The tale of a formula*

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Dedicated to E. B. Saff for his 70th birthday

Abstract

These are the extended notes of the plenary lecture on the conference *Constructive Functions 2014, Nashville, TN, USA*. It deals with the problem how much zeros on the boundary of a set raise the norm of polynomials compared to the minimal norms.

1 The formula

The formula in question is

$$\mu_n = \cos\left(\frac{\pi}{2(n+1)}\right)^{-n-1}. \tag{1}$$

To understand what it means, let C_1 be the unit circle, and recall that if $P_n(z) = a_n z^n + \cdots + 1$ is a polynomial, then

$$\|P_n\|_{C_1} \geq 1,$$

where we used the notation

$$\|f\|_E = \sup_{z\in E} |f(z)|$$

for the supremum norm. Indeed, since $P_n(0) = 1$, this follows from the maximum principle, or from the formula

$$1 = \left|\frac{1}{2\pi i}\int_{C_1} \frac{P_n(\xi)}{\xi} d\xi\right| \leq \|P_n\|_{C_1}.$$

Now what happens if, in addition, P_n has a zero somewhere on the unit circle? In this case we claim that

$$\|P_n\|_{C_1} \geq 1 + \frac{1}{8\pi n},$$

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i.e. the norm must increase by a universal factor $1+1/30n$. To see that we may assume without loss of generality that $P_n(1)=0$ and $\|P_n\|_{C_1} \le 2$ (if this latter is not true, then the claim holds). By Riesz's inequality [11] for the derivative of a polynomial we have

$$\|P_n'\|_{C_1} \le n\|P_n\|_{C_1} \le 2n.$$

Hence, if $\xi = e^{ix} \in C_1$, $|x| \le 1/4n$, then (the integration is along the corresponding arc of the unit circle)

$$|P_n(\xi)| = \left|\int_1^\xi P_n'(u)du\right| \le \int_1^\xi |P_n'(u)||du| \le \frac{1}{4n}2n = \frac{1}{2},$$

therefore,

$$1 = \left|\frac{1}{2\pi i}\int_{C_1} \frac{P_n(\xi)}{\xi}d\xi\right| \le \int_{C_1} |P_n| \le \|P_n\|_{C_1}\frac{2\pi - 1/2n}{2\pi} + \frac{1}{2}\frac{1/2n}{2\pi}.$$

Now if we rearrange this inequality it follows that

$$\|P_n\|_{C_1} \ge \frac{2\pi - 1/4n}{2\pi - 1/2n} \ge 1 + \frac{1}{8\pi n}.$$

In the opposite direction G. Halász [6] proved in 1983 that there is a $P_n(z)$ with $P_n(1)=0$ such that

$$\|P_n\|_{C_1} \le e^{2/n},$$

and he asked to determine

$$\mu_n = \inf_{P_n(0)=1,\ P_n(1)=0} \|P_n\|_{C_1}.$$

This problem was solved in the paper [7] by M. Lachance, E. B. Saff and R. Varga, who proved the formula (1).

The topics in this paper are related to formula (1): they discusses several situations where zeros on the boundary raise the minimal norm.

2 More zeros

In this section, we briefly describe what happens if there are more than one zero on the unit circle.

Let us agree that whenever we write P_n (or R_n etc.), then it is understood that the degree of P_n (of R_n etc.) is at most n.

Theorem 2.1 *There is an absolute constant $c>0$ such that if $P_n(0)=1$ and P_n has k_n zeros on C_1, then*

$$\|P_n\|_{C_1} \ge 1 + c\frac{k_n}{n}.$$

Theorem 2.2 *There is an absolute constant $c > 0$ such that if $P_n(0) = 1$ and P_n has $n|J|/2\pi + k_n$ zeros on a subarc $J = J_n$ of the unit circle, then*

$$\|P_n\|_{C_1} \geq \exp(ck_n^2/n).$$

See [19] by V. Totik and P. Varjú.

