

FINDING NEEDLES IN A HAYSTACK

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ABSTRACT. Convex polygons are distinguishable among the piecewise C^∞ convex domains by comparing their visual angle functions on any surrounding circle. This is a consequence of our main result, that every segment in a C^∞ multicurve can be reconstructed from the masking function of the multicurve given on any surrounding circle.

1. INTRODUCTION

Given a compact convex domain \mathcal{K} inside a circle, we say it *distinguishable* if no other convex disc exists in the circle that subtends the same angle at each point of the circle as \mathcal{K} does. Green proved in [3] that a compact convex domain inside a circle which subtends a constant(!) angle ν at each point of the circle is not distinguishable if and only if ν/π is a rational number with even denominator in its smallest terms.

However it is proved in [4] and [8], that two polygons are always distinguishable from each other, so the question

Are convex polygons distinguishable among convex domains? (1.1) (1)

emerged naturally in [5, Question 3.2], where it is proved that all triangles [5, Corollary 2.4], and the regular octagon surrounded by the regular star octagon inscribed in the circle [5, Example 2.5] are distinguishable. Further, the midpoint square of the inscribed square are distinguishable too by [6].

In this article we prove in Theorem 4.1, that every segment in a C^∞ multicurve can be determined by knowing the masking function of the multicurve on any surrounding circle.

This implies an affirmative answer for (1.1) formulated in Corollary 4.2, i.e. that every convex polygon is distinguishable among the convex domains of piecewise C^∞ boundary.

2. NOTATIONS AND PRELIMINARIES

We work in the plane \mathbb{R}^2 , the open unit ball centered at $(0, 0)$ is \mathcal{B}^2 , and its boundary, the unit circle is $\mathcal{S}^1 = \partial\mathcal{B}^2$. Unit vectors are shorthanded as $\mathbf{u}_\beta = (\cos \beta, \sin \beta)$.

The linear map \cdot^\perp on \mathcal{S}^1 is defined by $\mathbf{u}_\beta^\perp = \mathbf{u}_{\beta+\pi/2} = (-\sin \beta, \cos \beta)$. The Euclidean multiplication is denoted by $\langle \cdot, \cdot \rangle$.

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Let $\mathbf{r}: [a, b] \rightarrow \mathbb{R}^2$ be a differentiable curve parameterized by arc-length parameter s . The trace $\text{Tr } \mathbf{r}$ of \mathbf{r} is the set of points in \mathbb{R}^2 that is in the range of the function \mathbf{r} . A non-degenerate segment of the form $\mathbf{r}([s_0, s_1])$ ($s_0 < s_1$) in $\text{Tr } \mathbf{r}$ is said *traced*. We call also a straight line *traced* if there is a traced segment on it.

A *multicurve* $\mathbf{r}_{\mathcal{J}}$ is a finite set of differentiable curves $\mathbf{r}_j: [a_j, b_j] \rightarrow \mathbb{R}^2$ ($j \in \mathcal{J} \subset \mathbb{N}$), the *members of the multicurve*, such that the members are of finite length and do not intersect each other in open arc. The trace $\text{Tr } \mathbf{r}_{\mathcal{J}}$ of a multicurve $\mathbf{r}_{\mathcal{J}}$ is the union $\bigcup_{j \in \mathcal{J}} \text{Tr } \mathbf{r}_j$. A multicurve is said to have a property if each of its members satisfies that property.

Let \mathfrak{C} be the set of curves that

- are twice differentiable,
- are not self-intersecting,
- are parameterized by arc-length on a finite closed interval,
- are intersecting every straight line in only finitely many closed (maybe degenerate) segments,
- have only finitely many tangents through any point of its exterior,
- have only finitely many points of vanishing curvature beside a finite set of traced straight lines, and
- have only finitely many multiple tangent lines.

A multicurve is called *regular* if all of its members are in \mathfrak{C} . A multicurve is a *multisegment* if all of its members are segments. A multisegment is obviously regular. To avoid long analytic technicalities, *we confine ourselves to considering only regular multicurves*.

