Finding Needles in a Haystack

Determining the segments of a multi-curve by masking function

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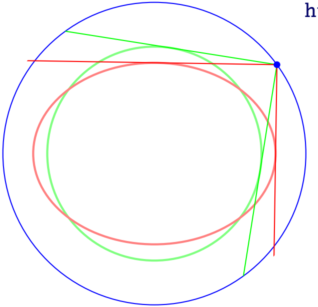
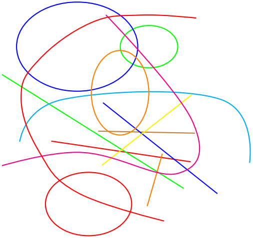
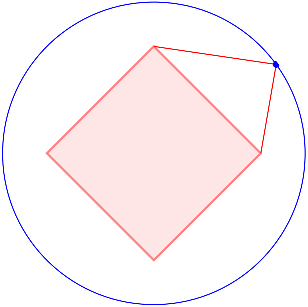
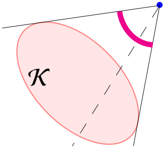




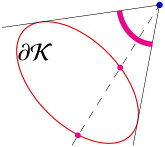


Discrete Geometry Days^2 at Technical University in Budapest, Hungary on July 10, 2019, 2019, 16:25.
The work of the author is supported by grants K 116451, KH_18 129630, and 20391-3/2018/FEKUSTRAT.
For the animations Adobe PDF Reader is necessary.

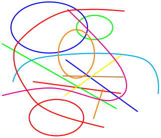
In this talk everything is in the plane.



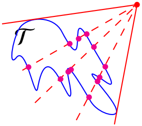
The *visual angle*[1] of a convex, bounded domain $\mathcal{K}$ at an outer point P is the measure of the angle $\mathcal{K}$ subtends at P. This is the measure of the set of straight lines through P intersecting $\partial\mathcal{K}$. We denote this by $M_{\mathcal{K}}(P)$, where $M_{\mathcal{K}}$ is called the *visual angle function*.



A *finite set $\boldsymbol{r}_{\mathcal{J}}$ of parametric curves $\boldsymbol{r}_j\colon [a_j, b_j] \to \mathbb{R}^n$* ($j \in \mathcal{J}$) of finite length without common arcs is called *multicurve*. The curves are the *members* of the multi-curve.



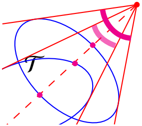
We say that a multi-curve has a property if its every member has that property[2]. The *trace of a multi-curve $\boldsymbol{r}_{\mathcal{J}}$* is the union of the traces of its members: $\operatorname{Tr}\boldsymbol{r}_{\mathcal{J}} := \bigcup_{j\in\mathcal{J}} \operatorname{Tr}\boldsymbol{r}_j$.



The *masking number*[3] of the trace $\mathcal{T} = \operatorname{Tr}\boldsymbol{r}_{\mathcal{J}}$ is defined by

$$M_{\mathcal{T}}(P) = \frac{1}{2}\int_{\mathbb{S}^{n-1}} \#(\mathcal{T} \cap \ell(P, \boldsymbol{w}))d\boldsymbol{w},$$

where straight line $\ell(P, \boldsymbol{w})$ through P has direction $\boldsymbol{w} \in \mathbb{S}^{n-1}$, and # is the counting measure[4]. $M_{\mathcal{T}}$ is the *masking function*.



The problem is to *determine the multi-curve from a given set of its masking numbers*.

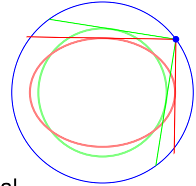
This is in its origin an old subject [12, 4] of the (now so called) *geometric tomography* [3].

[1] This is called the *point projection* in book [3] of Gardner, and it is called *shadow picture* in article [7] of Kincses and Kurusa.

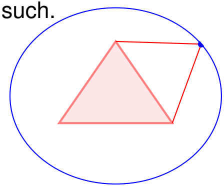
[2] A *multi-segment* (*multi-circle*) is a multi-curve such that the member curves are segments (circles) exclusively.

[3] The [2, cross integral], the [11, weighted back projection] and the [9, generalized visual angle] are in fact masking numbers.

[4] In our cases these are finite almost everywhere.