As an immediate corollary we obtain that if $P_n(0) = 1$ and $\|P_n\|_{C_1} = 1+o(1)$, then

(i) P_n have $o(n)$ zeros on C_1,

(ii) P_n have at most $n|J|/2\pi + o(\sqrt{n})$ zeros on any subarc $J = J_n$ of the unit circle.

In particular, if such polynomials have zeros somewhere on the unit circle, then the multiplicity of those zeros is necessarily $o(\sqrt{n})$ (as $n \to \infty$). Let us note that, on the other hand, $\|P_n\|_{C_1} = O(1)$ is already compatible with a zero on C_1 of multiplicity $\sqrt{n}$.

Next, we show that Theorem 2.2 and its corollary are sharp disregarding the constant c. First of all, we mention

Theorem 2.3 *If $z_1, \ldots, z_{k_n}$ are $k_n \leq n/2$ points on the unit circle, then there is a $P_n(z) = a_n z^n + \cdots + 1$ such that z_j are its zeros and*

$$\|P_n\|_{C_1} \leq \exp(4k_n^2/n).$$

Indeed, we have already mentioned Halász' theorem: for every m there is an R_m with $R_m(0) = 1$, $R_m(1) = 0$ such that

$$\|R_m\|_{C_1} \leq e^{2/m}.$$

Now all we need to do is to set

$$P_n(z) = \prod_{j=1}^{k_n} R_{[n/k_n]}(z/z_{n,j}).$$

The sharpness of Theorem 2.1 is somewhat more subtle. The first result in this direction was in [19], but the correct statement is due to V. Andrievskii and H.-P. Blatt [3]:

Theorem 2.4 *Let $\alpha > 1$, and for each n let X_n be a set of k_n points on the unit circle such that the distance between different points of X_n is at least $\alpha 2\pi/n$. Then there are polynomials $P_n(z) = a_n z^n + \cdots + 1$ such that P_n vanishes at each point of X_n and*

$$\|P_n\|_{C_1} \leq 1 + D_\alpha k_n/n.$$

Note that here the condition $\alpha > 1$ is necessary. Indeed, if $\alpha < 1$, then consider the $\alpha 2\pi/n$-spaced sequence X_n of k_n points consisting of

$$e^{ij\alpha 2\pi/n}, \qquad j = 0, 1, \ldots, k_n - 1,$$

and let $J = J_n$ be the (counterclockwise) arc on the unit circle from 1 to $e^{ik_n\alpha 2\pi/n}$. Now if P_n is a polynomial with $P_n(0) = 1$ such that it has a zero at every point of X_n, then there are $\geq (1-\alpha)k_n$ excess zeros of P_n on J_n compared to $n|J_n|/2\pi$, therefore, it follows from Theorem 2.2 that

$$\|P_n\|_{C_1} \geq \exp(c(1-\alpha)^2 k_n^2/n),$$

which is much bigger than $1 + D_\alpha k_n/n$ if $k_n \to \infty$.

3 General curves

The preceding results formulated for the unit circle have extensions to Jordan curves. To state them we need the concept of the equilibrium measure of a compact set $E \subset \mathbf{C}$ (see [4] or [10] for more details). It is the unique measure μ_E on E that minimizes the logarithmic energy

$$\int\int \log \frac{1}{|z-t|} d\mu(z) d\mu(t)$$

among all unit Borel-measures on E (provided there is a measure on E at all for which this energy is finite).

Examples:

- $d\mu_{[-1,1]}(x) = \frac{1}{\pi\sqrt{1-x^2}} dx$.
- $d\mu_{C_1}(e^{it}) = \frac{1}{2\pi} dt$.

