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Spherical floating bodies

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Abstract. Several affirmative answers are given in any dimension for Ulam's question about bodies floating stable in every direction if the body floats like a ball and its floating body is spherical.

1. Introduction

S. M. Ulam asked in [17] if the sphere is the only homogeneous body of density $\delta \in [0, 1]$ that can float in water in every direction in equilibrium.[1] There are known counterexamples[2] some of which are convex. The only general affirmative answer the authors are aware of is given in [7] and [13] for $\delta = 0$. There are some more positive results imposing more conditions. One of such results [6, Theortem 4] says that if a centrally symmetric body of revolution with $\delta = \frac{1}{2}$ floats indifferently stable in every direction, then it is a sphere. Another one in [2, Theorem 5] states that in dimension 2 the only figure that floats in equilibrium in every position and has perimetral density[3] $\frac{1}{3}$ or $\frac{1}{4}$ is the circle (a more general result in this style can be found in [14]).

In this article, after preliminaries and some calculation on flotation (Sections 2 and 3), we approach Ulam's problem in any dimension with an integral geometric method which is presented in Section 5. The main point of this is Lemma 5.3 which allows one to deduce the incidence of convex bodies by simply comparing their volumes given by two measures.

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[1] Some restriction on the body is requested to avoid trivial counterexamples.

[2] In dimension 2 for $\delta = \frac{1}{2}$ given by [1, 14] and for $\delta \in (0, \frac{1}{2})$ by [18, 19]. In dimension 3 for $\delta \in (0, \frac{1}{2}]$ by [21]. In arbitrary dimensions for $\delta = \frac{1}{2}$ by [22].

[3] In the plane the waterline divides the border of the body in constant ratio, and the ratio of the smaller part to the whole perimeter is called the perimetral density.

Section 6 is devoted to our results on Ulam's problem in general dimensions considering only those bodies that have spherical floating body.[4]

In Theorem 6.1 we prove that if a convex homogeneous solid body $\mathcal{K}$ floats in water indifferently stable in every direction, and has the same density $\delta \in (0, \frac{1}{2})$ and volume as the ball $\bar{r}\mathcal{B}$ of radius $\bar{r}$, and submerges so that its centre of buoyancy and that of $\bar{r}\mathcal{B}$ are in the same depth under the water, then $\mathcal{K} \equiv \bar{r}\mathcal{B}$.[5]

The other two results of Section 6 consider Ulam's question if its condition, the stable flotation in every direction, is valid not only for the water, but for an other, more dense liquid too.

It is proved in Theorem 6.2, that if a convex homogeneous solid body $\mathcal{K}$ floats indifferently stable in every direction in water and in a more dense liquid too, and its floating bodies and those of the ball $\bar{r}\mathcal{B}$ of radius $\bar{r}$ coincide, respectively, and its centres of buoyancy and those of $\bar{r}\mathcal{B}$ are in the same depth under the liquid's level, respectively, then $\mathcal{K} \equiv \bar{r}\mathcal{B}$.

Finally it turns out in Theorem 6.3 that if the volume of a body $\mathcal{K}$ is the same as that of the ball $\bar{r}\mathcal{B}$ of radius $\bar{r}$ and the floating bodies of $\mathcal{K}$ and $\bar{r}\mathcal{B}$ are the same ball for two different liquids, respectively, then $\mathcal{K} \equiv \bar{r}\mathcal{B}$.

For further information on the subject we refer the reader to [3, 5, 6, 18].

2. Preliminaries

We work with the n-dimensional real space $\mathbb{R}^n$, its unit ball is $\mathcal{B} = \mathcal{B}^n$ (in the plane the unit disc is $\mathcal{D}$), its unit sphere is $\mathbb{S}^{n-1}$ and the set of its hyperplanes is $\mathbb{H}$. The ball (resp. disc) of radius $\bar{r} > 0$ centred at the origin is denoted by $\bar{r}\mathcal{B} = \bar{r}\mathcal{B}^n$ (resp. $\bar{r}\mathcal{D}$).

Using the spherical coordinates $\boldsymbol{\xi} = (\xi_1, \dots, \xi_{n-1})$ every unit vector can be written in the form $\boldsymbol{u}_{\boldsymbol{\xi}} = (\cos\xi_1, \sin\xi_1\cos\xi_2, \sin\xi_1\sin\xi_2\cos\xi_3, \dots)$, the i-th coordinate of which is $u^i_{\boldsymbol{\xi}} = (\prod_{j=1}^{i-1}\sin\xi_j)\cos\xi_i$ ($\xi_n := 0$). In the plane we even use the $\boldsymbol{u}_\xi = (\cos\xi, \sin\xi)$ notation.

A hyperplane $\hbar \in \mathbb{H}$ is parametrized so that $\hbar(\boldsymbol{u}_{\boldsymbol{\xi}}, r)$ means the one that is orthogonal to the unit vector $\boldsymbol{u}_{\boldsymbol{\xi}} \in \mathbb{S}^{n-1}$ and contains the point $r\boldsymbol{u}_{\boldsymbol{\xi}}$, where $r \in \mathbb{R}$[6]. For convenience we also frequently use $\hbar(P, \boldsymbol{u}_{\boldsymbol{\xi}})$ to denote the hyperplane through the point $P \in \mathbb{R}^n$ with normal vector $\boldsymbol{u}_{\boldsymbol{\xi}} \in \mathbb{S}^{n-1}$. For instance, $\hbar(P, \boldsymbol{u}_{\boldsymbol{\xi}}) = \hbar(\boldsymbol{u}_{\boldsymbol{\xi}}, \langle\overrightarrow{OP}, \boldsymbol{u}_{\boldsymbol{\xi}}\rangle)$, where $O = \mathbf{0}$ is the origin and $\langle.,.\rangle$ is the usual inner product.

[4] Although this seems a very restrictive condition we could not find better results in the literature for higher dimensions.

[5] More is proved in Theorem 4.1 for dimension 2.

[6] Athough $\hbar(\boldsymbol{u}_{\boldsymbol{\xi}}, r) = \hbar(-\boldsymbol{u}_{\boldsymbol{\xi}}, -r)$ this parametrization is locally bijective.

By a convex body we mean a convex compact set $\mathcal{K} \subseteq \mathbb{R}^n$ with non-empty interior $\mathcal{K}^\circ$ and with piecewise C^1 boundary $\partial\mathcal{K}$. For a convex body $\mathcal{K}$ we let $p_{\mathcal{K}}\colon \mathbb{S}^{n-1} \to \mathbb{R}$ denote the support function of $\mathcal{K}$, which is defined by $p_{\mathcal{K}}(\boldsymbol{u_\xi}) = \sup_{\boldsymbol{x}\in\mathcal{K}}\langle \boldsymbol{u_\xi}, \boldsymbol{x}\rangle$. Let $\mathcal{F}_{\mathcal{K}}(\boldsymbol{u_\xi})$ denote the set of those points of the boundary $\partial\mathcal{K}$, where the outer normal is $\boldsymbol{u_\xi}$. It is well known [15, Theorem 1.7.2], that

$$p_{\mathcal{F}_{\mathcal{K}}(\boldsymbol{u_\xi})}(\boldsymbol{u_\psi}) = \partial_{\boldsymbol{u_\psi}} p_{\mathcal{K}}(\boldsymbol{u_\xi}), \tag{2.1}$$

where $\partial_{\boldsymbol{u_\psi}}$ means directional derivative.

We also use the notation $\hbar_{\mathcal{K}}(\boldsymbol{u}) = \hbar(\boldsymbol{u}, p_{\mathcal{K}}(\boldsymbol{u}))$. If the origin is in $\mathcal{K}^\circ$, another useful function of a convex body $\mathcal{K}$ is its *radial function* $\varrho_{\mathcal{K}}\colon \mathbb{S}^{n-1} \to \mathbb{R}_+$ which is defined by $\varrho_{\mathcal{K}}(\boldsymbol{u}) = |\{r\boldsymbol{u} : r > 0\} \cap \partial K|$.

We introduce the notation $|\mathbb{S}^k| := 2\pi^{k/2}/\Gamma(k/2)$ as the standard surface measure of the k-dimensional sphere, where Γ is Euler's Gamma function.

