Projective-metric spaces with Ceva or Menelaus property

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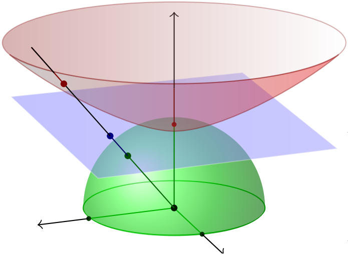

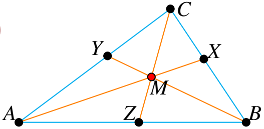

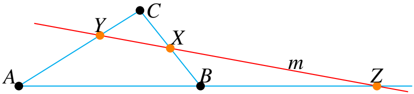

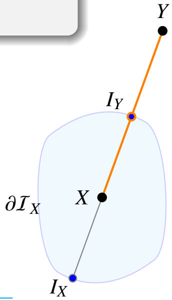

Talk given on the Erasmus trip to University of Basilicata at Potenza on 16th July 2019

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For the animations Adobe PDF Reader is necessary.

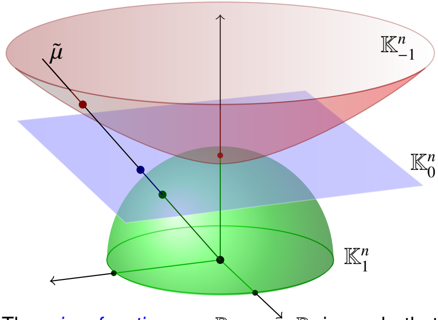
In todays' language *Hilbert's IV. problem* [8] was to give all the metrics, the *projective metrics* [6], on every projective space $\mathbb{P}^n$, $n \in \mathbb{N}$, that satisfy the strict triangle inequality, and then investigate those geometries given by these metrics.

Hamel [9] proved that, according to the domain $\mathcal{D}$ of the projective metric d, there are exactly three kinds of them:

hyperbolic type ($\mathcal{D} \subsetneq \mathbb{P}^n \setminus \mathbb{R}^n$ convex)
parabolic type ($\mathcal{D} = \mathbb{P}^n \setminus \mathbb{R}^n$) and
elliptic type ($\mathcal{D} = \mathbb{P}^n$) .

A pair $(\mathcal{D}, d)$ of an open convex domain $\mathcal{D}$ and a projective metric $d\colon \mathcal{D} \times \mathcal{D} \to \mathbb{R}_+$ is called *projective-metric space* if the geodesics (the chords of $\partial\mathcal{D}$) are isometric to a Euclidean circle (for elliptic type) or to a Euclidean straight line (for the *straight types*, i.e. either the parabolic or the hyperbolic type).

Spaces of constant curvature show that there are important projective-metric spaces.



The *size function* $\nu_\kappa\colon \mathbb{R}_+ \to \mathbb{R}$ is such that $\nu_\kappa(r)\mathcal{S}^{n-1}$ is isometric to a metric sphere of radius $r > 0$ in $\mathbb{K}^n_\kappa$. The *projection function* $\mu_\kappa\colon [0, \mathring{\imath}_\kappa) \to \mathbb{R}_+$ gives the geodesic correspondence $\tilde{\mu}_\kappa\colon \mathrm{Exp}_O(r\omega) \mapsto \mu_\kappa(r)\omega$.

$\mathbb{K}^n_\kappa$	κ	ν_κ	μ_κ	$\mathring{\imath}_\kappa$
$\mathbb{H}^n$	-1	$\sinh r$	$\tanh r$	∞
$\mathbb{R}^n$	0	r	r	∞
$\mathbb{S}^n$ ($\mathbb{P}^n$)	$+1$	$\sin r$	$\tan r$	$\pi/2$

Here we identified the space $\mathcal{T}_O\mathbb{K}^n_\kappa$ with $\mathbb{R}^n$ by the natural way, and used $\omega \in \mathcal{S}^{n-1}$ in both senses.

Blaschke [2] proved that Crofton's formula [7] gives projective metrics from measures on the Grassmannian.