Let now K be a smooth Jordan curve (homeomorphic image of the unit circle) and z_0 a fixed point inside K. The following result was proven by Andrievskii and Blatt [3]: If K is an analytic Jordan curve and P_n with $P_n(z_0) = 1$ has k_n zeros on K, then with $\alpha > 0$-separation (in terms of the conformal map of the exterior onto the exterior of C_1) of these zeros

$$\|P_n\|_K \geq 1 + c\frac{k_n}{n},$$

and with $\alpha > 1$-separation it is possible to have

$$\|P_n\|_K \leq 1 + C_\alpha \frac{k_n}{n}.$$

Here, in the first part, the separation condition and the analyticity of K can be omitted (see [16]):

Theorem 3.1 *If K is a C^{1+} smooth Jordan curve and if P_n with $P_n(z_0) = 1$ has k_n zeros on K, then*

$$\|P_n\|_K \geq 1 + c\frac{k_n}{n}.$$

As before, here $c>0$ is a positive constant depending only on K.

[16] also has the full analogue of Theorem 2.2:

Theorem 3.2 *Under the assumptions of Theorem 3.1 if $P_n(z_0)=1$ and P_n has $n\mu_K(J)+k_n$ zeros on a subarc $J=J_n$ of K then*

$$\|P_n\|_K \geq \exp(ck_n^2/n).$$

This is sharp:

Theorem 3.3 *If $w_1,\ldots,w_{k_n}$ is a set of $k_n \leq n$ points on K, then there is a P_n such that $P_n(z_0)=1$, all w_j are zeros of P_n and*

$$\|P_n\|_K \leq \exp(Ck_n^2/n).$$

Here the constants $c, C>0$ depend only on K.

4 Widom's conjecture

We started this paper with polynomials $P_n(z)=a_nz^n+\cdots+1$ on the unit circle. Now $Q_n(z)=z^nP_n(1/z)=z^n+\cdots$ has leading coefficient 1 and $|Q_n(z)|=|P_n(z)|$ on C_1, so the results for the circle about polynomials with constant term 1 have a direct translation for polynomials with leading coefficient 1. The situation is different regarding results on Jordan curves that we have just discussed.

To deal with general curves, we need to introduce the notion of the logarithmic capacity of a compact set $K\subset\mathbf{C}$ (see [4] or [10] for more details). If

$$I(K)=\int\int \log\frac{1}{|z-t|}d\mu_K(z)d\mu_K(t)$$

is the minimal energy on K for all unit Borel measures on K (see the preceding section), then $\mathrm{cap}(K)=\exp(-I(K))$ is called the logarithmic capacity of K (if μ_K does not exist, i.e. when all unit Borel measures on K have infinite energy, then we set $\mathrm{cap}(K)=0$).

Examples:

A segment of length ℓ has capacity $\ell/4$, in particular $\mathrm{cap}([-1,1])=1/2$.

A disk/circle of radius r has capacity r, in particular $\mathrm{cap}(C_1)=1$.

There is a related quantity, the so called Chebyshev constant $t(K)$ associated with K. The number

$$t_n(K)=\inf\|z^n+\cdots\|_K,$$

where the infimum is taken for all monic polynomials of degree n, is called the n-th Chebyshev number of K. It is easy to show that there is a unique minimizing polynomial $T_n(z)=z^n+\cdots$, called the Chebyshev polynomial of degree n for K.

Examples:

- If $K=[-1,1]$ then $t_n(K)=\frac{1}{2^{n-1}}$, $T_n(x)=\frac{1}{2^{n-1}}\cos(n\arccos x)$.
- If $K=C_1$, then $t_n(K)=1$, $T_n(z)=z^n$.

It is easy to see that the sequence $\{t_n(K)^{1/n}\}_{n=1}^{\infty}$ converges, and actually its limit equals its infimum. It is a basic fact due to M. Fekete, G. Szegő and A. Zygmund, that $\{t_n(K)^{1/n}\}_{n=1}^{\infty}$ converges to $\mathrm{cap}(K)$ (see e.g. [10, Corollary 5.5.5]). Hence, we always have (see also [10, Theorem 5.5.4])

