\times\mathcal{M}\to\mathbb{R}_+$ is determined on $\operatorname{Conv}\mathcal{A}$ by its restriction on pairs $(P,Q)\in\mathcal{A}\times\partial\mathcal{M}$.*



A bit more generality

Theorem. *Let* C_1 *and* C_2 *be systems of curves in the compact domain* $\mathcal{N}$ *such that*

1. *each curve* C *in* $\mathsf{C}_1 \cup \mathsf{C}_2$ *is injectively parameterized on a closed interval* $[a, b]$*, and* $C(a), C(b) \in \partial\mathcal{N}$*;*
2. *there is exactly one curve* $C \in \mathsf{C}_i$ $(i = 1, 2)$ *through any two given distinct points of* $\mathcal{M}$*;*
3. *the systems* C_1 *and* C_2 *fulfill the Desargues property.*

If there are two metrics d_1 *and* d_2 *on* $\mathcal{N}$ *such that* C_1 *and* C_2 *are their respective sets of geodesics, and* $d_1 \equiv d_2$ *on* $\partial\mathcal{N}^2$*, then there is a homeomorphism* $\chi\colon \mathcal{N} \to \mathcal{N}$ *such that* $d_2 \equiv d_1 \circ (\chi, \chi)$*.*

For, let $\mathcal{M}$ lay in the plane $z = 0$ of the space $\mathbb{R}^3$, and choose a homeomorphism ψ that makes $\mathcal{N}$ the unit circular disc of plane $z = 0$. Project $\psi(\mathcal{M})$ onto the upper half $\mathcal{S}^2_+$ of the unit sphere $\mathcal{S}^2$ by the projection $\varpi\colon (x, y, 0) \mapsto (x, y, \sqrt{1 - x^2 - y^2})$. Project $\mathcal{S}^2_+$ from the origin $O = (0, 0, 0)$ onto the plane $z = 1$ with $\hat{\varpi}\colon (x, y, \sqrt{1 - x^2 - y^2}) \mapsto (x/\sqrt{1 - x^2 - y^2}, y/\sqrt{1 - x^2 - y^2}, 1)$. On the plane $z = 1$ the image curves of C_i $(i = 1, 2)$ satisfy the condition of Busemann's [5, (11.2) Theorem], so plane $z = 1$ can be metrized as a straight G-plane $\mathcal{G}_i$ so that the curves in $\hat{\varpi}(\varpi(\psi(\mathsf{C}_i)))$ are the geodesics of $d_i \circ (\hat{\varpi} \circ \varpi \circ \psi, \hat{\varpi} \circ \varpi \circ \psi)$ $(i = 1, 2)$. From this, Busemann's [5, (13.1) Theorem], gives that $\mathcal{G}_i$ can be mapped topologically on an open convex domain $\mathcal{M}_i \subset \mathbb{R}^2$ in such a way that each geodesics in $\mathcal{G}_i$ goes into the intersection of $\mathcal{M}_i$ with a line in $\mathbb{R}^2$.

THANK YOU FOR YOUR ATTENTION!

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Identifying and characterizing unknown generalized Radon transforms by some knowledge about there behavior is a classical subject (see for example only some works of E. T. Quinto, A. Hertle, D. C. Solmon, F. Natterer, J. Boman, etc.). Identifying the Radon transform that integrates appropriate functions on the geodesics of a compact, simple Riemannian manifold with boundary, is a subject researched for a long time (see for example only some works of G. Herglotz, Ju. E. Anikonov, V. G. Romanov, R. G. Mukhometov). It revived nowadays in some important new results, called the boundary-distance rigidity of Riemannian manifolds, due to the works of R. Michel, C. Croke, G. Uhlmann, A. Vasy, P. Stefanov, etc..

Contemplating these results the feeling comes that the properties of the system of the curves over which the integration is performed probably play more important role than what differentiability allows to see. This feeling motivated the investigation of the boundary-metric of projective metrics.

A *projective metric* is a continuous metric defined on a convex, not necessarily proper subset $\mathcal{M}$ of the Euclidean space such that the geodesics are the chords of $\mathcal{M}$. The class of these metrics is really huge (this was observed by H. Busemann), but, by Beltrami's theorem, the only Riemannian projective metrics are those that have constant curvature.

Theorem. (Á. K. & T. Ódor, 2018) *Let $\mathcal{M}$ be a compact convex non-empty domain in the plane. If a continuous bounded metric $\delta\colon \partial\mathcal{M}\times\partial\mathcal{M}\to\mathbb{R}_+$ satisfies the quadrangle inequality*

$$\delta(P,R)+\delta(Q,S)-\delta(P,S)-\delta(Q,R)\geq 0$$

for any convex non-degenerate quadrangle $\square(PQRS)$, then δ uniquely extends to a projective metric $d\colon \mathcal{M}\times\mathcal{M}\to\mathbb{R}_+$.

The proof basically follows Busemann's integral geometric idea to generate all projective metrics from measures on the Grassmannian by the Crofton formula. Then the uniqueness comes from the uniqueness part of Carathéodory's extension theorem.

Unicity in the theorem remains valid in any dimension due to uniqueness in every plane section.

Structureof this talk

1. **Generalized Radon transforms**
   - Characterizations and identifications
2. **X-ray transform and boundary-rigidity**
3. **Projective metrics**
   - Definition and examples
   - Blaschke–Busemann construction
   - Boundary-metric
     - Extension theorem (in plane)
     - Uniqueness theorems (in space)
4. **Discussion**