Busemann conjectured [4] that all projective metrics can be constructed in this way. This was first proved by Pogorelov [13] and Szabó [14].

Beltrami's theorem [1] implies that the only Riemannian projective metrics are those of constant curvature.

The class of projective metrics and the class of projective-metric spaces are both so huge. Busemann noticed [5] that "... the second part of [Hilbert's] problem ... has inevitably been replaced by the investigation of special, or special classes of, interesting geometries."

Our general goal is to

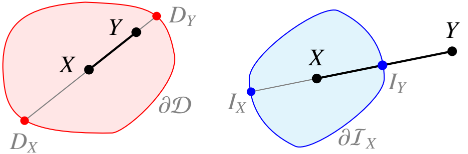
characterize the interesting geometries among projective-metric spaces.

This time we investigate the validity of the theorems of Ceva and Menelaus.

Hilbert metric $d\colon \mathcal{D}\times\mathcal{D}\to\mathbb{R}$ is defined by

$$d(X,Y)=\begin{cases}0, & \text{if } X=Y,\\ \frac{1}{2}\left|\ln(X,Y;D_X,D_Y)\right|, & \text{if } X\neq Y,\end{cases}$$

where $\mathcal{D}$ is an open, strictly convex, bounded domain in $\mathbb{R}^n$ and $\overline{D_XD_Y}=\mathcal{D}\cap XY$.



Minkowski metric $d_{\mathcal{I}}\colon \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ is defined by $d(X,Y)=(Y,I_Y;X)$, where $\mathcal{I}$, the *indicatrix*, is an open, strictly convex, bounded domain in $\mathbb{R}^n$ symmetric at O, $\mathcal{I}_X=\mathcal{I}+\overrightarrow{OX}$, and $\overline{I_XI_Y}=\mathcal{I}_X\cap XY$.

Elliptic metric $d\colon \mathbb{P}^n\times\mathbb{P}^n\to\mathbb{R}$ is defined by $d(X,Y)=\arccos|\langle\overrightarrow{OX},\overrightarrow{OY}\rangle|$, where the points of $\mathbb{P}^n$ are the diagonal point pairs of $\mathcal{S}^n$.

Although the theorems of Ceva and Menelaus basically belong to affine geometry, they can be formulated by the metric just as well. Below, if the projective-metric space is of elliptic type, then it is so meant that a straight line was removed a priori.

Let A, B be different points in a projective-metric space $(\mathcal{D}, d)$, and let $C \in (AB \cap \mathcal{D}) \setminus \{B\}$.

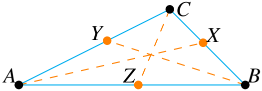
Then the *metric ratio* and the *size-ratio* of the triplet $(A, B; C)$ are

$$\langle A, B; C\rangle_d = \begin{cases} \frac{d(A,C)}{d(C,B)}, & \text{if } C \in \overline{AB}, \\ -\frac{d(A,C)}{d(C,B)}, & \text{otherwise,} \end{cases} \quad \text{and} \quad \langle A, B; C\rangle_d^\circ = \begin{cases} \frac{\nu(d(A,C))}{\nu(d(C,B))}, & \text{if } C \in \overline{AB}, \\ -\frac{\nu(d(A,C))}{\nu(d(C,B))}, & \text{otherwise,} \end{cases} \quad \text{respectively,}$$

where ν is the size function of the hyperbolic, Euclidean, or elliptic space according to the type of $(\mathcal{D}, d)$. Observe that in a constant curvature space $\mathbb{K}^n$ a size-ratio $\langle A, B; C\rangle_d^\circ$ is the affine ratio of the orthogonal projections of points A, B, C into the tangent space $T_C\mathbb{K}^n$.

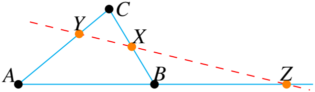
By a *triplet (Z, X, Y) of a non-degenerate triangle $ABC\triangle$* we mean three points Z, X and Y being respectively on the straight lines AB, BC and CA. It is called

- a *Ceva triplet* if lines AX, BY and CZ are concurrent, and
- a *Menelaus triplet* if Z, X and Y are collinear.