$$\|z^n+\cdots\|_K \geq \mathrm{cap}(K)^n.$$

Now it is a fundamental problem how close one can get to this theoretical lower limit, i.e. how small $t_n(K)/\mathrm{cap}(K)^n$ can be (it is always ≥ 1). For example, if K is a circle, then $t_n(K)/\mathrm{cap}(K)^n=1$, so in this case $t_n(K)/\mathrm{cap}(K)^n$ attains the smallest possible value. However, if $K=[-1,1]$, then, as we have just seen, $t_n(K)/\mathrm{cap}(K)^n=2$, therefore, in this case, the fraction $t_n(K)/\mathrm{cap}(K)^n$ stays away from the smallest possible value 1 by a factor 2. This latter fact is true for any real set, for K. Schiefermayr [12] proved that if $K\subset \mathbf{R}$, then $t_n(K)\geq 2\mathrm{cap}(K)^n$. A general upper estimate for $t_n(K)/\mathrm{cap}(K)^n$ was given by H. Widom [22] in 1969: if K consists of smooth Jordan curves and arcs (recall that a Jordan arc is the homeomorphic image of a segment), then $t_n(K)\leq C\mathrm{cap}(K)^n$ with some constant C that depends only on K.

Widom also proved that if K consists of $m\geq 2$ smooth Jordan curves, then $t_n(K)/\mathrm{cap}(K)^n$ does not have a limit, and its limit points typically (i.e. except for some special configurations when the limit points form a finite set) fill a whole interval $[1,\Gamma]$ with an explicitly given Γ. This non-convergence phenomenon had already been observed by N. I. Achiezer [1] in 1931 in the case when the set consisted of two intervals. His result was extended by Widom to the following form: if K consists of $m\geq 2$ intervals on the real line, then $t_n(K)/\mathrm{cap}(K)^n$ does not have a limit, and its limit points typically fill the whole interval $[2,2\Gamma]$, where Γ is the same quantity as before (just written up for the interval case). Regarding this result Widom conjectured that if K consists of C^{2+} smooth Jordan curves and arcs and there is at least one arc present, then

$$\liminf_{n\to\infty}\frac{t_n(K)}{\mathrm{cap}(K)^n}\geq 2. \tag{2}$$

Here one can observe again the phenomenon we are discussing in this paper: when K consist of Jordan curves, then the zeros of polynomials that minimize the norm tend to stay in the interior of the curves. However, when there is an arc present, that arc does not have an interior, and the zeros, that necessarily appear also around that arc component, need to stay on, or close to the boundary, and that is the reason why the norm is raised by a factor >1 compared to the theoretically possible lowest value $\mathrm{cap}(K)^n$.

Widom's conjecture (2) is not true: it was proved by J.-P. Thiran and C. Detaille [13] in 1989 that if K is a subarc on the unit circle of central angle 2α, then

$$t_n(K)\sim \mathrm{cap}(K)^n 2\cos^2\frac{\alpha}{4}$$

(here $\sim$ means that the ratio of the two sides tends to 1 as n tends to infinity). Now if $\alpha < \pi$ approaches π, then the right-hand side approaches

$$\mathrm{cap}(K)^n 2\cos^2\frac{\pi}{4} = \mathrm{cap}(K)^n,$$

so the limit of $t_n(K)/\mathrm{cap}(K)^n$ can be as close to 1 as one wishes.

However, it was proven in [17] that Widom's conjecture is partially true:

Theorem 4.1 *If K, consisting of C^{1+} smooth Jordan curves and arcs, contains at least one arc, then there is a $\beta > 0$ for which*

$$t_n(K) \geq (1+\beta)\mathrm{cap}(K)^n, \qquad n = 1, 2, \ldots. \tag{3}$$

Actually, in [22] Widom had a complete description of the behaviour of the Chebyshev numbers for unions of Jordan curves, namely he established that

$$t_n(K) \sim \mathrm{cap}(K)^n \nu_n$$

with a rather explicitly defined sequence $\{\nu_n\}$. He conjectured that if there is an arc present, then the formula changes by a factor 2, i.e. in that case

$$t_n(K) \sim 2\mathrm{cap}(K)^n \nu_n,$$

and he verified this conjecture when K consists of intervals on the real line. In [20] it was shown that the opposite is true.