A strictly positive integrable function $\omega\colon \mathbb{R}^n \setminus \mathcal{B} \to \mathbb{R}_+$ is called a *weight* and the integral

$$V_\omega(f) := \int_{\mathbb{R}^n\setminus\mathcal{B}} f(\boldsymbol{x})\omega(\boldsymbol{x})\, d\boldsymbol{x}$$

of an integrable function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called the *volume of f with respect to the weight ω* or simply the *ω-volume of f*. For the volume of the indicator function $\chi_{\mathcal{S}}$ of a set $\mathcal{S} \subseteq \mathbb{R}^n$ we use the notation $V_\omega(\mathcal{S}) := V_\omega(\chi_{\mathcal{S}})$ as a shorthand. If more weights are indexed by $i \in \mathbb{N}$, then we use the even shorter notation $V_i(\mathcal{S}) := V_{\omega_i}(\mathcal{S}) = V_i(\chi_{\mathcal{S}}) := V_{\omega_i}(\chi_{\mathcal{S}})$. Nevertheless the notations $V(\mathcal{S})$ and $V(f)$ with no subscript to V always denote the standard volume of $\mathcal{S}$ and f, respectively.

The *floating body* $\mathcal{K}_{[\vartheta]}$ ($\vartheta > 0$) of a convex body $\mathcal{K}$ as introduced by Dupin in [4] is a not necessarily convex body all of whose tangent hyperplanes cut off a part out of $\mathcal{K}$ of constant volume ϑ[7].

The *convex floating body* $\mathcal{K}_\vartheta$ ($\vartheta > 0$) of a convex body $\mathcal{K}$ is the intersection of all halfspaces whose defining hyperplanes cut off a part out of $\mathcal{K}$ of constant volume ϑ [16], which is more than half of the volume of $\mathcal{K}$.

We denote the density of a solid relative to the liquid it floats in by $\delta \in [0,1]$, and take the liberty to use the notations $\mathcal{K}_{[\delta]} := \mathcal{K}_{[V_\delta(\mathcal{K})]}$ and $\mathcal{K}_\delta := \mathcal{K}_{V_\delta(\mathcal{K})}$.

From now on we usually assume that $\delta \in (0, 1/2)$[8]. Let $\mathbb{H}_{\mathcal{K},\delta}$ denote the set of those hyperplanes $\hbar$ that divide $\mathcal{K}$ into two parts $\mathcal{K} \cap \hbar^+$ and $\mathcal{K} \cap \hbar^-$ so that $(1-\delta)V(\mathcal{K}\cap\hbar^+) = \delta V(\mathcal{K}\cap\hbar^-)$. Corresponding to the density δ then we have the convex floating body

$$\mathcal{K}_\delta = \bigcap_{\mathbb{H}_{\mathcal{K},\delta}} (\mathcal{K} \cap \hbar^-).$$

[7] For small ϑ this coincides with the convex floating body $\mathcal{K}_\vartheta$

[8] Allowing $\delta \in (0,1) \setminus \{1/2\}$ would not give more generality.

Finally we introduce a utility function χ that takes relations as argument and gives 1 if its argument fulfilled. For example $\chi(1>0)=1$, but $\chi(1\le 0)=0$ and $\chi(x>y)$ is 1 if $x>y$ and it is zero if $x\le y$. However we still use χ also as the indicator function of the set given in its subscript.

3. Physics of flotation in every position indifferently stable

Although most of the formulas in this section are known (see [22] for example) we decided to present these easy calculations for the sake of completeness and to establish terms and notations here.

Assume that the convex body $\mathcal{K}$ of density $\delta\in(0,\frac{1}{2})$ freely floats in the water in equilibrium in every position, and assume that a coordinate system $\mathbb{K}$ is attached to $\mathcal{K}$ so that its origin is in the centre of mass O of $\mathcal{K}$.

As $\mathcal{K}$ floats in the water one can represent the surface of the water as a hyperplane $\hbar(\boldsymbol{u},p(\boldsymbol{u}))$ in the coordinate system $\mathbb{K}$, where $\boldsymbol{u}\in\mathbb{S}^{n-1}$ and $p(\boldsymbol{u})>0$. Let $\hbar^-(\boldsymbol{u},p(\boldsymbol{u}))$ be the halfspace of $\hbar(\boldsymbol{u},p(\boldsymbol{u}))$ that contains O, and let $\hbar^+(\boldsymbol{u},p(\boldsymbol{u}))$ the other halfspace of $\hbar(\boldsymbol{u},p(\boldsymbol{u}))$, that in fact contains the water.

Since $\mathcal{K}$ floats, the absolute value of the weight of $\mathcal{K}$ is (by Archimedes' principle) equal to the absolute value of the buoyancy of the water displaced, hence we get[9]

$$V(\mathcal{K}\cap\hbar^+(\boldsymbol{u},p(\boldsymbol{u})))=V_\delta(\mathcal{K})=\delta V(\mathcal{K}). \tag{3.1}$$

Since $\mathcal{K}$ floats in equilibrium, the torque for the centre of mass O of $\mathcal{K}$ should vanish, hence the vector $\overrightarrow{OB_\delta(\boldsymbol{u})}$, where $B_\delta(\boldsymbol{u})$ is the centre of buoyancy[10], should be a positive real multiple of $\boldsymbol{u}$, say $\overrightarrow{OB_\delta(\boldsymbol{u})}=b_\delta(\boldsymbol{u})\boldsymbol{u}$. For the same reason the vector $\overrightarrow{OD_\delta(\boldsymbol{u})}$, where $D_\delta(\boldsymbol{u})$ is the centre of mass of the part of $\mathcal{K}$ above the water, should be a negative real multiple of $\boldsymbol{u}$, say $\overrightarrow{OD_\delta(\boldsymbol{u})}=d_\delta(\boldsymbol{u})\boldsymbol{u}$. The functions[11] $d_\delta(\boldsymbol{u})$ and $b_\delta(\boldsymbol{u})$ can be calculated as

$$b_\delta(\boldsymbol{u})=\frac{1}{V_\delta(\mathcal{K})}\int_{\mathcal{K}\cap\hbar^+(\boldsymbol{u},p(\boldsymbol{u}))}\langle\boldsymbol{x},\boldsymbol{u}\rangle\,d\boldsymbol{x}, \tag{3.2}$$

and $d_\delta(\boldsymbol{u})=\frac{1}{V_{1-\delta}(\mathcal{K})}\int_{\mathcal{K}\cap\hbar^-(\boldsymbol{u},p(\boldsymbol{u}))}\langle\boldsymbol{x},\boldsymbol{u}\rangle\,d\boldsymbol{x}$, hence

$$\delta b_\delta(\boldsymbol{u})+(1-\delta)d_\delta(\boldsymbol{u})=\frac{1}{V(\mathcal{K})}\Big\langle\int_{\mathcal{K}}\boldsymbol{x}\,d\boldsymbol{x},\boldsymbol{u}\Big\rangle=0. \tag{3.3}$$

[9] As the density of the water is 1.
[10] This is the same as the centre of mass of the submerged part of $\mathcal{K}$.
[11] Notice that $d_\delta(\boldsymbol{u})=b_{1-\delta}(\boldsymbol{u})$.

The potential energy $E_{\boldsymbol{u}}(\mathcal{K})$ of $\mathcal{K}$ with respect to the water level is the same in every position because $\mathcal{K}$ floats in every position in equilibrium. Let $E_\delta(\mathcal{K})$ be that constant potential energy of $\mathcal{K}$ with respect to the water level. Then

$$E_\delta(\mathcal{K}) = \delta V_{1-\delta}(\mathcal{K})(p(\boldsymbol{u}) - d_\delta(\boldsymbol{u})) + \delta V_\delta(\mathcal{K})(p(\boldsymbol{u}) - b_\delta(\boldsymbol{u})) - V_\delta(\mathcal{K})(p(\boldsymbol{u}) - b_\delta(\boldsymbol{u}))$$

that implies through (3.3) that

$$\begin{aligned}\frac{E_\delta(\mathcal{K})}{V_\delta(\mathcal{K})} &= (1-\delta)(p(\boldsymbol{u}) - d_\delta(\boldsymbol{u})) + \delta(p(\boldsymbol{u}) - b_\delta(\boldsymbol{u})) - (p(\boldsymbol{u}) - b_\delta(\boldsymbol{u}))\\ &= (p(\boldsymbol{u}) - (1-\delta)d_\delta(\boldsymbol{u}) - \delta b_\delta(\boldsymbol{u})) + (b_\delta(\boldsymbol{u}) - p(\boldsymbol{u})) = b_\delta(\boldsymbol{u}),\end{aligned} \tag{3.4}$$

i.e. $b_\delta(\boldsymbol{u})$, and correspondingly also $d_\delta(\boldsymbol{u})$, is constant, say b_δ and d_δ, respectively.