Following [9], the *masking number*¹ $M_{\mathcal{T}}(P)$ of the trace $\mathcal{T} = \text{Tr } \mathbf{r}_{\mathcal{J}}$ of the regular multicurve $\mathbf{r}_{\mathcal{J}}$ is $M_{\mathcal{T}}(P) = \frac{1}{2} \int_{\mathcal{S}^1} \#(\mathcal{T} \cap \ell(P, \mathbf{w})) d\mathbf{w}$, where $\ell(P, \mathbf{w})$ is the straight line through the point $P \in \mathbb{R}^2$ with direction $\mathbf{w} \in \mathcal{S}^1$ and $\#$ is the counting measure. If \mathcal{T} is a closed convex curve, then the masking number $M_{\mathcal{T}}(P)$ is twice of the *point projection* (see [2]) and the *shadow picture* (see [4]).

We define the *masking function* $M_{\mathbf{r}_{\mathcal{J}}}: \mathbb{R}^2 \rightarrow \mathbb{R}$ of a regular multicurve $\mathbf{r}_{\mathcal{J}}$ by

$$M_{\mathbf{r}_{\mathcal{J}}}(X) = \int_0^\pi \#(\text{Tr } \mathbf{r}_{\mathcal{J}} \cap \ell(X, \mathbf{u}_\alpha)) d\alpha.$$

We clearly have $M_{\mathbf{r}_{\mathcal{J}}}(X) = \sum_{j \in \mathcal{J}} M_{\mathbf{r}_j}(X)$ and also $M_{\text{Tr } \mathbf{r}_{\mathcal{J}}} = M_{\mathbf{r}_{\mathcal{J}}}$.

Proposition 2.1 ([9, Proposition 3.2]). *If $\mathbf{r}: [0, h] \ni s \mapsto \mathbf{r}(0) + s\mathbf{v}$ ($\mathbf{v} \in \mathcal{S}^1$), then*

$$\partial_{\mathbf{w}} M_{\mathbf{r}}(X) = \begin{cases} -|\langle \mathbf{v}, \mathbf{w}^\perp \rangle| (\frac{1}{x} + \frac{1}{h-x}) & \text{if } X = \mathbf{r}(0) + x\mathbf{v} \text{ and } x \in \mathbb{R} \setminus \{0, h\}, \\ -\partial_{-\mathbf{w}} M_{\mathbf{r}}(X) & \text{if } X \notin \ell(\mathbf{r}(0), \mathbf{v}), \end{cases}$$

where $\mathbf{w} \in \mathcal{S}^1$ and $\partial_{\mathbf{w}}$ denotes the one sided directional derivation.

¹This number is finite at almost every point because of the regularity condition.

Lemma 2.2 ([9, Lemma 4.1]). *Let $\mathbf{r}_{\mathcal{J}}$ be a regular multicurve, and let $\mathbf{w} \in \mathcal{S}^1$ be not parallel² to any traced segment.*

- (1) *If no traced line goes through X , then $\partial_{\mathbf{w}}M_{\mathbf{r}_{\mathcal{J}}}(X) + \partial_{-\mathbf{w}}M_{\mathbf{r}_{\mathcal{J}}}(X) = 0$.*
- (2) *If $X \notin \text{Tr } \mathbf{r}_{\mathcal{J}}$, then $\partial_{\mathbf{w}}M_{\mathbf{r}_{\mathcal{J}}}(X) + \partial_{-\mathbf{w}}M_{\mathbf{r}_{\mathcal{J}}}(X) \geq 0$, and it is positive if and only if X is on a traced straight line.*
- (3) *Except for finitely many points X of a traced segment of $\mathbf{r}_{\mathcal{J}}$ we have $\partial_{\mathbf{w}}M_{\mathbf{r}_{\mathcal{J}}}(X) + \partial_{-\mathbf{w}}M_{\mathbf{r}_{\mathcal{J}}}(X) \neq 0$.*

For multicurves of class C^k ($k \in \mathbb{N}$) item (1) can be obviously replaced with

- (1') *if no traced line goes through X , then $\partial_{\mathbf{w}}^k M_{\mathbf{r}_{\mathcal{J}}}(X) = (-1)^k \partial_{-\mathbf{w}}^k M_{\mathbf{r}_{\mathcal{J}}}(X)$.*

Finally, the following known result on harmonic functions is displayed here for the sake of completeness, and because it is crucial in what follows.