Theorem 4.2 *If K consists of C^{2+} smooth Jordan curves and arcs and there is at least one Jordan curve present, then*

$$\limsup \frac{t_n(K)}{\mathrm{cap}(K)^n \nu_n} < 2.$$

For a more precise statement let K_{arc} be the union of the arc components of K.

Theorem 4.3 *If K consists of C^{2+} smooth Jordan curves and arcs and K is symmetric with respect to the real line, then the limit points of $t_n(K)/\mathrm{cap}(K)^n$ lie in the interval*

$$\left[2^{\mu_K(K_{\mathrm{arc}})}, 2^{\mu_K(K_{\mathrm{arc}})}\Gamma\right] \tag{4}$$

and typically fill this interval.

In the last sentence "typically fill this interval" means that this is the case except when there is a special rational relation between the harmonic measures of the components, see [22] for more details.

In this theorem Γ is the quantity mentioned before, and though we do not define it explicitly, we want to point out that if $K_{\mathrm{arc}} = \emptyset$ then the interval in (4) becomes $[1, \Gamma]$, while if $K_{\mathrm{arc}} = K$ (i.e. K lies on the real line) then (4) becomes

$[2, 2\Gamma]$. Theorem 4.3 in these two cases had been established by Widom, and Theorem 4.3 sort of connects these two extreme situations.

Next, we mention the following related results from [15]. Recall that if K consists of $m \geq 2$ smooth Jordan curves, then Widom's results imply that necessarily there is an infinite sequence $\mathcal{N}'$ of the natural numbers such that for $n \in \mathcal{N}'$

$$t_n(K) \geq (1+\beta)\mathrm{cap}(K)^n$$

with some $\beta > 0$.

Theorem 4.4 *Let K be the union of $m \geq 2$ analytic Jordan curves lying exterior to each other. There is an infinite sequence $\mathcal{N}$ such that for $n \in \mathcal{N}$*

$$t_n(K) \leq \left(1 + \frac{C}{n^{1/(m-1)}}\right) \mathrm{cap}(K)^n.$$

Theorem 4.5 *There is a K which is the union of m circles such that for any n*

$$t_n(K) \geq \left(1 + \frac{c}{n^{1/(m-1)}}\right) \mathrm{cap}(K)^n.$$

When K consists of m real intervals, then the right-hand sides must be multiplied by two, see [14]:

Theorem 4.6 *Let K be the union of $m \geq 2$ disjoint intervals on the real line.*

- *There is a sequence $\mathcal{N}'$ such that for $n \in \mathcal{N}'$*

$$t_n(K) \geq 2(1+\beta)\mathrm{cap}(K)^n.$$

- *There is another sequence $\mathcal{N}$ such that for $n \in \mathcal{N}$*

$$t_n(K) \leq 2\left(1 + \frac{C}{n^{1/(m-1)}}\right) \mathrm{cap}(K)^n.$$

- *There is a K which is the union of m intervals such that for any n*

$$t_n(K) \geq 2\left(1 + \frac{c}{n^{1/(m-1)}}\right) \mathrm{cap}(K)^n.$$

Let us explain what is happening here and what is the difficulty in getting close to $\mathrm{cap}(K)^n$ by the norm of a monic polynomial of degree n. If

$$P_n(z) = z^n + \cdots = (z - z_1)\cdots(z - z_n),$$

then

$$\log|P_n(z)| = \sum_{j=1}^{n} \log|z - z_j| = \int \log|z - t| d\nu_n(t),$$

where ν_n is the counting measure on the zeros of P_n (taking into account multiplicity). We want this expression to be not much bigger than $n \log \operatorname{cap}(K)$ for $z \in K$. Now if μ_K is the equilibrium measure of K, then