Finally we deduce from (3.1) that

$$\int_{\mathcal{K}\cap\hbar^+(\boldsymbol{u},p(\boldsymbol{u}))} 1\, d\boldsymbol{x} = V_\delta(\mathcal{K}), \tag{3.5}$$

and from (3.4) with the paragraph before (3.2) that

$$E_\delta(\mathcal{K})\boldsymbol{u} = V_\delta(\mathcal{K})b_\delta(\mathcal{K})\boldsymbol{u} = V_\delta(\mathcal{K})\overrightarrow{OB_\delta(\mathcal{K})} = \int_{\mathcal{K}\cap\hbar^+(\boldsymbol{u},p(\boldsymbol{u}))} \boldsymbol{x}\, d\boldsymbol{x}. \tag{3.6}$$

4. Some consequences

Observe, that —by equation (A.3)— differentiating (3.5) with respect to spherical coordinates leads to

$$\int_{\hbar(\boldsymbol{u},p(\boldsymbol{u}))} \chi_{\mathcal{K}}(\boldsymbol{x})(\langle \boldsymbol{x}, \boldsymbol{u}^\perp\rangle - p_{\mathcal{F}_{\mathcal{M}}(\boldsymbol{u})}(\boldsymbol{u}^\perp))\, d\boldsymbol{x}_{\hbar(\boldsymbol{u},p(\boldsymbol{u}))} = 0 \tag{4.1}$$

for any unit vectors $\boldsymbol{u}, \boldsymbol{u}^\perp \in \mathbb{S}^{n-1}$ that are orthogonal to each other.

The derivative of (3.6) —by equations (A.5) and (4.1)— leads to

$$E_\delta(\mathcal{K}) = \int_{\hbar(\boldsymbol{u},p(\boldsymbol{u}))} \chi_{\mathcal{K}}(\boldsymbol{x})(\langle \boldsymbol{x}, \boldsymbol{u}^\perp\rangle - p_{\mathcal{F}_{\mathcal{M}}(\boldsymbol{u})}(\boldsymbol{u}^\perp))^2\, d\boldsymbol{x}_{\hbar(\boldsymbol{u},p(\boldsymbol{u}))}. \tag{4.2}$$

An immediate implication of these formulas in the plane is the following result.

Theorem 4.1. *Assume that the convex body $\mathcal{K} \subset \mathbb{R}^2$ of volume 1 floats in equilibrium in every direction and its floating body is convex. If $\mathcal{K}_{[\delta]} = \mathcal{K}_\delta$ and $E_\delta(\mathcal{K})$ are known, then $\mathcal{K}$ can be uniquely determined.*

Proof. In the plane we can use $\boldsymbol{u}_\xi = (\cos\xi, \sin\xi)$. Fix the origin $\mathbf{0}$ in $\mathcal{K}^\circ_{[\delta]}$ and let $p(\xi) = p_{\mathcal{K}_{[\delta]}}(\boldsymbol{u}_\xi)$ denote the support function of $\mathcal{K}_{[\delta]}$. Let $\boldsymbol{a}(\xi)$ and $\boldsymbol{b}(\xi)$ be the two intersections of $\hbar(p(\xi), \boldsymbol{u}_\xi)$ and $\partial\mathcal{K}$ taken so that $\boldsymbol{a}(\xi) = p(\xi)\boldsymbol{u}_\xi + a(\xi)\boldsymbol{u}_\xi^\perp$, and $\boldsymbol{b}(\xi) = p(\xi)\boldsymbol{u}_\xi - b(\xi)\boldsymbol{u}_\xi^\perp$.

If $\hbar(p(\xi), \boldsymbol{u}_\xi)$ touches $\mathcal{K}_{[\delta]}$ in a unique point $\boldsymbol{h}(\xi)$, then by (2.1) we have $\boldsymbol{h}(\xi) - p(\xi)\boldsymbol{u}_\xi = p'(\xi)\boldsymbol{u}_\xi^\perp$, hence $a(\xi) - p'(\xi)$ and $b(\xi) + p'(\xi)$ are positive. For these values (4.1) and (4.2) give

$$\begin{aligned}(a(\xi) - p'(\xi))^2 - (b(\xi) + p'(\xi))^2 &= 0,\\ (a(\xi) - p'(\xi))^3 + (b(\xi) + p'(\xi))^3 &= 3E_\delta(\mathcal{K}).\end{aligned}$$

From the first one of these equations $a(\xi) - p'(\xi) = b(\xi) + p'(\xi)$ follows, hence the second one gives $(b(\xi) + p'(\xi))^3 = \frac{3}{2}E_\delta(\mathcal{K})$.

This clearly determines $b(\xi)$, hence $\mathcal{K}$ can be reconstructed. ■

If in this proof the floating body is a disc, then p' vanishes, hence $b^3 = a^3 = \frac{3}{2}E_\delta(\mathcal{K})$ is a constant. This proves that a convex body $\mathcal{K} \subset \mathbb{R}^2$ of volume 1 floating in equilibrium in every direction is a disc if and only if its floating body is a disc. However this is already proved in [10, Theorem 3.2] without the condition of indifferent flotation.

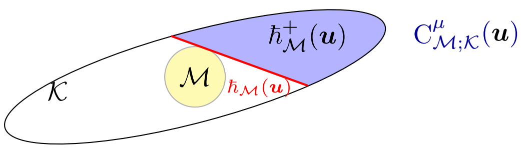
5. Measures of convex bodies

The forms of (3.5) and (3.6) suggest to consider the following setup.

Let $\mathcal{M}$ and $\mathcal{K}$ be convex bodies such that $\mathcal{M} \subseteq \mathcal{K}^\circ$. Let $\mu\colon \mathbb{H} \to C^1(\mathbb{R}^n)$ be *functions of weights*, that is, $\mu_\hbar$ is a weight for every $\hbar \in \mathbb{H}$.

Assuming that $\mathcal{M}$ contains the origin, we define the *weighted cap function* of $\mathcal{K}$ with respect to $\mathcal{M}$, the so-called *kernel*[12], as

$$\mathrm{C}^\mu_{\mathcal{M};\mathcal{K}}(\boldsymbol{u}) = \int_{\langle \boldsymbol{x},\boldsymbol{u}\rangle \ge p_{\mathcal{M}}(\boldsymbol{u})} \chi_{\mathcal{K}}(\boldsymbol{x})\mu_{\hbar_{\mathcal{M}}(\boldsymbol{u})}(\boldsymbol{x})\, d\boldsymbol{x}. \tag{5.1}$$



[12] Notice that the kernel body $\mathcal{M}$ may happen to be a convex floating body of $\mathcal{K}$.

The function $\mu\colon \mathbb{H} \to C^1(\mathbb{R}^n)$ of weights is called *rotationally symmetric* if $\mu_{U\hbar}(U\boldsymbol{x}) = \mu_{\hbar}(\boldsymbol{x})$ for every $\hbar \in \mathbb{H}$, $U \in SO(n)$ ([13]) and $\boldsymbol{x} \in \mathbb{R}^n$. Assume that $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ are not vanishing and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{S}^{n-1}$. If $|\boldsymbol{x}| = |\boldsymbol{y}|$ and $\langle \boldsymbol{x}, \boldsymbol{u}\rangle = \langle \boldsymbol{y}, \boldsymbol{v}\rangle$, then there is a $D \in SO(n)$, that $D\boldsymbol{x} = \boldsymbol{y}$ and $D\boldsymbol{u} = \boldsymbol{v}$, hence we have the following immediate consequence.