Theorem 2.3 ([1, I.4. Theorem (c)]). *If the function f is harmonic on the open subset \mathcal{D} of \mathbb{R}^n , and f has a continuous extension to $\mathcal{D} \cup \partial\mathcal{D}$, then the supremum and infimum of the extension are attained on $\partial\mathcal{D}$.*

Here the space \mathbb{R}^n is with the Euclidean topology and is compactified by the ideal point at infinity, which is not included in \mathbb{R}^n , but is included in the boundary of every unbounded subset of \mathbb{R}^n .

3. UTILITIES

Let $\phi_j(\mathbf{p}) \in (-\pi, \pi)$ be the oriented visual angle of the j th member segment $\mathbf{r}_j: [0, h] \ni s \mapsto \mathbf{r}(0) + s\mathbf{v}$ ($\mathbf{v} \in \mathcal{S}^1$) of the multisegment $\mathbf{r}_{\mathcal{J}}$ at $\mathbf{p} \in \mathbb{R}^2 \setminus \text{Tr } \mathbf{r}_{\mathcal{J}}$, i.e.

$$(\cos \phi_j(\mathbf{p}), \sin \phi_j(\mathbf{p})) = \left(\frac{\langle \mathbf{r}_j(0) - \mathbf{p}, \mathbf{r}_j(h) - \mathbf{p} \rangle}{|\mathbf{r}_j(0) - \mathbf{p}| \cdot |\mathbf{r}_j(h) - \mathbf{p}|}, \frac{\langle (\mathbf{r}_j(0) - \mathbf{p}) \times (\mathbf{r}_j(h) - \mathbf{p}), \mathbf{m} \rangle}{|\mathbf{r}_j(0) - \mathbf{p}| \cdot |\mathbf{r}_j(h) - \mathbf{p}|} \right), \quad (3.1) \quad (4)$$

where \mathbf{m} is a fixed unit normal vector of the plane. Observe that ϕ_j changes sign if the parameterization of \mathbf{r}_j was reversed, so the definition

$$\Phi_{\mathbf{r}_{\mathcal{J}}} : \mathbb{R}^2 \ni \mathbf{p} \mapsto \sum_{j \in \mathcal{J}} \phi_j(\mathbf{p})$$

is valid only if a choice of the parameterization was fixed for every segment.

We need the following slight generalization of the observation given in the proof of [4, Lemma 2.2].

Lemma 3.1. *Let $\mathbf{r}_{\mathcal{J}}$ be a multisegment.*

- (1) *If no traced line of $\mathbf{r}_{\mathcal{J}}$ goes through \mathbf{p} , then $M_{\mathbf{r}_{\mathcal{J}}}$ is real analytic around \mathbf{p} .*
- (2) *The function $\Phi_{\mathbf{r}_{\mathcal{J}}}$ is real analytic on $\mathbb{R}^2 \setminus \text{Tr } \mathbf{r}_{\mathcal{J}}$.*

²This condition is mistakenly missing in [9, Lemma 4.1].

Proof. As $M_{\mathbf{r}_{\mathcal{J}}}$ and $\Phi_{\mathbf{r}_{\mathcal{J}}}$ are locally the sum of the functions $\pm\phi_j$, we can assume without loss of generality that $\mathbf{r}_{\mathcal{J}}$ is the segment $\mathbf{r}: [0, h] \ni s \mapsto \mathbf{r}(0) + s\mathbf{v}$ ($\mathbf{v} \in \mathcal{S}^1$).