A 3-tuple (α, β, γ) of real numbers is

- of *Ceva type* if $\alpha \cdot \beta \cdot \gamma = +1$, and
- of *Menelaus type* if $\alpha \cdot \beta \cdot \gamma = -1$.



The *Ceva or Menelaus property* of a projective-metric space means that any triplet (Z, X, Y) of any non-degenerate triangle $ABC\triangle$ is Menelaus or Ceva if and only if the 3-tuple $(\langle A, B; Z\rangle_d^\circ, \langle B, C; X\rangle_d^\circ, \langle C, A; Y\rangle_d^\circ)$ is of Menelaus or Ceva type, respectively.

A projective-metric space is called *Cevian* or *Menelausian* if it has the appropriate property. Known examples are the *Minkowski geometries*, as the size-function is the identity, and the *constant curvature spaces*, where appropriate trigonometry applies [12].

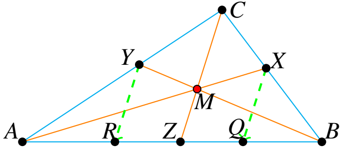
Non-Cevian and *non-Menelausian* spaces are the *non-hyperbolic Hilbert geometries*[1]. This was proved in [10, Theorem 3.1] by showing that both the Ceva and the Menelaus properties fail for some triangles if the boundary of the Hilbert geometry is not an ellipsoid.

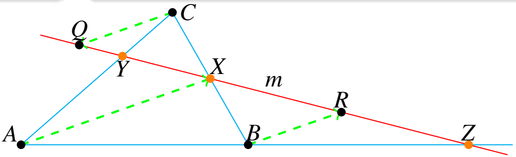
Lemma. *If five collinear points A, R, Z, Q, B in a Cevian projective-metric space satisfies $(Z, A; R)(B, Z; Q)(A, B; Z) = 1$ in order $A \prec R \prec Z \prec Q \prec B$, then*

$$\langle Z, A; R\rangle^\circ_d \langle B, Z; Q\rangle^\circ_d \langle A, B; Z\rangle^\circ_d = 1. \tag{2.1}$$

Lemma. *If five collinear points Q, Y, X, R, Z in a Menelausian projective-metric space satisfies $(X, R; Z)(R, Q; X)(Q, X; Y) = -1$ in order $Q \prec Y \prec X \prec R \prec Z$, then*

$$\langle X, R; Z\rangle^\circ_d \langle R, Q; X\rangle^\circ_d \langle Q, X; Y\rangle^\circ_d = -1. \tag{2.2}$$





Relabeling the points $Q \prec Y \prec X \prec R \prec Z$ as $Q \mapsto B$, $Y \mapsto Q$, $X \mapsto Z$, $R \mapsto R$, and $Z \mapsto A$ shows that (2.2) is equivalent[2] to (2.1).

[1] We say that a Hilbert geometry is hyperbolic or is the hyperbolic geometry if it is a Cayley–Klein modell of the hyperbolic geometry.

[2] By projective duality it is not a surprise that the Ceva and the Menelaus properties boil down to the same equation.

Theorem. (Á. K. 2019 [11]).

A projective-metric space is Cevian or Menelausian if and only if

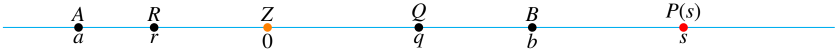
- *it is a Cayley–Klein model of the hyperbolic geometry, or*
- *it is a Minkowski geometry, or*
- *it is the elliptic geometry.*

For proving this result we use the equivalency

$$\frac{\overrightarrow{ZB}-\overrightarrow{ZQ}}{\overrightarrow{ZA}-\overrightarrow{ZR}}\frac{\overrightarrow{RZ}}{\overrightarrow{ZQ}}\frac{\overrightarrow{AZ}}{\overrightarrow{ZB}}=1 \iff \frac{\nu(d(Z,B)-d(Z,Q))}{\nu(d(A,Z)-d(R,Z))}\frac{\nu(d(R,Z))}{\nu(d(Z,Q))}\frac{\nu(d(A,Z))}{\nu(d(Z,B))}=1, \tag{2.3}$$

that follows from (2.1) by the additivity of the metric d. Although this equivalency is quite different in every type of the projective-metric spaces, each case leads to Cauchy's functional equation [15].