$$n \log \operatorname{cap}(K) = n \int \log |z-t| d\mu_K(t),$$

so we want

$$\int \log |z-t| d\nu_n(t) - \int \log |z-t| d(n\mu_K)(t) \tag{5}$$

to be as small as possible for all $z \in K$. Note that here two measures of equal masses n are involved. Let $K_1, \ldots, K_m$ be the connected components of K. The numbers $\mu_K(K_j)$ are called the harmonic measures of these components. Now (5) being small for all $z \in K$ implies (this is not trivial) that all $|\nu_n(K_j) - n\mu_K(K_j)|$, $j = 1, 2, \ldots, m$, are small. However, $\nu_n(K_j)$ is always an integer, while $n\mu_K(K_j)$ need not be close to an integer, and this is the reason why, for all n, $t_n(K)/\operatorname{cap}(K)^n$ cannot be too close to 1 in general, and too close to 2 if K lies on the real line. In fact, we can see that a simultaneous Diophantine approximation problem emerges: one should approximate all harmonic densities $\mu_K(K_j)$, $j = 1, \ldots, m$, by numbers of the form p_j/n with a common denominator. Kronecker's theorem tells us that this is possible for certain n's with error $\leq C/n^{1/(m-1)}$, and this is the reason for the appearance of the terms $c/n^{1/(m-1)}$ in Theorems 4.4–4.6.

For sets on the real line this heuristics can be made very precise. Indeed, let $\{x\}$ denote the distance of an $x \in \mathbf{R}$ from the nearest integer, and set

$$\kappa_n = \min\Big\{ \{n\mu_K(K_j)\} \,\Big|\, j = 1, \ldots, m \Big\}.$$

The proof of Theorems 3 and 4 in [14] can be easily modified to show the following theorem.

Theorem 4.7 *Let K be the union of $m \geq 2$ disjoint intervals on the real line. There are constants $c, C > 0$ depending only on K such that for all n we have*

$$2\left(1 + c\frac{\kappa_n}{n}\right) \operatorname{cap}(K)^n \leq t_n(K) \leq 2\left(1 + C\frac{\kappa_n}{n}\right) \operatorname{cap}(K)^n.$$

Now in special circumstances it may happen that $\kappa_n = 0$ for certain n's (namely when all $\mu_K(K_j)$ are rational), and then $t_n(K)/\operatorname{cap}(K)^n$ assumes its minimal value 2. But note that if $m \geq 2$, then it cannot happen that κ_n is small for all n. Indeed, there are infinitely many n's for which $\{n\mu_K(K_1)\} \geq 1/3$ (consider the rational and irrational cases for $\mu_K(K_1)$ separately).

We close this section by an analogue of Theorems 2.1 and 2.2, see [18].

Theorem 4.8 *Let K be a family of C^{1+} smooth Jordan curves lying exterior to each other. If $P_n = z^n + \cdots$ has k_n zeros on K, then*

$$\|P_n\|_K \geq (1 + ck_n/n)\operatorname{cap}(K)^n.$$

Note that here we could allow arc components, as well, since an arc component automatically implies (3) in view of Theorem 4.1.

Theorem 4.9 *Let K be a family of C^{1+} smooth Jordan curves or arcs lying exterior to each other. If $P_n = z^n + \cdots$ has $n\mu_K(J) + k_n$ zeros on a subarc $J = J_n$ of K, then*

$$\|P_n\|_K \geq \exp(ck_n^2/n)\mathrm{cap}(K)^n.$$

In particular, if

$$\|P_n\|_K = (1 + o(1))\mathrm{cap}(K)^n$$

along a sequence $n \in \mathcal{N}$, then P_n have $o(n)$ zeros on K, and P_n cannot have a zero on K of multiplicity $\geq c\sqrt{n}$.

These imply that if all zeros of P_n are on K (like for Fekete polynomials), then there is a $\beta > 0$ such that

$$\|P_n\|_K \geq (1 + \beta)\mathrm{cap}(K)^n$$

even if K is a single C^{1+} smooth Jordan curve. Here the smoothness of K is necessary, without it these results are not true.

Theorem 4.10 *There is a Jordan curve K and $P_n(z) = z^n + \cdots$, $n = 1, 2, \ldots$, with all their zeros on K such that*

$$\liminf_{n\to\infty} \frac{\|P_n\|_K}{\mathrm{cap}(K)^n} = 1.$$

5 Discrepancy theorems

The problem we are dealing with is related to some classical discrepancy theorems, the first of which was proved by P. Erdős and P. Turán in 1950.