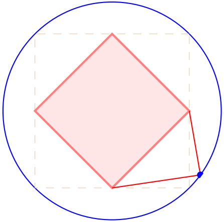
The *ν-isoptic* of a bounded convex domain $\mathcal{K}$ is the set of points P, where $\nu = M_{\mathcal{K}}(P)$. *Green* proved in 1950 [4] that there are non-circular discs with a circle as its ν-isoptic if and only if $1-\nu/\pi$ is a rational number that has odd nominator in its simplest form. For $\pi/2$ a suitable ellipse is such.



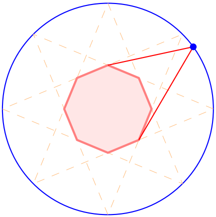
The *equioptic* set of two convex domains is the set of points where their visual angles are equal.

Kincses & Kurusa proved in 1995 [7] that if the equioptic of two convex polygons contains an analytic curve that surrounds the union of the convex polygons, then the polygons coincide.

A convex domain in the unit disc is called *distinguishable among the convex domains* if there is no other convex domain with the same visual angle function on the unit circle.



Kincses proved that all *triangles* [5], the *midpoint square* of the inscribed square [6], and the *regular octagon* surrounded by the regular inscribed star octagon [5] are *distinguishable* among the convex domains, so he put [5, Question 3.2]

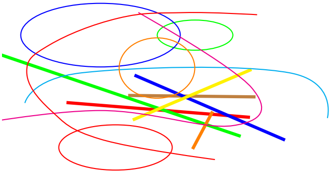


Are the convex polygons distinguishable among convex domains?

We change the scene to answer the question of Kincses. We consider multi-curves and find the segments in it; so to say, *we find the needles in a haystack!*

Main result. (ÁK: [8, Theorem 4.1]).

The traced segments of a multi-curve of class C^∞ can be reconstructed if the masking function is given on any rounding circle.



This answers the question of Kincses affirmatively:

Corollary. *The visual angle function of a convex domain on a surrounding circle determines every segment in the border of the convex domain, hence every convex polygon is reconstructable from its visual angle function on any surrounding circle.*

For the sake of accuracy: we confine ourselves for such curves in the multi-curves that are twice differentiable, are not self-intersecting, are parametrized by arc-length on a closed interval, are intersecting every straight line in only finitely many closed (maybe degenerate) segments, have only finitely many tangents through any point of its exterior, have only finitely many points of vanishing curvature beside a finite set of traced straight lines, and have only finitely many multiple tangent lines. This kind of multi-curves constitutes *class* $\mathfrak{C}$.

A *traced segment* of a given multi-curve $\boldsymbol{r}_{\mathcal{J}}$ is a non-degenerate segment of the form $\operatorname{Tr}\boldsymbol{r}_j([s_0, s_1])$ $(s_0 < s_1)$, where $\boldsymbol{r}_j$ is a member curve of $\boldsymbol{r}_{\mathcal{J}}$.
A *traced straight line* is a straight line containing a traced segment.

Lemma. (ÁK: [10, Proposition 3.2 and Lemma 4.1]).

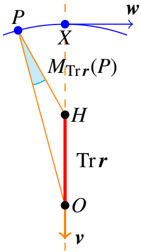
If $\boldsymbol{r}\colon [0,h] \ni s \mapsto \boldsymbol{r}(0) + s\boldsymbol{v}$ ($\boldsymbol{v} \in \mathcal{S}^2$), *then for any* $\boldsymbol{w} \in \mathcal{S}^2$ *we have*

$$\partial_{\boldsymbol{w}} M_{\mathrm{Tr}\,\boldsymbol{r}}(X) = \begin{cases} -|\langle \boldsymbol{v}, \boldsymbol{w}^{\perp}\rangle|\left(\frac{1}{x} + \frac{1}{h-x}\right), & \text{if } X = \boldsymbol{r}(0) + x\boldsymbol{v} \text{ and } x \in \mathbb{R}\setminus\{0,h\}, \\ -\partial_{-\boldsymbol{w}} M_{\mathrm{Tr}\,\boldsymbol{r}}(X), & \text{if } X \notin \ell(\boldsymbol{r}(0), \boldsymbol{v}). \end{cases}$$

So, if $X \notin \mathrm{Tr}\,\boldsymbol{r}_{\mathcal{J}}$ *of a multi-curve* $\boldsymbol{r}_{\mathcal{J}}$ *of class* $\mathfrak{C}$, *then*

(1) $\partial_{\boldsymbol{w}} M_{\mathrm{Tr}\,\boldsymbol{r}_{\mathcal{J}}}(X) + \partial_{-\boldsymbol{w}} M_{\mathrm{Tr}\,\boldsymbol{r}_{\mathcal{J}}}(X) = 0$ *if no traced line goes through* X,

(2) $\partial_{\boldsymbol{w}} M_{\mathrm{Tr}\,\boldsymbol{r}_{\mathcal{J}}}(X) + \partial_{-\boldsymbol{w}} M_{\mathrm{Tr}\,\boldsymbol{r}_{\mathcal{J}}}(X) > 0$ *if a traced line goes through* X.