As a general setup we parameterize the five collinear points $A \prec R \prec Z \prec Q \prec B$ by the linear function $P\colon \mathbb{R} \to RQ$ so that $Z = P(0)$, $A = P(a)$, $R = P(r)$, $Q = P(q)$, $B = P(b)$, where $a < r < 0 < q < b$.



Further, we introduce *$\ell\colon RQ \to \mathbb{R}$ defined by $\ell(s) = \nu(d(P(s),Z))$.*

Cevian $(\mathcal{D}, d)$ is of elliptic type.

The geodesics of a projective-metric space of elliptic type have equal lengths, so we can set their length to π by simply multiplying the projective metric with an appropriate positive constant. Hence $\nu(\cdot) = \sin(\cdot)$, and so $\ell(s) = \sin(d(P(s), Z))$.

Equivalency (2.3) with the addition formulas for sine give

$$\frac{b-q}{b}\,\frac{-a}{r-a}\,\frac{-r}{q} = 1 \;\Leftrightarrow\; \frac{\ell(b)\cos(d(Z,Q)) - \cos(d(Z,B))\ell(q)}{\ell(a)\cos(d(R,Z)) - \cos(d(A,Z))\ell(r)}\,\frac{\ell(r)}{\ell(q)}\,\frac{\ell(a)}{\ell(b)} = 1.$$

After some easy simplifications this becomes

$$\frac{1}{q} - \frac{1}{b} = \frac{1}{a} - \frac{1}{r} \;\Leftrightarrow\; \cot(d(Z,Q)) - \cot(d(Z,B)) = \cot(d(R,Z)) - \cot(d(A,Z)). \tag{2.4}$$

Letting $b \to \infty$ and $a \to -\infty$ implies that $q \to -r$ by the left-hand equation of (2.4). The right-hand equation of (2.4) gives that $\cot(d(Z,Q)) = \cot(d(R,Z))$, hence $d(Z,Q) = d(R,Z)$. Thus, $q = -r$ is equivalent to $d(Z,Q) = d(R,Z)$, hence ℓ is an even function.

Let function $f\colon \mathbb{R} \to \mathbb{R}_+$ be defined by $f(x) := \cot(d(Z, P(x)))$. Then (2.4) reads as

$$f\Big(\frac{abr}{ar+br-ab}\Big) = f(b) + f(r) - f(a).$$

Putting $r = -b$ (hence accepting $a < -b$ too!), this gives

$$f\Big(\frac{ab}{2a+b}\Big) = 2f(b) - f(a), \tag{2.5}$$

because f is an even function due to the evenness of ℓ.

Define the odd function

$$g(x) = \begin{cases} f(1/x), & \text{if } x > 0, \\ -f(1/x), & \text{if } x < 0. \end{cases}$$

Then, as $2a + b < a < 0 < b$, (2.5) gives

$$g\Big(\frac{2}{b} + \frac{1}{a}\Big) = 2g\Big(\frac{1}{b}\Big) + g\Big(\frac{1}{a}\Big). \tag{2.6}$$

For the moment let $b = -a/2$. Then (2.6) gives $g(\frac{-3}{a}) = 2g(\frac{-2}{a}) + g(\frac{1}{a})$. So $g(0) = 0$ follows from $a \to -\infty$ by the continuity of g. Now, $a \to -\infty$ in (2.6) gives by the continuity of g that $g(2/b) = 2g(1/b)$. Substituting this into (2.6) we arrive at Cauchy's functional equation [15] for the continuous function g, so we obtain that $g(x) = cx$ for some $c > 0$ and every x. By the definition of g and f this gives $d(P(s), P(0)) = |\arctan(cs)|$ which implies $c = 1$.