Lemma 5.1. *The function μ of weights is rotationally symmetric if and only if there is a function $\bar{\mu}\colon \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that $\mu_{\hbar(\boldsymbol{u},r)}(\boldsymbol{x}) = \bar{\mu}\big(r, \langle \boldsymbol{x}, \boldsymbol{u}\rangle, |\boldsymbol{x}|\big)$.*

If the kernel body is a ball, i.e. $\varrho\mathcal{B}$, we use the notation $\mathrm{C}^{\mu}_{\varrho;\mathcal{K}} := \mathrm{C}^{\mu}_{\varrho\mathcal{B};\mathcal{K}}$ as a shorthand.

Lemma 5.2. *Let the convex body $\mathcal{K}$ contain in its interior the ball $\varrho\mathcal{B}$. Then for any rotationally symmetric function μ of weights we have*

$$\int_{\mathbb{S}^{n-1}} \mathrm{C}^{\mu}_{\varrho;\mathcal{K}}(\boldsymbol{u}_{\boldsymbol{\xi}})d\boldsymbol{\xi} = |\mathbb{S}^{n-2}| \int_{\mathcal{K}\setminus\varrho\mathcal{B}} \int_{\varrho/|\boldsymbol{x}|}^{1} \bar{\mu}(\varrho, \lambda|\boldsymbol{x}|, |\boldsymbol{x}|)(1-\lambda^2)^{\frac{n-3}{2}}\, d\lambda\, d\boldsymbol{x}. \tag{5.2}$$

Proof. We have

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} \mathrm{C}^{\mu}_{\varrho;\mathcal{K}}(\boldsymbol{u}_{\boldsymbol{\zeta}})d\boldsymbol{\zeta} &= \int_{\mathbb{S}^{n-1}} \int_{\langle \boldsymbol{x}, \boldsymbol{u}_{\boldsymbol{\zeta}}\rangle \geq \varrho} \mu_{\hbar(\boldsymbol{u}_{\boldsymbol{\zeta}},\varrho)}(\boldsymbol{x})\chi_{\mathcal{K}}(\boldsymbol{x})\, d\boldsymbol{x}\, d\boldsymbol{\zeta}\\ &= \int_{\mathcal{K}\setminus\varrho\mathcal{B}} \int_{\langle \boldsymbol{x}, \boldsymbol{u}_{\boldsymbol{\zeta}}\rangle \geq \varrho} \mu_{\hbar(\boldsymbol{u}_{\boldsymbol{\zeta}},\varrho)}(\boldsymbol{x})\, d\boldsymbol{\zeta}\, d\boldsymbol{x}.\end{aligned}$$

Using that μ is rotationally symmetric, and letting $|\boldsymbol{x}|\boldsymbol{u}_{\boldsymbol{\xi}} = \boldsymbol{x}$, where $\boldsymbol{u}_{\boldsymbol{\xi}} \in \mathbb{S}^{n-1}$, we can continue as

$$\begin{aligned}&\int_{\mathbb{S}^{n-1}} \mathrm{C}^{\mu}_{\varrho;\mathcal{K}}(\boldsymbol{u}_{\boldsymbol{\zeta}})\, d\boldsymbol{\zeta}\\ &\quad= \int_{\mathcal{K}\setminus\varrho\mathcal{B}} \int_{\langle \boldsymbol{x}, \boldsymbol{u}_{\boldsymbol{\zeta}}\rangle \geq \varrho} \bar{\mu}(\varrho, \langle \boldsymbol{x}, \boldsymbol{u}_{\boldsymbol{\zeta}}\rangle, |\boldsymbol{x}|)\, d\boldsymbol{\zeta}\, d\boldsymbol{x}\\ &\quad= \int_{\mathcal{K}\setminus\varrho\mathcal{B}} \int_{\langle \boldsymbol{u}_{\boldsymbol{\xi}}, \boldsymbol{u}_{\boldsymbol{\zeta}}\rangle \geq \varrho/|\boldsymbol{x}|} \bar{\mu}(\varrho, |\boldsymbol{x}|\langle \boldsymbol{u}_{\boldsymbol{\xi}}, \boldsymbol{u}_{\boldsymbol{\zeta}}\rangle, |\boldsymbol{x}|)\, d\boldsymbol{\zeta}\, d\boldsymbol{x}\\ &\quad= \int_{\mathcal{K}\setminus\varrho\mathcal{B}} \int_{\varrho/|\boldsymbol{x}|}^{1} \int_{\mathbb{S}^{n-2}} \bar{\mu}(\varrho, |\boldsymbol{x}|\lambda, |\boldsymbol{x}|)(1-\lambda^2)^{\frac{n-2}{2}}\, d\boldsymbol{\psi}\, (1-\lambda^2)^{\frac{-1}{2}}\, d\lambda\, d\boldsymbol{x}.\end{aligned}$$

This proves the lemma. ∎

[13] $SO(n)$ is the group of rotations around the origin $\mathbf{0}$.

The following slight generalization of [10, Lemma 4.3] will be needed.

Lemma 5.3. *Let ω_i $(i = 1, 2)$ be weights, let $\mathcal{K}$ and $\mathcal{L}$ be convex bodies containing the unit ball $\mathcal{B}$, and let $c \geq 1$. If there is a constant $c_{\mathcal{L}}$ such that $\omega_2 = c_{\mathcal{L}}\omega_1$ may occur only in a set of measure zero then*

$$cV_1(\mathcal{L}) \leq V_1(\mathcal{K}) \text{ and } \begin{Bmatrix} \omega_2(X) \leq c_{\mathcal{L}}\omega_1(X) & \text{for } X \in \mathcal{L}, \\ \omega_2(X) \geq c_{\mathcal{L}}\omega_1(X) & \text{for } X \notin \mathcal{L}, \end{Bmatrix} \text{ imply } cV_2(\mathcal{L}) \leq V_2(\mathcal{K}), \tag{5.3}$$

$$V_1(\mathcal{K}) \leq cV_1(\mathcal{L}) \text{ and } \begin{Bmatrix} \omega_2(X) \geq c_{\mathcal{L}}\omega_1(X) & \text{for } X \in \mathcal{L}, \\ \omega_2(X) \leq c_{\mathcal{L}}\omega_1(X) & \text{for } X \notin \mathcal{L}, \end{Bmatrix} \text{ imply } V_2(\mathcal{K}) \leq cV_2(\mathcal{L}), \tag{5.4}$$

and in both cases equality happens if and only if $\mathcal{K} = \mathcal{L}$ and $c = 1$.

Proof. In both statements $\mathcal{K}\triangle\mathcal{L} = \emptyset$ implies $V_1(\mathcal{K}) = V_1(\mathcal{L})$, hence $c = 1$ and $V_1(\mathcal{K}) = V_1(\mathcal{L})$.

Assume from now on that $\mathcal{K}\triangle\mathcal{L} \neq \emptyset$.

Having (5.3) we proceed as

$$\begin{aligned}
&V_2(\mathcal{K}) - cV_2(\mathcal{L}) \\
&= V_2(\mathcal{K}) - V_2(\mathcal{L}) + (1-c)V_2(\mathcal{L}) = V_2(\mathcal{K}\setminus\mathcal{L}) - V_2(\mathcal{L}\setminus\mathcal{K}) + (1-c)V_2(\mathcal{L}) \\
&= \int_{\mathcal{K}\setminus\mathcal{L}} \frac{\omega_2(x)}{\omega_1(x)}\omega_1(x)dx - \int_{\mathcal{L}\setminus\mathcal{K}} \frac{\omega_2(x)}{\omega_1(x)}\omega_1(x)dx + (1-c)V_2(\mathcal{L}) \\
&> c_{\mathcal{L}}(V_1(\mathcal{K}\setminus\mathcal{L}) - V_1(\mathcal{L}\setminus\mathcal{K})) + (1-c)V_2(\mathcal{L}) = c_{\mathcal{L}}(V_1(\mathcal{K}) - V_1(\mathcal{L})) + (1-c)V_2(\mathcal{L}) \\
&\geq (c-1)(c_{\mathcal{L}}V_1(\mathcal{L}) - V_2(\mathcal{L})) = (c-1)\Big(\int_{\mathcal{L}} \Big(c_{\mathcal{L}} - \frac{\omega_2(x)}{\omega_1(x)}\Big)\omega_1(x)dx\Big) \geq 0
\end{aligned}$$

that implies $V_2(\mathcal{K}) - cV_2(\mathcal{L}) > 0$.