Since both the functions arccos and arcsin are analytic on $(-1, 1)$, the equality of the first coordinates in (3.1) proves both statements if the traced line avoids \mathbf{p} , and the equality of the second coordinates in (3.1) proves the second statement if the traced line goes through \mathbf{p} . \square

The following lemma is an obvious extension of [9, (1) of Theorem 6.1].

Lemma 3.2. *For every multisegment $\mathbf{r}_{\mathcal{J}}$ the functions $M_{\mathbf{r}_{\mathcal{J}}}$ and $\Phi_{\mathbf{r}_{\mathcal{J}}}$ are locally harmonic, where they are differentiable.*

For a more direct proof one observes that $M_{\mathbf{r}_{\mathcal{J}}}$ and $\Phi_{\mathbf{r}_{\mathcal{J}}}$ are locally the sum of the functions $\pm\phi_j$, hence one can assume without loss of generality that $\mathbf{r}_{\mathcal{J}}$ is a segment for which [7, (2) of Lemma A.1] implies the harmonicity directly.

4. FINDING THE NEEDLES IN A HAYSTACK

Theorem 4.1. *The traced segments of a regular multicurve of class C^∞ can be reconstructed if the masking function is given on any rounding circle.*

Proof. Let $\text{Tr } \mathbf{r}_{\mathcal{J}}$ be in \mathcal{B}^2 , and suppose that $M_{\mathbf{r}_{\mathcal{J}}}$ is given on \mathcal{S}^1 .

By (1) and (2) of Lemma 2.2 the set of the intersections of \mathcal{S}^1 with the traced lines is $\{\mathbf{u}_\xi : \partial_{\mathbf{u}_\xi^\perp} M_{\mathbf{r}_{\mathcal{J}}}(\mathbf{u}_\xi) + \partial_{-\mathbf{u}_\xi^\perp} M_{\mathbf{r}_{\mathcal{J}}}(\mathbf{u}_\xi) > 0\}$. This is a finite set, so we can enumerate its elements in anticlockwise order: $\mathbf{u}_{\xi_0}, \dots, \mathbf{u}_{\xi_i}, \dots, \mathbf{u}_{\xi_n}$.

Let $\mathbf{p}_{\mathcal{I}}$ be the multisegment of all the traced segments in $\mathbf{r}_{\mathcal{J}}$. Let $\mathbf{p}_{\mathcal{I}_i}$ be the multisegment of the segments in $\mathbf{p}_{\mathcal{I}}$ that are collinear with \mathbf{u}_{ξ_i} ($i = 0, \dots, n$). Let $\mathbf{u}_{\xi_i^0} = \mathbf{u}_{\xi_i}$, and let $\mathbf{u}_{\xi_i^1}, \dots, \mathbf{u}_{\xi_i^m}, \dots, \mathbf{u}_{\xi_i^{p_i}}$ be the remaining intersections of \mathcal{S}^1 with the traced lines of $\mathbf{p}_{\mathcal{I}_i}$ enumerated in anticlockwise order.

Let $\mathbf{p}_{\mathcal{I}_i^m}$ be the multisegment of the segments in $\mathbf{p}_{\mathcal{I}_i}$ lying on the straight line $\mathbf{u}_{\xi_i} \mathbf{u}_{\xi_i^m}$, and let \mathcal{A}_i^m be the counterclockwise arc $\widehat{\mathbf{u}_{\xi_i^m} \mathbf{u}_{\xi_i^{m+1}}}$ of \mathcal{S}^1 for every $m = 1, \dots, p_i$, where $p_i + 1$ is understood as 0. (See Figure 4.1.)