This proves the theorem for projective-metric spaces of elliptic type.

■

Cevian $(\mathcal{D}, d)$ is of parabolic type.

We have $\nu(\cdot) = \cdot$, so $\ell(s) = d(P(s), Z)$, hence (2.3) gives

$$\frac{b-q}{b}\frac{-a}{r-a}\frac{-r}{q} = 1 \iff \frac{\ell(b)-\ell(q)}{\ell(a)-\ell(r)}\frac{\ell(r)}{\ell(q)}\frac{\ell(a)}{\ell(b)} = 1.$$

After some easy simplifications this becomes

$$\frac{1}{q} - \frac{1}{b} = \frac{1}{a} - \frac{1}{r} \iff \frac{1}{\ell(q)} - \frac{1}{\ell(b)} = \frac{1}{\ell(r)} - \frac{1}{\ell(a)}. \tag{2.7}$$

Letting $a \to -\infty$ and $b \to \infty$ equation (2.7) gives

$$\frac{1}{q} = -\frac{1}{r} \iff \frac{1}{\ell(q)} = \frac{1}{\ell(r)},$$

so the affine and the d-metric midpoint of any segment coincide.

Thus, according to Busemann [3, page 94], d is a Minkowski metric, hence the theorem for projective-metric spaces of parabolic type.

■

Cevian $(\mathcal{D}, d)$ is of hyperbolic type.

We have $\nu(\cdot) = \sinh(\cdot)$, so $\ell(s) = \sinh(d(P(s), Z))$, and (2.3) with the addition formulas for the hyperbolic sine give

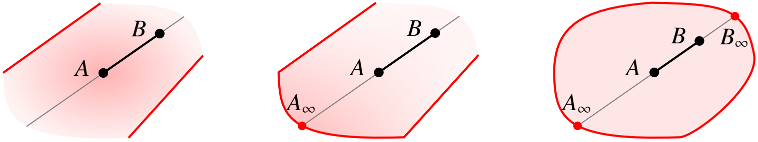
$$\frac{b-q}{b}\frac{-a}{r-a}\frac{-r}{q} = 1 \Leftrightarrow \frac{\ell(b)\cosh(d(Z,Q)) + \cosh(d(Z,B))\ell(q)}{\ell(a)\cosh(d(R,Z)) + \cosh(d(A,Z))\ell(r)}\frac{\ell(r)}{\ell(q)}\frac{\ell(a)}{\ell(b)} = 1.$$

After some easy simplifications this shows

$$\frac{1}{q} - \frac{1}{b} = \frac{1}{a} - \frac{1}{r} \Leftrightarrow \coth(d(Z,Q)) + \coth(d(Z,B)) = \cot(d(R,Z)) + \cot(d(A,Z)). \tag{2.8}$$

The intersection $e := AB \cap \mathcal{D}$ of line AB and the domain $\mathcal{D}$ can be of three types:

- a whole affine line AB,
- a ray $\overline{A_\infty B}$, or
- a segment $\overline{A_\infty B_\infty}$.



- $e = AB$.

Letting $b \to \infty$ and $a \to -\infty$, implies that $q \to -r$ by the left-hand equation of (2.4). From the right-hand equation of (2.4) we get that $\coth(d(Z,Q)) = \coth(d(R,Z))$, hence $d(Z,Q) = d(R,Z)$. Thus, $q = -r$ is equivalent to $d(Z,Q) = d(R,Z)$, hence ℓ is an even function. Thus the map $\rho_{d;e;z}\colon P(z-x) \leftrightarrow P(z+x)$ is a d-isometric point reflection of e for every $P(z) \in e$, hence $\tau_{d;e;z,t} := \rho_{d;e;t} \circ \rho_{d;e;z}\colon P(y) \to P(2z-y) \to P(2(t-z)+y))$ is a d-isometric translation.