As (5.4) can be easily proved in the same way, it is left to the reader. ∎

6. Spherical convex floating bodies in any dimension

In this section we consider convex bodies that float in equilibrium in every direction and have spherical floating body. If $\mathcal{K} \subset \mathbb{R}^n$ is such a convex body, then $\mathcal{K}_{[\delta]}$ is the ball $\varrho\mathcal{B}$, that means $p_{\mathcal{K}_{[\delta]}} \equiv \varrho$, hence $\mathcal{K}_{[\delta]} = \mathcal{K}_\delta = \bigcap_{\boldsymbol{u}\in\mathbb{S}^{n-1}} \hbar^-(\boldsymbol{u}, \varrho)$. Note however that in [20] there are shown numerous $\mathcal{K}$ such that $\mathcal{K}$ and $\mathcal{K}_{[\delta]}$ are strictly convex have smooth boundaries and are non-circular.

Equations (3.5) and (3.6) can be reformulated using the terms and notations of Section 5. Define the functions $\lambda_{\hbar(\boldsymbol{u},r)}(\boldsymbol{x}) := 1$ and $\mu_{\hbar(\boldsymbol{u},r)}(\boldsymbol{x}) := \langle \boldsymbol{x}, \boldsymbol{u}\rangle$ of weights and consider the weighted cap functions

$$\mathrm{C}^{\lambda}_{\varrho;\mathcal{K}}(\boldsymbol{u}) = \int_{\mathcal{K}\cap\hbar^+(\boldsymbol{u},\varrho)} 1\,d\boldsymbol{x} \qquad \text{(constant } V_\delta(\mathcal{K}) \text{ by (3.5))}, \tag{6.1}$$

$$\mathrm{C}^{\mu}_{\varrho;\mathcal{K}}(\boldsymbol{u}) = \int_{\mathcal{K}\cap\hbar^+(\boldsymbol{u},\varrho)} \langle \boldsymbol{x}, \boldsymbol{u}\rangle\,d\boldsymbol{x}, \qquad \text{(constant } E_\delta(\mathcal{K}) \text{ by (3.6))}. \tag{6.2}$$

Observe that according to Lemma 5.1, the functions λ and μ of weights are rotationally symmetric.

For the next two results it is worth noting that for convex bodies $\mathcal{K}$ and $\mathcal{L}$ floating indifferently stable in every position, the equation $E_\delta(\mathcal{K}) = E_\delta(\mathcal{L})$ means that their centre of buoyancies are in the same distance from their centres of mass, respectively.

Theorem 6.1. *Let the convex body $\mathcal{K}$ and the ball $\bar{r}\mathcal{B}$ have unit volume. If $\mathcal{K}$ floats indifferently stable in every position, $\mathcal{K}_{[\delta]} = (\bar{r}\mathcal{B})_\delta$ and $E_\delta(\mathcal{K}) = E_\delta(\bar{r}\mathcal{B})$, then $\mathcal{K} \equiv \bar{r}\mathcal{B}$.*

Proof. Let $\varrho \in \mathbb{R}$ be the radius of $(\bar{r}\mathcal{B})_\delta$. By the conditions there is an $\bar{r} \in \mathbb{R}$ such that

$$\mathrm{C}^{\lambda}_{\varrho;\bar{r}\mathcal{B}} = \mathrm{C}^{\lambda}_{\varrho;\mathcal{K}} \quad \text{and} \quad \mathrm{C}^{\mu}_{\varrho;\bar{r}\mathcal{B}} = \mathrm{C}^{\mu}_{\varrho;\mathcal{K}}\,.$$

Then Lemma 5.2 implies

$$\frac{1}{|\mathbb{S}^{n-2}|}\int_{\mathbb{S}^{n-1}} \mathrm{C}^{\lambda}_{\varrho;\mathcal{K}}(\boldsymbol{u}_{\boldsymbol{\xi}})d\boldsymbol{\xi} = \int_{\mathcal{K}\setminus\varrho\mathcal{B}}\int_{\varrho/|\boldsymbol{x}|}^{1} (1-y^2)^{\frac{n-3}{2}}\,dy\,d\boldsymbol{x}, \tag{6.3}$$

$$\frac{1}{|\mathbb{S}^{n-2}|}\int_{\mathbb{S}^{n-1}} \mathrm{C}^{\mu}_{\varrho;\mathcal{K}}(\boldsymbol{u}_{\boldsymbol{\xi}})d\boldsymbol{\xi} = \int_{\mathcal{K}\setminus\varrho\mathcal{B}}\int_{\varrho/|\boldsymbol{x}|}^{1} y|\boldsymbol{x}|(1-y^2)^{\frac{n-3}{2}}\,dy\,d\boldsymbol{x}. \tag{6.4}$$

Define the weights

$$\bar{\omega}_1(r) = \int_{\varrho/r}^{1} (1-y^2)^{\frac{n-3}{2}}\,dy \quad \text{and} \quad \bar{\omega}_2(r) = r\int_{\varrho/r}^{1} y(1-y^2)^{\frac{n-3}{2}}\,dy \tag{6.5}$$

on $\mathbb{R}_{>\varrho}$. By the conditions we have

$$\int_{\bar{r}\mathcal{B}\setminus\varrho\mathcal{B}} \omega_1(\boldsymbol{x})\,d\boldsymbol{x} = \frac{1}{|\mathbb{S}^{n-2}|}\int_{\mathbb{S}^{n-1}} \mathrm{C}^{\lambda}_{\varrho;\mathcal{K}}(\boldsymbol{u}_{\boldsymbol{\xi}})d\boldsymbol{\xi} = \int_{\mathcal{K}\setminus\varrho\mathcal{B}} \omega_1(\boldsymbol{x})\,d\boldsymbol{x}$$

and

$$\int_{\bar{r}\mathcal{B}\setminus\varrho\mathcal{B}} \omega_2(\boldsymbol{x})\,d\boldsymbol{x} = \frac{1}{|\mathbb{S}^{n-2}|}\int_{\mathbb{S}^{n-1}} \mathrm{C}^{\mu}_{\varrho;\mathcal{K}}(\boldsymbol{u}_{\boldsymbol{\xi}})d\boldsymbol{\xi} = \int_{\mathcal{K}\setminus\varrho\mathcal{B}} \omega_2(\boldsymbol{x})\,d\boldsymbol{x},$$

for the weights $\omega_1(\boldsymbol{x}) := \bar{\omega}_1(|\boldsymbol{x}|)$ and $\omega_2(\boldsymbol{x}) := \bar{\omega}_2(|\boldsymbol{x}|)$. In terms of Lemma 5.3 these mean that $V_1(\mathcal{K}) = V_1(\bar{r}\mathcal{B})$ and $V_2(\mathcal{K}) = V_2(\bar{r}\mathcal{B})$.

Making the substitution $y = \sqrt{1-z}$ in the integrals (6.5) we get

$$\bar{\omega}_1(r) = \frac{1}{2}\int_0^{1-\frac{\varrho^2}{r^2}} z^{\frac{n-3}{2}}(1-z)^{-1/2}\,dz = \frac{(r^2-\varrho^2)^{\frac{n-1}{2}}}{2r^{n-2}}\int_0^1 \frac{x^{\frac{n-3}{2}}}{((1-x)r^2+x\varrho^2)^{1/2}}\,dx$$

and

$$\bar{\omega}_2(r) = \frac{r}{2}\int_0^{1-\varrho^2/r^2} z^{\frac{n-3}{2}}\,dz = \frac{(r^2-\varrho^2)^{\frac{n-1}{2}}}{(n-1)r^{n-2}}. \tag{6.6}$$

These imply that

$$\frac{\bar{\omega}_1(r)}{\bar{\omega}_2(r)} = \frac{n-1}{2}\int_0^1 \frac{x^{\frac{n-3}{2}}}{((1-x)r^2+x\varrho^2)^{1/2}}\,dz$$

is strictly decreasing.