Let $P_n(x) = x^n + \cdots$, and assume that all zeros of P_n are real and

$$\|P_n\|_{[-1,1]} \leq A_n/2^n.$$

Theorem 5.1 (Erdős-Turán, 1950) *For any $-1 \leq a < b \leq 1$*

$$\left| \frac{\#\{x_j \in (a,b)\}}{n} - \int_a^b \frac{1}{\pi\sqrt{1-x^2}}dx \right| \leq 8\sqrt{\frac{\log A_n}{n}}. \tag{6}$$

Introduce the normalized zero distribution:

$$\nu_n = \frac{1}{n}\sum_j \delta_{x_j},$$

where $\{z_j\}$ is the zero set for P_n, with which an equivalent form of (6) is the following: with the Chebyshev distribution

$$d\mu_{[-1,1]}(x) = \frac{1}{\pi\sqrt{1-x^2}}dx$$

for any interval $I \subset [-1,1]$ we have

$$\left|\nu_n(I) - \mu_{[-1,1]}(I)\right| \le 8\sqrt{\frac{\log A_n}{n}}.$$

This discrepancy theorem has been extended to very general situations. To state one extension, let K be a finite union of smooth Jordan arcs, and let J be a subarc on K. A "neighborhood" J^* of J is depicted on Figure 5

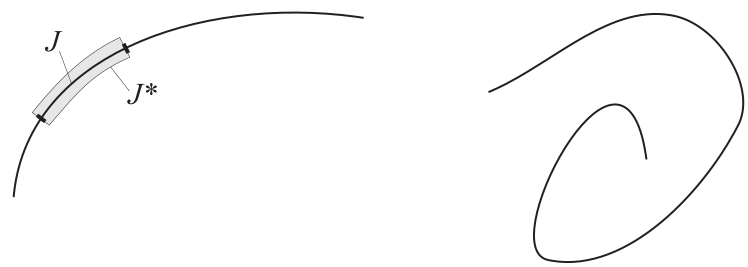

Figure 1: A set J^* associated with J

The following theorem is due to Andrievskii and Blatt, see [2, Theorem 2.4.2].

Theorem 5.2 *Let K be the union of finitely many C^{2+} smooth Jordan arcs, and $P_n(z) = z^n + \cdots$ monic polynomials such that*

$$\|P_n\|_K \le A_n \mathrm{cap}(K)^n.$$

If ν_n is the normalized zero distribution of P_n, then for any subarc $J \subset K$ we have

$$|\nu_n(J^*) - \mu_K(J^*)| \le C\sqrt{\frac{\log A_n}{n}}$$

with some constant C that depends only on K.

Note that this implies the following analogue of Theorem 2.2: If there are $n\mu_K(J) + k_n$ zeros on J, then

$$\frac{k_n}{n} \le |\nu_n(J^*) - \mu_K(J^*)| \le C\sqrt{\frac{\log A_n}{n}},$$

which, after rearrangement gives

$$A_n \ge \exp(ck_n^2/n),$$

i.e.

$$\|P_n\|_K \ge \exp(ck_n^2/n)\mathrm{cap}(K)^n.$$

6 A problem of Erdős

We started this paper with the observation (see also the beginning of Section 4) that if $P_n(z) = z^n + \cdots$, then zeros of P_n on the boundary of the unit circle imply that the norm cannot be too close to 1. In particular, if all the zeros of P_n are on the unit circle, then (this is an excercise for the reader)

$$\|P_n\|_{C_1} \geq 2.$$

The example $z^n - 1$ shows that here the constant 2 is the correct one, but note that the zeros of $z^n - 1$ are the n-th roots of unity, and this zero set changes completely when we move from n to $n+1$. Erdős conjectured that if the zero sets for different n are nested, then boundedness cannot happen, i.e. if $\{z_n\} \subset C_1$ is any sequence of points on the unit circle and