For multi-curves of class C^k ($k \in \mathbb{N}$) one can replace (1) with

(1') $\partial^k_{\boldsymbol{w}} M_{\mathrm{Tr}\,\boldsymbol{r}_{\mathcal{J}}}(X) = (-1)^k \partial^k_{-\boldsymbol{w}} M_{\mathrm{Tr}\,\boldsymbol{r}_{\mathcal{J}}}(X)$ if no traced line passes X.

Theorem. (ÁK: [10, Proposition 6.1]).

Except at the points where it is not differentiable

(1) *the masking function of every multi-segment is locally harmonic, i.e.* $\Delta M_{\mathrm{Tr}\,\boldsymbol{r}_{\mathcal{J}}} \equiv 0$;

(2) *the masking function of every multi-curve is locally subharmonic, i.e.* $\Delta M_{\mathrm{Tr}\,\boldsymbol{r}_{\mathcal{J}}} \geq 0$.

Lemma. (ÁK: [10, Theorem A.1]).

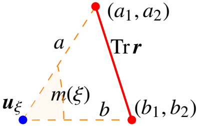
Assume that the function $f\colon (a,b) \to \mathbb{R}\cup\{\infty,-\infty\}$ *can be written in the form* $f(x) = p(x) + \sum_{i=1}^{k} \frac{c_i}{(x-d_i)^{e_i}}$ *for a polynomial* p *and some* $k, e_i \in \mathbb{R}$ *and* $c_i, d_i \in \mathbb{R}$ *such that* $d_i \neq d_j$ *for every* $i,j = 1,\dots,k$. *Then the set* $\{p\} \cup \{(c_i,d_i,e_i)\}_{i=1}^{k}$ *is unique.*

Lemma. *Let $\boldsymbol{r}_{\mathcal{J}}$ be a multi-segment in the open unit disc $\mathcal{D}$. The function $m\colon \xi \to M_{\mathrm{Tr}\,\boldsymbol{r}_{\mathcal{J}}}(\boldsymbol{u}_\xi)$ is analytic around α if no traced line goes through $\boldsymbol{u}_\alpha = (\cos\alpha, \sin\alpha)$.*

If $\mathrm{Tr}\,\boldsymbol{r}$ is the segment $\overline{(a_1,a_2)(b_1,b_2)}$, then

$$\cos M_{\mathrm{Tr}\,\boldsymbol{r}}(\boldsymbol{u}_\xi) = \frac{(a_1-\cos\xi)(b_1-\cos\xi)+(a_2-\sin\xi)(b_2-\sin\xi)}{\sqrt{(a_1-\cos\xi)^2+(a_2-\sin\xi)^2}\sqrt{(b_1-\cos\xi)^2+(b_2-\sin\xi)^2}}.$$

This proves the lemma, because the $\arccos$ function is analytic and invertible on $(0,\pi)$,.



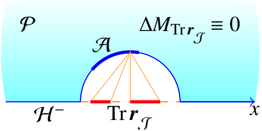
Key proposition. (ÁK: [8, Lemma 3.2]).

If a multi-segment has only one traced line, then it is reconstructible by its masking function given on any open arc of a surrounding circle.