Let $\mathcal{L} = \bar{r}\mathcal{B}$, $c = 1$, and let $c_{\mathcal{L}}$ be the constant value of ω_2/ω_1 on $\partial(\bar{r}\mathcal{B}) \equiv \bar{r}\mathbb{S}^{n-1}$. As ω_2/ω_1 is strictly increasing, (5.3) of Lemma 5.3 implies from $V_1(\mathcal{K}) \le V_1(\bar{r}\mathcal{B})$, that $V_2(\mathcal{K}) \le V_2(\bar{r}\mathcal{B})$, where equality is allowed if and only if $\mathcal{K} = \bar{r}\mathcal{B}$. But $V_2(\mathcal{K}) = V_2(\bar{r}\mathcal{B})$, therefore $\mathcal{K} = \bar{r}\mathcal{B}$ follows and the theorem is proved. ∎

Theorem 6.2. *Let the convex body $\mathcal{K}$ float in equilibrium in every position for both of the densities $0 < \delta_1 < \delta_2 < \frac{1}{2}$. If there is a ball $\bar{r}\mathcal{B}$ satisfying $\mathcal{K}_{[\delta_i]} \equiv (\bar{r}\mathcal{B})_{[\delta_i]}$ and $E_{\delta_i}(\mathcal{K}) = E_{\delta_i}(\bar{r}\mathcal{B})$, where $i = 1, 2$, then $\mathcal{K}$ is the ball $\bar{r}\mathcal{B}$.*

Proof. Let ϱ_i be the radius of the ball $(\bar{r}\mathcal{B})_{\delta_i}$ $(i = 1, 2)$, and observe that $\varrho_2 < \varrho_1$. Reformulating the conditions using (6.2), we obtain

$$\mathrm{C}^{\mu}_{\varrho_i;\mathcal{K}}(\boldsymbol{u}) = E_{\delta_i}(\mathcal{K}) = E_{\delta_i}(\bar{r}\mathcal{B}) = \mathrm{C}^{\mu}_{\varrho_i;\bar{r}\mathcal{B}}(\boldsymbol{u}) \qquad (i = 1, 2).$$

Using (6.4) we deduce

$$\int_{\mathcal{K}\setminus\varrho_i\mathcal{B}}\int_{\varrho_i/|\boldsymbol{x}|}^1 y|\boldsymbol{x}|(1-y^2)^{\frac{n-3}{2}}\,dy\,d\boldsymbol{x} = \int_{\bar{r}\mathcal{B}\setminus\varrho_i\mathcal{B}}\int_{\varrho_i/|\boldsymbol{x}|}^1 y|\boldsymbol{x}|(1-y^2)^{\frac{n-3}{2}}\,dy\,d\boldsymbol{x}. \tag{6.7}$$

Define the weights

$$\bar{\omega}_i(r) = r\int_{\varrho_i/r}^1 y(1-y^2)^{\frac{n-3}{2}}\,dy \overset{(6.6)}{=} \frac{(r^2-\varrho_i^2)^{\frac{n-1}{2}}}{(n-1)r^{n-2}} \qquad (i = 1, 2)$$

on $\mathbb{R}_{>\varrho_1}$. Then (6.7) implies

$$\int_{\mathcal{K}\setminus\varrho\mathcal{B}} \omega_i(\boldsymbol{x})\, d\boldsymbol{x} = \int_{\bar{r}\mathcal{B}\setminus\varrho\mathcal{B}} \omega_i(\boldsymbol{x})\, d\boldsymbol{x}, \qquad (i=1,2),$$

for the weights $\omega_i(\boldsymbol{x}) := \bar{\omega}_i(|\boldsymbol{x}|)$ $(i=1,2)$. In terms of Lemma 5.3 this reads $V_i(\mathcal{K}) = V_i(\bar{r}\mathcal{B})$ $(i=1,2)$.

Let $\mathcal{L} = \bar{r}\mathcal{B}$ and $c=1$. As

$$\frac{\bar{\omega}_2(r)}{\bar{\omega}_1(r)} = \Big(\frac{r^2-\varrho_2^2}{r^2-\varrho_1^2}\Big)^{\frac{n-1}{2}}$$

is strictly decreasing and is a constant $c_{\mathcal{L}}$ on $\partial(\bar{r}\mathcal{B}) \equiv \bar{r}\mathbb{S}^{n-1}$, statement (5.4) of Lemma 5.3 gives from $V_1(\mathcal{K}) \le V_1(\bar{r}\mathcal{B})$ that $V_2(\mathcal{K}) \le V_2(\bar{r}\mathcal{B})$, where equality is allowed if and only if $\mathcal{K} = \bar{r}\mathcal{B}$. Since $V_2(\mathcal{K}) = V_2(\bar{r}\mathcal{B})$, the theorem is proved. ■

Theorem 6.3. *Let the convex body $\mathcal{K}$ and the ball $\bar{r}\mathcal{B}$ have unit volume. If there are $0<\delta_1<\delta_2<\frac{1}{2}$ such that $\mathcal{K}_{[\delta_1]} \equiv (\bar{r}\mathcal{B})_{\delta_1}$ and $\mathcal{K}_{[\delta_2]} \equiv (\bar{r}\mathcal{B})_{\delta_2}$, then $\mathcal{K}$ is the ball $\bar{r}\mathcal{B}$.*

Proof. Let ϱ_i be the radius of the ball $(\bar{r}\mathcal{B})_{\delta_i}$ $(i=1,2)$. Reformulating the conditions using (6.1), we obtain

$$\mathrm{C}^{\lambda}_{\varrho_i;\mathcal{K}}(\boldsymbol{u}) = V_{\delta_i}(\mathcal{K}) = V_{\delta_i}(\bar{r}\mathcal{B}) = \mathrm{C}^{\lambda}_{\varrho_i;\bar{r}\mathcal{B}}(\boldsymbol{u}) \qquad (i=1,2).$$

This implies the statement of our theorem by [10, Theorem 5.1]. ■

As we already noted, it is enough to request $\mathcal{K}_{[\delta]} \equiv (\bar{r}\mathcal{B})_{\delta}$ for only one δ to achieve the same result in the plane [10, Theorem 3.2].

7. Discussion

Observing Theorem 6.3 one may ask what can be said about two convex bodies having a common floating body, or how many convex floating bodies of a convex body should one know to be able to reconstruct the body?

Other problem that clearly raises in Section 6 is to find a good description of those pairs (ϱ_1, ϱ_2) of positive numbers for which there is a radius $\bar{r}$ and there are densities $\delta_1, \delta_2 \in (0, \frac{1}{2})$ so that $(\bar{r}\mathcal{B})_{\delta_i} = \varrho_i\mathcal{B}$ $(i=1,2)$.

Observing (6.1) and (6.2) it is natural to introduce the *floating momentums* of a convex body $\mathcal{K}$ as

$$M_{\mathcal{K},\delta,n}(\boldsymbol{u}) := \int_{\mathcal{K}\cap\hbar^+(\boldsymbol{u},p(\boldsymbol{u}))} \langle \boldsymbol{x}, \boldsymbol{u}\rangle^n\, d\boldsymbol{x} \qquad (n\in\mathbb{N}). \tag{7.1}$$

Observe that the first two momentums of a convex body $\mathcal{K}$ which can float in the water indifferently stable in every direction are the constants $M_{\mathcal{K},\delta,0} \equiv V_\delta(\mathcal{K})$ and $M_{\mathcal{K},\delta,1} \equiv E_\delta(\mathcal{K})$, hence it is natural to consider those convex bodies $\mathcal{K}$ that have constant $M_{\delta,n}(\mathcal{K}) := M_{\mathcal{K},\delta,n}(\boldsymbol{u})$ momentum for every $n \in \mathbb{N}$. We say that these convex bodies float in the water *hyper stable* in every direction, or *indifferently hyper stable*.