So $d(P(x), P(y)) = d(P(0), P(y-x))$, hence
$d(P(0), P(y-x)) + d(P(0), P(z-y)) = d(P(x), P(y)) + d(P(y), P(z)) = d(P(x), P(z)) = d(P(0), P(z-x))$.
Thus the continuous function $f(x) = d(P(0), P(x))$ satisfies Cauchy's functional equation [15], hence a constant $c_e > 0$ exists such that $d(P(x), P(y)) = c_e|x-y|$ for every $x, y \in \mathbb{R}$.

- $e = \overline{A_\infty B}$.

Let $A_\infty = P(a_\infty)$. Letting $b \to \infty$ and $a \to a_\infty$ implies that

$$\frac{1}{q} = \frac{1}{a_\infty} - \frac{1}{r} \Leftrightarrow \coth(d(Z,Q)) = \coth(d(R,Z)) \tag{2.9}$$

by (2.8). Reparameterizing ray e by the linear map $\bar{P}\colon \mathbb{R} \to RQ$ such that $\bar{A}_\infty = \bar{P}(0)$, $R = \bar{P}(r)$, $Z = \bar{P}(z)$, $Q = \bar{P}(q)$, we can reformulate equivalency (2.9) to

$$\frac{1}{q-z} = \frac{1}{-z} - \frac{1}{r-z} \Leftrightarrow d(Z,Q) = d(R,Z),$$

where $0 < r < z < q$. Thus, the map $\rho_{d;e;z}\colon P(r) \leftrightarrow P(z^2/r)$ is a d-isometric point reflection on ray e for every $P(z) \in e$, hence $\tau_{d;e;z,t} := \rho_{d;e;t} \circ \rho_{d;e;z}\colon P(r) \to P(z^2/r) \to P(rt^2/z^2)$ is a d-isometric translation.

So $d(P(r), \tau_{d;e;z,t}(P(r)))$ does not depend on r, hence it is a real function δ of t/z. As d is additive, this implies $\delta(x) + \delta(y) = \delta(xy)$, so, by the solution of Cauchy's functional equation [15], we have a constant $\bar{c}_e > 0$ such that $\delta(x) = 2c_e|\ln(x)|$. Thus

$$d(P(x), P(y)) = d(P(x), \tau_{d;e;1,\sqrt{y/x}}(P(x))) = \delta(\sqrt{y/x}) = \bar{c}_e|\ln(y/x)| \quad \text{for every } x, y \in \mathbb{R}.$$

This means $d(P(x), P(y)) = \bar{c}_e|\ln(A_\infty, \infty; P(y), P(x))|$, i.e. a Hilbert metric on ray e.

- $e = \overline{A_\infty B_\infty}$.

Let $A_\infty = P(a_\infty)$ and $B_\infty = P(b_\infty)$. Letting $b \to b_\infty$ and $a \to a_\infty$ implies that

$$\frac{1}{q} - \frac{1}{b_\infty} = \frac{1}{a_\infty} - \frac{1}{r} \iff \coth(d(Z,Q)) = \coth(d(R,Z)).$$

by (2.8). Reparameterizing segment e by the linear map $\bar{P}\colon \mathbb{R} \to RQ$ such that $\bar{A}_\infty = \bar{P}(0)$, $R = \bar{P}(r)$, $Z = \bar{P}(z)$, $Q = \bar{P}(q)$, and $\bar{B}_\infty = \bar{P}(1)$ we can reformulate the equivalency in (2.9) to

$$\frac{1}{q-z} - \frac{1}{1-z} = \frac{1}{-z} - \frac{1}{r-z} \iff d(Z,Q) = d(R,Z),$$

where $0 < r < z < q < 1$. Thus, the map $\rho_{d;e;z}\colon P(r) \leftrightarrow P(\frac{z^2(1-r)}{z^2-r(2z-1)})$ is a d-isometric point reflection on segment e for every $P(z) \in e$, hence