$$P_n(z) = (z - z_1) \cdots (z - z_n)$$

are the polynomials with zeros in the first n terms of the given sequence, then

$$\sup_n \|P_n\|_{C_1} = \infty,$$

i.e. $\|P_{n_k}\|_{C_1} \to \infty$ for some sequence $\{n_k\}$ (note that the full sequence $\{\|P_n\|_{C_1}\}_{n=1}^{\infty}$ may not converge to ∞, as an easy example based on 2^k-th roots of unity with $k = 1, 2, \ldots$ shows). Erdős' conjecture was verified by Wagner [21] in 1980. The strongest result so far is due to J. Beck [5], who proved

Theorem 6.1 *There is a $\theta > 0$ such that, under the preceding assumptions,*

$$\|P_n\|_{C_1} \geq n^{\theta}$$

for infinitely many n.

There had been an earlier conjecture, namely that perhaps even $\|P_n\|_{C_1} \geq n + 1$ is true for infinitely many n, but that was disproven by C. N. Linden [9] in 1977: There is a sequence $\{z_n\} \subset C_1$ and a $\theta^* < 1$ such that

$$\|P_n\|_{C_1} \leq n^{\theta^*}, \qquad n \geq n_0.$$

What happens if here, instead of the unit circle, we consider some other compact set $K \subset \mathbf{C}$ and an arbitrary sequence $\{z_n\}$ from K? Recall that in this case $P_n(z) = z^n + \cdots$, and hence

$$\|P_n\|_K \geq \operatorname{cap}(K)^n,$$

so the analogue of Erdős' question is if $\|P_n\|_K / \operatorname{cap}(K)^n$ is necessarily unbounded or not. However, if K is the unit disk and the sequence $\{z_n\}$ is the identically zero sequence, i.e. $z_n \equiv 0$, then

$$P_n(z) = (z - z_1) \cdots (z - z_n) \equiv z^n,$$

and in this case the norm is identically $1 = \mathrm{cap}(K)^n$, i.e. the minimal possible norm is achieved. This example shows that to find the correct analogue of Wagner's theorem, one should restrict the sequence to lie on the other boundary of K (which is the boundary of the unbounded component of the complement of K). Now Wagner's proof can be modified to show that if K consists of smooth Jordan curves and arcs, then for any $\{z_n\} \subset K$ and $P_n(z) = (z - z_1) \cdots (z - z_n)$ we have

$$\sup \frac{\|P_n\|_K}{\mathrm{cap}(K)^n} = \infty.$$

It is not clear if this is true without the smoothness assumption, i.e. if this statement is true for all compact K (and for any $\{z_n\}$ on the outer boundary of K).

7 High order zeros/incomplete polynomials

The motivation for this paper was a result of Lachance, Saff and Varga, so let us finish with another theorem of them.

Let K be a family of disjoint smooth Jordan arcs on the plane. We have already mentioned in Section 5 that Theorem 5.2 implies the following: if $P_n = z^n + \cdots$ has $n\mu_K(J) + k_n$ zeros on a subarc $J = J_n$ of K (e.g. it has a zero somewhere of multiplicity k_n), then

$$\|P_n\|_K \geq \exp(ck_n^2/n)\mathrm{cap}(K)^n.$$

In particular, if P_n has a zero at some point of K of multiplicity $k_n \sim \lambda n$, then

$$\|P_n\|_K^{1/n}/\mathrm{cap}(K) \geq \exp(c_\lambda).$$

In connection with incomplete polynomials, in the paper [8] Lachance, Saff and Varga answered the following question: what is the best asymptotic lower bound $\Theta(\lambda)$ for

$$\|P_n\|_{[-1,1]}^{1/n}/\mathrm{cap}([-1,1])$$

if P_n has a zero of order $k_n \sim \lambda n$ at 1? They proved the formula

Theorem 7.1

$$\Theta(\lambda) = (1+\lambda)^{1+\lambda}(1-\lambda)^{1-\lambda}.$$

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