Question 7.1. Is the ball the only homogeneous body of density $\delta \in (0,1)$ that can float indifferently hyper stable in water? ▲

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A. Appendix: Differentiating some geometric integrals

Let $\mathcal{M}$ and $\mathcal{K}$ be convex bodies such that $\mathcal{M} \subseteq \mathcal{K}^\circ$, $\mathbf{0} \in \mathcal{M}^\circ$, and let p be the support function of $\mathcal{M}$.

First, we consider the function

$$f(\boldsymbol{u}) := \int_{\mathcal{K}\cap\hbar^+(\boldsymbol{u},p(\boldsymbol{u}))} 1\,d\boldsymbol{x} \tag{A.1}$$

on the unit sphere. Directional derivative of this function was already calculated in [12] and later in [9] but with different notations and purpose, so we decided to present the following short calculation here.

Fix an arbitrary unit vector $\boldsymbol{u}$ and choose arbitrarily an other unit vector $\boldsymbol{u}^\perp$ orthogonal to $\boldsymbol{u}$. Define $\boldsymbol{u}(\alpha) = \cos\alpha\boldsymbol{u} + \sin\alpha\boldsymbol{u}^\perp$. Differentiating $f(\boldsymbol{u}(\alpha))$ by α at $\alpha = 0$ from the right results in

$$\begin{aligned}
&\frac{df(\boldsymbol{u}(\alpha))}{d\alpha}(0+)\\
&\qquad= \lim_{0<\alpha\to 0}\frac{V(\mathcal{K}\cap\hbar^+(\boldsymbol{u}(\alpha),p(\boldsymbol{u}(\alpha))))) - V(\mathcal{K}\cap\hbar^+(\boldsymbol{u}(0),p(\boldsymbol{u}(0)))))}{\alpha}\\
&\qquad= \lim_{0<\alpha\to 0}\frac{V(\mathcal{K}\cap(\hbar^+(\boldsymbol{u}(\alpha),p(\boldsymbol{u}(\alpha)))\setminus\hbar^+(\boldsymbol{u}(0),p(\boldsymbol{u}(0)))))}{\alpha} -\\
&\qquad\qquad- \lim_{0<\alpha\to 0}\frac{V(\mathcal{K}\cap(\hbar^+(\boldsymbol{u}(0),p(\boldsymbol{u}(0)))\setminus\hbar^+(\boldsymbol{u}(\alpha),p(\boldsymbol{u}(\alpha)))))}{\alpha}\\
&\qquad=: \lim{}_1 - \lim{}_2.
\end{aligned} \tag{A.2}$$

Let us introduce the notations $\hbar^+_\alpha = \hbar^+(\boldsymbol{u}(\alpha), p(\boldsymbol{u}(\alpha)))$ and define $\mathbb{R}^{n-2}_\alpha = \hbar(\boldsymbol{u}(\alpha), p(\boldsymbol{u}(\alpha)))) \cap \hbar(\boldsymbol{u}(0), p(\boldsymbol{u}(0)))$, an $(n-2)$-dimensional affine subspace, for $\alpha > 0$. We need also $\mathbb{R}^{n-2}_{\boldsymbol{u},\boldsymbol{u}^\perp} := \lim_{0<\alpha\to 0} \mathbb{R}^{n-2}_\alpha$.

The limits in equation (A.2) then can be calculated using the substitution $\boldsymbol{x} = \boldsymbol{y} + r\boldsymbol{u}(\xi)$, where $\boldsymbol{y} \in \mathbb{R}^{n-2}_\alpha$. The first integral becomes

$$\begin{aligned}\lim{}_1 &= \lim_{0<\alpha\to 0} \frac{\int_{\mathbb{R}^{n-2}_\alpha} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\alpha} \int_0^\infty \chi_{\mathcal{K}\cap(\hbar^+_\alpha\setminus\hbar^+_0)}(\boldsymbol{y} + r\boldsymbol{u}(\xi)) r\,dr\,d\xi\,d\boldsymbol{y}}{\alpha}\\ &= \int_{\mathbb{R}^{n-2}_{\boldsymbol{u},\boldsymbol{u}^\perp}} \int_0^\infty \lim_{0<\alpha\to 0} \frac{\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\alpha} \chi_{\mathcal{K}\cap(\hbar^+_\alpha\setminus\hbar^+_0)}(\boldsymbol{y} + r\boldsymbol{u}(\xi))\,d\xi}{\alpha} r\,dr\,d\boldsymbol{y}\\ &= \int_{\mathbb{R}^{n-2}_{\boldsymbol{u},\boldsymbol{u}^\perp}} \int_0^\infty \chi_{\mathcal{K}\cap\hbar(\boldsymbol{u},p(\boldsymbol{u}))}(\boldsymbol{y} + r\boldsymbol{u}^\perp)\, r\,dr\,d\boldsymbol{y}\end{aligned}$$

and the second one gets the form

$$\lim{}_2 = \int_{\mathbb{R}^{n-2}_{\boldsymbol{u},\boldsymbol{u}^\perp}} \int_0^\infty \chi_{\mathcal{K}\cap\hbar(\boldsymbol{u},p(\boldsymbol{u}))}(\boldsymbol{y} - r\boldsymbol{u}^\perp) r\,dr\,d\boldsymbol{y}.$$

All these together and (2.1) imply

$$\begin{aligned}\partial_{\boldsymbol{u}^\perp} f(\boldsymbol{u}) &= \frac{df(\boldsymbol{u}(\alpha))}{d\alpha}(0+) = \int_{\mathbb{R}^{n-2}_{\boldsymbol{u},\boldsymbol{u}^\perp}} \int_{-\infty}^\infty \chi_{\mathcal{K}\cap\hbar(\boldsymbol{u},p(\boldsymbol{u}))}(\boldsymbol{y} + t\boldsymbol{u}^\perp) t\,dt\,d\boldsymbol{y}\\ &= \int_{\hbar(\boldsymbol{u},p(\boldsymbol{u}))} \chi_{\mathcal{K}}(\boldsymbol{x})(\langle \boldsymbol{x}, \boldsymbol{u}^\perp\rangle - p_{\mathcal{F}_{\mathcal{M}}(\boldsymbol{u})}(\boldsymbol{u}^\perp))\,d\boldsymbol{x}_{\hbar(\boldsymbol{u},p(\boldsymbol{u}))}\end{aligned} \tag{A.3}$$

for any pair of unit orthogonal vectors $\boldsymbol{u}, \boldsymbol{u}^\perp \in \mathbb{S}^{n-1}$.

Next we consider the derivative of the function

$$g(\boldsymbol{u}) := \int_{\mathcal{K}\cap\hbar^+(\boldsymbol{u},p(\boldsymbol{u}))} \boldsymbol{x}\,d\boldsymbol{x} \tag{A.4}$$

defined on the unit sphere. By taking the derivative of $g(\boldsymbol{u}(\alpha))$ with respect to α at 0 from the right we get

$$\begin{aligned}\frac{d(g(\boldsymbol{u}(\alpha)))}{d\alpha}(0+) &= \lim_{0<\alpha\to 0} \frac{\int_{\mathcal{K}\cap\hbar^+(\boldsymbol{u}(\alpha),p(\boldsymbol{u}(\alpha)))} \boldsymbol{x}\,d\boldsymbol{x} - \int_{\mathcal{K}\cap\hbar^+(\boldsymbol{u}(0),p(\boldsymbol{u}(0)))} \boldsymbol{x}\,d\boldsymbol{x}}{\alpha}\\ &= \lim_{0<\alpha\to 0} \frac{\int_{\mathcal{K}\cap(\hbar^+(\boldsymbol{u}(\alpha),p(\boldsymbol{u}(\alpha)))\setminus\hbar^+(\boldsymbol{u}(0),p(\boldsymbol{u}(0))))} \boldsymbol{x}\,d\boldsymbol{x}}{\alpha} -\\ &\quad - \lim_{0<\alpha\to 0} \frac{\int_{\mathcal{K}\cap(\hbar^+(\boldsymbol{u}(0),p(\boldsymbol{u}(0)))\setminus\hbar^+(\boldsymbol{u}(\alpha),p(\boldsymbol{u}(\alpha))))} \boldsymbol{x}\,d\boldsymbol{x}}{\alpha}\\ &=: \lim{}_a - \lim{}_b\ .\end{aligned}$$