$$\tau_{d;e;z,t} := \rho_{d;e;t} \circ \rho_{d;e;z}\colon P(r) \to P\Big(\frac{z^2(1-r)}{z^2-r(2z-1)}\Big) \to P\Big(\frac{1}{1+\frac{1-r}{r}\frac{z^2}{(1-z)^2}\frac{(1-t)^2}{t^2}}\Big)$$

is a d-isometric translation. So $d(P(r), \tau_{d;e;z,t}(P(r)))$ does not depend on r, hence it is a real function δ of $\frac{z^2}{(1-z)^2}\frac{(1-t)^2}{t^2}$. As d is additive, this implies $\delta(x) + \delta(y) = \delta(xy)$ so, by the solution of Cauchy's functional equation [15], we have a constant $\bar{c}_e > 0$ such that $\delta(x) = 2c_e|\ln(x)|$. Thus

$$d(P(x), P(y)) = d(P(x), \tau_{d;e;1,\frac{x}{1-x}\frac{1-y}{y}}(P(x))) = \delta\Bigg(\sqrt{\frac{x}{1-x}\frac{1-y}{y}}\Bigg) = \bar{c}_e\Big|\ln\Big(\frac{x}{1-x}\frac{1-y}{y}\Big)\Big|.$$

This means $d(P(x), P(y)) = \bar{c}_e|\ln(A_\infty, B_\infty; P(y), P(x))|$, i.e. a Hilbert metric on segment e.

Having the metric for every possible domain of a projective-metric space of hyperbolic type, we are ready to step forward by considering the properties of the domain $\mathcal{D}$.

If $\mathcal{D}$ contains a whole affine line, then by [6, Exercise [17.8]] it is either a half plane or a strip bounded by two parallel lines, because it is not the whole plane. Thus, domain $\mathcal{D}$ is

$$\text{either } \mathcal{P}_{(0,\infty)} := \{(x,y)\in\mathbb{R}^2 : 0<x\} \text{ or } \mathcal{P}_{(0,b)} := \{(x,y)\in\mathbb{R}^2 : 0<x<b\}$$

in suitable linear coordinates. As the perspective projectivity $\varpi\colon (x,y)\mapsto (\frac{x}{x+1},\frac{y}{x+1})$ maps $\mathcal{P}_{(0,\infty)}$ onto $\mathcal{P}_{(0,1)}$ bijectively, it is enough to consider the case $\mathcal{D}=\mathcal{P}_{(0,1)}$.

By the above, we know that $d((x,y),(x,z)) = c(x)|z-y|$ for a continuous $c\colon (0,1)\to\mathbb{R}_+$, and

$$d((x,\lambda+\sigma x),(\mu x,\lambda+\mu\sigma x)) = \bar{c}(\lambda,\sigma)\Big|\ln\Big(0,\frac{1}{x};1,\mu\Big)\Big| = \bar{c}(\lambda,\sigma)\Big|\ln\frac{1-\mu x}{\mu(1-x)}\Big|,$$

where $\bar{c}\colon \mathbb{R}\times\mathbb{R}_+\to\mathbb{R}_+$ is also a continuous function. Putting these together gives

$$d((x,0),(s,y)) = \begin{cases}\bar{c}(\frac{-yx}{s-x},\frac{y}{s-x})\Big|\ln\frac{x(1-s)}{s(1-x)}\Big|, & \text{if } x\neq s,\\ c(x)|y|, & \text{if } x=s,\end{cases}$$

for every $x,s\in(0,1)$ and $y\in\mathbb{R}$. Letting $y=k(s-x)>0$ where $k\geq 0$, we get

$$kc(x) = \lim_{s\to x}\frac{d((x,0),(x,s-x))}{s-x} = \bar{c}(-kx,k)\lim_{s\to x}\Bigg|\frac{\ln\frac{x(1-s)}{s(1-x)}}{s-x}\Bigg|$$

$$= \bar{c}(-kx,k)\lim_{s\to x}\Bigg|\frac{\ln\big(1-\frac{1}{s(1-x)/(s-x)}\big)^{s(1-x)/(s-x)}}{s(1-x)}\Bigg| = \frac{\bar{c}(-kx,k)}{x(1-x)}.$$

This gives $0=\lim_{k\to 0}\bar{c}(-kx,k)$, and by continuity $\bar{c}(0,0)=0$, a contradiction.