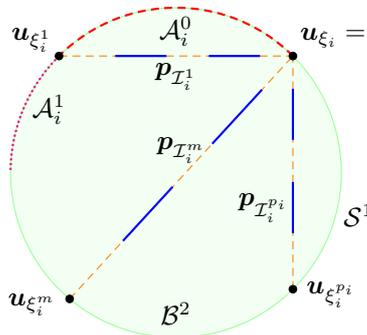


Figure 4.1. Traced segments, traced lines and the “first” arc they determine

With these notations in hand (1') of Lemma 2.2 gives for every $k \in \mathbb{N}$ that

$$\begin{aligned} \partial_{\mathbf{u}_\xi^\perp}^k M_{\mathbf{r}_\mathcal{J}}(\mathbf{u}_\xi) + (-1)^{k+1} \partial_{-\mathbf{u}_\xi^\perp}^k M_{\mathbf{r}_\mathcal{J}}(\mathbf{u}_\xi) \\ = \begin{cases} 2\partial_{\mathbf{u}_{\xi_i}^\perp}^k M_{\mathbf{p}_{\mathcal{I}_i}}(\mathbf{u}_{\xi_i}) & \text{if } \xi = \xi_i \text{ for an } i \in \{0, \dots, n\}, \\ 0 & \text{if } \xi \neq \xi_i \text{ for every } i \in \{0, \dots, n\}. \end{cases} \end{aligned} \tag{4.1} \tag{5}$$

According to (1) of Lemma 3.1 function $M_{\mathbf{p}_{\mathcal{I}_i}}$ is analytic on the arc \mathcal{A}_i^0 , so the values in (4.1) determine the function $M_{\mathbf{p}_{\mathcal{I}_i}}$ on the arc \mathcal{A}_i^0 .

Take the parameterization \mathbf{r}_e on every segment in $\mathbf{p}_{\mathcal{I}_i}$ so that $\phi_e > 0$ on \mathcal{A}_i^0 . Then $\Phi_{\mathbf{p}_{\mathcal{I}_i}} \equiv M_{\mathbf{p}_{\mathcal{I}_i}}$ on \mathcal{A}_i^0 so (2) of Lemma 3.1 gives $\Phi_{\mathbf{p}_{\mathcal{I}_i}}$ all over \mathcal{S}^1 . From this Theorem 2.3 implies that $\Phi_{\mathbf{p}_{\mathcal{I}_i}}$ is determined all over the exterior of \mathcal{B}^2 , because $\Phi_{\mathbf{p}_{\mathcal{I}_i}}$ vanishes at infinity and is harmonic in the exterior of \mathcal{B}^2 by Lemma 3.2.

In the other hand, $\Phi_{\mathbf{p}_{\mathcal{I}_i}}$ is real analytic on $\mathbb{R}^2 \setminus \text{Tr } \mathbf{p}_{\mathcal{I}_i}$ by (2) of Lemma 3.1 so it is determined on $\mathbb{R}^2 \setminus \text{Tr } \mathbf{p}_{\mathcal{I}_i}$ by the unique analytic extension from the outside of \mathcal{B}^2 . However, $\Phi_{\mathbf{p}_{\mathcal{I}_i}}$ can not extend continuously to $\text{Tr } \mathbf{p}_{\mathcal{I}_i}$ because it has different limits from different sides of the traced line of every segment in the multisegment $\mathbf{p}_{\mathcal{I}_i}$. Thus $\mathbf{p}_{\mathcal{I}_i}$ is determined as the set of points where $\Phi_{\mathbf{p}_{\mathcal{I}_i}}$ can not extend to.

Considering the difference $M_{\mathbf{r}_\mathcal{J}} - M_{\mathbf{p}_{\mathcal{I}_i}}$ puts us into the same situation as at the start of the proof, but with less traced straight lines, so repeating our procedure over and over again will lead to the determination of all traced segments. \square

As a special case we obtain from Theorem 4.1 the following nice result.

Corollary 4.2. *Every segment of the piecewise C^∞ boundary of a convex domain \mathcal{D} is determined by the visual angle function of \mathcal{D} given on a surrounding circle.*

Since a point of a surrounding circle can be collinear with at most two traced segments of the boundary of a convex domain, the following comes up.

Conjecture 4.3. *In the interior of \mathcal{B}^2 any closed convex polygon and any closed convex disc with piecewise C^5 boundary can be distinguished from each other by their visual angle functions restricted to \mathcal{S}^1 .*

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