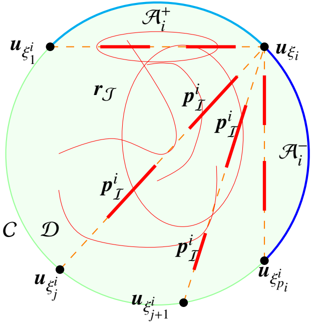
Let the x-axis be traced, and let $\{\overline{(s_j,0)(t_j,0)} : j \in \mathcal{J}\}$ be the traced segments in the unit open disc $\mathcal{D}$. Let $\mathcal{A}$ be an arc of the circle $\partial\mathcal{D}$ in the open upper half-plane $\mathcal{H}^+$. If $M_{\mathrm{Tr}\,\boldsymbol{r}_{\mathcal{J}}}$ is known on $\mathcal{A}$, then its unique analytic continuation gives $M_{\mathrm{Tr}\,\boldsymbol{r}_{\mathcal{J}}}$ on $\partial\mathcal{D}\cap\mathcal{H}^+$. However $M_{\mathrm{Tr}\,\boldsymbol{r}_{\mathcal{J}}}$ vanishes on the x-axis outside $\mathcal{D}$, and it tends to zero at infinity, so $M_{\mathrm{Tr}\,\boldsymbol{r}_{\mathcal{J}}}$ is known on the border $\partial\mathcal{P}$ of $\mathcal{P} = \mathcal{H}^+ \setminus \mathcal{D}$. Thus the unique harmonic extension [1, I.4. Theorem (c)] gives $M_{\mathrm{Tr}\,\boldsymbol{r}_{\mathcal{J}}}$ on $\mathcal{P}$, so one can determine

$$\partial_{(0,1)} M_{\mathrm{Tr}\,\boldsymbol{r}_{\mathcal{J}}}(\cdot,0)\colon (1,\infty) \ni x \mapsto \sum_{j\in\mathcal{J}}\left(\frac{1}{x-t_j} - \frac{1}{x-s_j}\right) \in \mathbb{R}.$$

This determines $\{s_j, t_j : j \in \mathcal{J}\}$, hence the proposition.



To prove our Main result let $\boldsymbol{r}_{\mathcal{J}}$ be a multi-curve of class C^∞ in the open unit disc $\mathcal{D}$, and assume that $m\colon \xi \to M_{\mathrm{Tr}\,\boldsymbol{r}_{\mathcal{J}}}(\boldsymbol{u}_\xi)$ is given on the unit circle $\mathcal{C} = \partial\mathcal{D}$. Let $\boldsymbol{p}_{\mathcal{I}}$ be the multi-segment of all the traced segments in $\boldsymbol{r}_{\mathcal{J}}$.



The set of the intersections of $\mathcal{C}$ with the traced lines is $\mathcal{L} := \{\boldsymbol{u}_\xi : m'(\xi+) \neq m'(\xi-)\}$. This is a finite set, so we have a sequence $0 < \xi_0 < \cdots < \xi_n < 2\pi$ such that $\mathcal{L} = \{\boldsymbol{u}_{\xi_i} : i = 0, \ldots, n\}$.

Let $\boldsymbol{p}_{\mathcal{I}}^i$ be the multi-segment of the segments in $\boldsymbol{p}_{\mathcal{I}}$ that are collinear with $\boldsymbol{u}_{\xi_i}$ $(i = 0, \ldots, n)$.

Let $\boldsymbol{u}_{\xi_j^i}$ $(j = 1, \ldots, p_i)$ be the points in anticlockwise order that are on traced lines of $\boldsymbol{p}_{\mathcal{I}}^i$.

Then we have

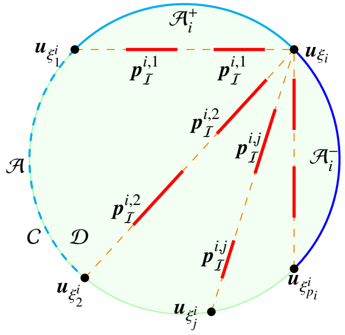
$$\frac{m^{(k)}(\xi+) + (-1)^k m^{(k)}(\xi-)}{2} = \begin{cases} \partial^k_{\boldsymbol{u}^\perp_{\xi_i}} M_{\mathrm{Tr}\,\boldsymbol{p}_{\mathcal{I}}^i}(\boldsymbol{u}_{\xi_i}), & \text{if } \xi = \xi_i \text{ for an } i = 0, \ldots, n, \\ 0, & \text{if } \xi \neq \xi_i \text{ for every } i = 0, \ldots, n. \end{cases}$$

These derivatives determine $M_{\mathrm{Tr}\,\boldsymbol{p}_{\mathcal{I}}^i}(\boldsymbol{u}_\xi)$ on the arcs $\mathcal{A}_i^+ = \widehat{\boldsymbol{u}_{\xi_i}\boldsymbol{u}_{\xi_1^i}}$ and $\mathcal{A}_i^- = \widehat{\boldsymbol{u}_{\xi_{p_i}^i}\boldsymbol{u}_{\xi_i}}$, because $M_{\mathrm{Tr}\,\boldsymbol{p}_{\mathcal{I}}^i}$ is analytic on these arcs.