Thus $\mathcal{D}$ does not contain a whole affine line, so it is either bounded or contains some rays. The metric on every chord $\ell \cap \mathcal{D}$ cut out by the straight lines ℓ from $\mathcal{D}$ is of the form $c_\ell \delta$, where δ is the Hilbert metric on $\mathcal{D}$. Multiplier c_ℓ depends on ℓ continuously because d and δ are continuous. Given non-collinear points $A, B, C \in \mathcal{D}$ the strict triangle inequalities give that $|\delta(A,C) - \delta(B,C)| < \delta(A,B)$ and $|c_{AC}\delta(A,C) - c_{BC}\delta(B,C)| = |d(A,C) - d(B,C)| < d(A,B) = c_{AB}\delta(A,B)$. These imply

$$\left|\frac{\delta(A,C)}{\delta(B,C)} - 1\right| < \frac{\delta(A,B)}{\delta(B,C)}, \quad \text{and} \quad \left|c_{AC}\frac{\delta(A,C)}{\delta(B,C)} - c_{BC}\right| < c_{AB}\frac{\delta(A,B)}{\delta(B,C)}.$$

If C tends to a point ∞ on the boundary $\partial\mathcal{D}$ of $\mathcal{D}$, then the first inequality implies $\frac{\delta(A,C)}{\delta(B,C)} \to 1$, so from the second inequality $c_{A\infty} = c_{B\infty}$ follows. Thus c_ℓ is the same for every line that goes through the same point of $\partial\mathcal{D}$. This clearly implies that c_ℓ does not depend on ℓ, i.e. constant, hence $(\mathcal{D}, d)$ is a Hilbert geometry.

However, [10, Theorem 3.1] proves that a Hilbert geometry which has the Ceva property is hyperbolic, hence the theorem for projective-metric spaces of hyperbolic type.

■

To make versions of Ceva's or Menelaus' theorems valid in more projective-metric spaces more freedom should be allowed for the ratios.

Let A, B be different points in a projective-metric space $(\mathcal{M}, d)$, and let $C \in (AB \cap \mathcal{M}) \setminus \{B\}$. Then the real number

$$\langle A, B; C\rangle_d^{\dagger} = \begin{cases} \frac{\lambda(d(A,C))}{\lambda(d(C,B))}, & \text{if } C \in \overline{AB}, \\ -\frac{\lambda(d(A,C))}{\lambda(d(C,B))}, & \text{otherwise}, \end{cases} \tag{3.1}$$

is called the *λ-ratio* of the triplet (A, B, C), where λ is a non-negative strictly increasing function of the positive real numbers.

The question arises if there is a projective-metric space in which Ceva's or Menelaus' theorems are valid with a λ-ratio. The answer to this question is negative for the Hilbert geometries $(\mathcal{M}, d)$.

For, just choose five points on $\partial\mathcal{M}$, and fit an ellipse $\mathcal{E}$ through these points. Then $\mathcal{E}$ intersects $\partial\mathcal{M}$ in at least six points in a circumcise order $M_1, M_2, M_3, M_4, M_5, M_6$. The chords $\overline{M_1M_4}$, $\overline{M_2M_5}$, and $\overline{M_3M_6}$ in general intersect each other in three points, say in A, B, and C. Now, on the side-lines of trigon $ABC\triangle$ the hyperbolic metric is given, hence Ceva's and Menelaus' theorems are valid with $\lambda(\cdot) \equiv \sinh(\cdot)$. For the hyperbolic geometry only the hyperbolic sine function can be a good choice, and we know from the results of the previous slides that it just does not work for more general Hilbert geometries.


Thank you for your attention!

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Projective-metric spaces

with Ceva or Menelaus property