The limits in the last equation can be calculated by making again the substitution $\boldsymbol{x} = \boldsymbol{y} + r\boldsymbol{u}(\xi)$, where $\boldsymbol{y} \in \mathbb{R}^{n-2}_{\alpha}$. For the first integral we obtain

$$\begin{aligned}
\lim{}_a &= \lim_{0<\alpha\to 0} \frac{\int_{\mathbb{R}^{n-2}_{\alpha}} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\alpha} \int_0^\infty \chi_{\mathcal{K}\cap(\hbar^+_\alpha\setminus\hbar^+_0)}(\boldsymbol{y}+r\boldsymbol{u}(\xi))(\boldsymbol{y}+r\boldsymbol{u}(\xi)) r\,dr\,d\xi\,d\boldsymbol{y}}{\alpha} \\
&= \lim_{0<\alpha\to 0} \frac{\int_{\mathbb{R}^{n-2}_{\alpha}} \int_0^\infty \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\alpha} \chi_{\mathcal{K}\cap(\hbar^+_\alpha\setminus\hbar^+_0)}(\boldsymbol{y}+r\boldsymbol{u}(\xi))\boldsymbol{y}\,d\xi\,r\,dr\,d\boldsymbol{y}}{\alpha} + \\
&\qquad + \lim_{0<\alpha\to 0} \frac{\int_{\mathbb{R}^{n-2}_{\alpha}} \int_0^\infty \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\alpha} \chi_{\mathcal{K}\cap(\hbar^+_\alpha\setminus\hbar^+_0)}(\boldsymbol{y}+r\boldsymbol{u}(\xi))\boldsymbol{u}(\xi)\,d\xi\,r^2\,dr\,d\boldsymbol{y}}{\alpha} \\
&= \int_{\mathbb{R}^{n-2}_{\boldsymbol{u},\boldsymbol{u}^\perp}} \int_0^\infty \lim_{0<\alpha\to 0} \frac{\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\alpha} \chi_{\mathcal{K}\cap(\hbar^+_\alpha\setminus\hbar^+_0)}(\boldsymbol{y}+r\boldsymbol{u}(\xi))\,d\xi}{\alpha}\, r\,dr\boldsymbol{y}\,d\boldsymbol{y} + \\
&\qquad + \int_{\mathbb{R}^{n-2}_{\boldsymbol{u},\boldsymbol{u}^\perp}} \int_0^\infty \lim_{0<\alpha\to 0} \frac{\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\alpha} \chi_{\mathcal{K}\cap(\hbar^+_\alpha\setminus\hbar^+_0)}(\boldsymbol{y}+r\boldsymbol{u}(\xi))\boldsymbol{u}(\xi)\,d\xi}{\alpha}\, r^2\,dr\,d\boldsymbol{y} \\
&= \int_{\mathbb{R}^{n-2}_{\boldsymbol{u},\boldsymbol{u}^\perp}} \int_0^\infty \chi_{\mathcal{K}\cap\hbar(\boldsymbol{u},p(\boldsymbol{u}))}(\boldsymbol{y}+r\boldsymbol{u}^\perp)\, r\,dr\,\boldsymbol{y}\,d\boldsymbol{y} + \\
&\qquad + \int_{\mathbb{R}^{n-2}_{\boldsymbol{u},\boldsymbol{u}^\perp}} \int_0^\infty \chi_{\mathcal{K}\cap\hbar(\boldsymbol{u},p(\boldsymbol{u}))}(\boldsymbol{y}+r\boldsymbol{u}^\perp) \lim_{0<\alpha\to 0} \frac{\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\alpha} \boldsymbol{u}(\xi)\,d\xi}{\alpha}\, r^2\,dr\,d\boldsymbol{y} \\
&= \int_{\mathbb{R}^{n-2}_{\boldsymbol{u},\boldsymbol{u}^\perp}} \int_0^\infty \chi_{\mathcal{K}\cap\hbar(\boldsymbol{u},p(\boldsymbol{u}))}(\boldsymbol{y}+r\boldsymbol{u}^\perp)(\boldsymbol{y}+r\boldsymbol{u}^\perp)\, r\,dr\,d\boldsymbol{y}
\end{aligned}$$

and for the second one a very similar calculation gives

$$\lim{}_b = \int_{\mathbb{R}^{n-2}_{\boldsymbol{u},\boldsymbol{u}^\perp}} \int_0^\infty \chi_{\mathcal{K}\cap\hbar(\boldsymbol{u},p(\boldsymbol{u}))}(\boldsymbol{y}-r\boldsymbol{u}^\perp)(\boldsymbol{y}-r\boldsymbol{u}^\perp) r\,dr\,d\boldsymbol{y}.$$

Summing these up and using (2.1) implies

$$\begin{aligned}
\partial_{\boldsymbol{u}^\perp} g(\boldsymbol{u}) &= \frac{d(g(\boldsymbol{u}(\alpha)))}{d\alpha}(0+) \\
&= \int_{\mathbb{R}^{n-2}_{\boldsymbol{u},\boldsymbol{u}^\perp}} \int_{-\infty}^\infty \chi_{\mathcal{K}\cap\hbar(\boldsymbol{u},p(\boldsymbol{u}))}(\boldsymbol{y}+t\boldsymbol{u}^\perp)(\boldsymbol{y}+t\boldsymbol{u}^\perp) t\,dt\,d\boldsymbol{y} \\
&= \int_{\hbar(\boldsymbol{u},p(\boldsymbol{u}))} \chi_{\mathcal{K}}(\boldsymbol{x})\boldsymbol{x}(\langle \boldsymbol{x}, \boldsymbol{u}^\perp\rangle - p_{\mathcal{F}_{\mathcal{M}}(\boldsymbol{u})}(\boldsymbol{u}^\perp))\,d\boldsymbol{x}_{\hbar(\boldsymbol{u},p(\boldsymbol{u}))}
\end{aligned}$$

that can be written as

$$\begin{aligned}
&\partial_{\boldsymbol{u}^\perp} g(\boldsymbol{u})\\
&= \int_{\hbar(\boldsymbol{u},p(\boldsymbol{u}))} \chi_{\mathcal{K}}(\boldsymbol{x})\langle \boldsymbol{x},\boldsymbol{u}\rangle(\langle \boldsymbol{x},\boldsymbol{u}^\perp\rangle - p_{\mathcal{F}_{\mathcal{M}}(\boldsymbol{u})}(\boldsymbol{u}^\perp))\, d\boldsymbol{x}_{\hbar(\boldsymbol{u},p(\boldsymbol{u}))}\boldsymbol{u}+\\
&\qquad + \int_{\hbar(\boldsymbol{u},p(\boldsymbol{u}))} \chi_{\mathcal{K}}(\boldsymbol{x})\langle \boldsymbol{x},\boldsymbol{u}^\perp\rangle(\langle \boldsymbol{x},\boldsymbol{u}^\perp\rangle - p_{\mathcal{F}_{\mathcal{M}}(\boldsymbol{u})}(\boldsymbol{u}^\perp))\, d\boldsymbol{x}_{\hbar(\boldsymbol{u},p(\boldsymbol{u}))}\boldsymbol{u}^\perp\\
&= (p(\boldsymbol{u}) + p_{\mathcal{F}_{\mathcal{M}}(\boldsymbol{u})}(\boldsymbol{u}^\perp))\partial_{\boldsymbol{u}^\perp} f(\boldsymbol{u})\boldsymbol{u}+\\
&\qquad + \int_{\hbar(\boldsymbol{u},p(\boldsymbol{u}))} \chi_{\mathcal{K}}(\boldsymbol{x})(\langle \boldsymbol{x},\boldsymbol{u}^\perp\rangle - p_{\mathcal{F}_{\mathcal{M}}(\boldsymbol{u})}(\boldsymbol{u}^\perp))^2\, d\boldsymbol{x}_{\hbar(\boldsymbol{u},p(\boldsymbol{u}))}\boldsymbol{u}^\perp.
\end{aligned} \tag{A.5}$$

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