Let $\widetilde{\mu}_i^\pm(\xi)$ be the unique analytic continuation of $M_{\mathrm{Tr}\,\boldsymbol{p}_{\mathcal{I}}^i}(\boldsymbol{u}_\xi)$ from $\mathcal{A}_i^\pm$ to the whole circle $\mathcal{C}$.

If $\mathcal{A}_i^+ \cup \mathcal{A}_i^- = \mathcal{C}$ (i.e. $p_i = 1$), then the Key proposition gives $\mathrm{Tr}\,\boldsymbol{p}_{\mathcal{I}}^i$ from either one of $\widetilde{\mu}_i^\pm$.

Assume now that $\mathcal{A}_i^+ \cup \mathcal{A}_i^-$ does not cover $\mathcal{C}$, hence $p_i > 1$.



Observe that

$\widetilde{\mu}_i^{\pm}$ is the sum of the *signed*(!) visual angles

of the traced segments in $\boldsymbol{p}_{\mathcal{I}}^i$. The sign of the visual angle of a segment $\overline{AB}$ of $\boldsymbol{p}_{\mathcal{I}}^i$ in $\widetilde{\mu}_i^{\pm}(\xi)$ is
'+' if $\mathcal{A}_i^{\pm}$ is on the same side of line AB as $\boldsymbol{u}_{\xi}$ is, and
'−' otherwise.

Let $\boldsymbol{p}_{\mathcal{I}}^{i,j}$ be the multi-segment of the segments in $\boldsymbol{p}_{\mathcal{I}}^i$ lying on the line $\boldsymbol{u}_{\xi_i}\boldsymbol{u}_{\xi_j^i}$ ($j = 1, \dots, p_i$).

Then

$\widetilde{\mu}_i^+ = M_{\mathrm{Tr}\,\boldsymbol{p}_{\mathcal{I}}^{i,1}} + \sum_{j=2}^{p_i} M_{\mathrm{Tr}\,\boldsymbol{p}_{\mathcal{I}}^{i,j}}$ on arc $\mathcal{A}_i^+$, and

$\widetilde{\mu}_i^- = M_{\mathrm{Tr}\,\boldsymbol{p}_{\mathcal{I}}^{i,1}} - \sum_{j=2}^{p_i} M_{\mathrm{Tr}\,\boldsymbol{p}_{\mathcal{I}}^{i,j}}$ on the arc $\mathcal{A} = \widehat{\boldsymbol{u}_{\xi_1^i}\boldsymbol{u}_{\xi_2^i}}$.

Thus, we obtain $M^{(k)}_{\mathrm{Tr}\,\boldsymbol{p}_{\mathcal{I}}^{i,1}}(\boldsymbol{u}_{\xi_1^i-}) = ((\widetilde{\mu}_i^+)^{(k)}(\xi_1^i-) + (-1)^k(\widetilde{\mu}_i^-)^{(k)}(\xi_1^i-))/2$ for the derivatives of order $k = 0, 1, \dots$. As $M_{\mathrm{Tr}\,\boldsymbol{p}_{\mathcal{I}}^{i,1}}(\boldsymbol{u}_{\xi})$ is analytic on (ξ_i, ξ_1^i), these derivatives determine $M_{\mathrm{Tr}\,\boldsymbol{p}_{\mathcal{I}}^{i,1}}$ on (ξ_i, ξ_1^i), so the Key proposition gives $\mathrm{Tr}\,\boldsymbol{p}_{\mathcal{I}}^{i,1}$.

Considering the difference $M_{\mathrm{Tr}\,\boldsymbol{r}_{\mathcal{J}}} - M_{\mathrm{Tr}\,\boldsymbol{p}_{\mathcal{I}}^{i,1}}$, we get into the same situation as at the start of the proof, but with one less traced straight lines, so an induction on the number of the traced straight lines gives our Main result.

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1. **Visual angle and masking number**
2. **Distinguishability of convex domains**
3. **Reconstructibility of multi-segments**
   - The result
   - Preliminaries for the proof
   - Preparation and the key of the proof
   - Sketch of the proof

Abstract: Convex polygons are distinguishable among the convex domains by comparing their visual angle functions on any surrounding circle. This is a consequence of our main result that every segment in a multicurve can be reconstructed if the masking function of the multicurve is known on any surrounding circle.

This work was supported by the NFSR of Hungary (NKFIH) under grant numbers K 116451, KH_18 129630, and by the Ministry of Human Capacities, Hungary grant number 20391-3/2018/FEKUSTRAT.