A converse to a theorem of Salem and Zygmund*

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February 11, 2016

Abstract

By proving a converse to a theorem of Salem and Zygmund the paper gives a full description of the sets E of points x where the integral $\int_0^1 (F(x+t) - F(x-t))/t dt$ is infinite for a continuous and nondecreasing function F. It is shown that for this it is necessary and sufficient that E is a G_δ set of zero logarithmic capacity. Several corollaries are derived concerning boundary values of univalent functions.

1 Introduction and main results

In this paper we shall use some of the notions of logarithmic potential theory, see e.g. [1] or [9]. In particular, we shall need the concept of logarithmic capacity of a compact set. Then it is usual to define the logarithmic capacity of a Borel set as the supremum of the capacities of its compact subsets. We shall use several times Choquet's capacitability theorem [5, Section 5.8]: the capacity of a Borel set H equals the infimum of the capacities of the open sets containing H. The logarithmic capacity of a Borel set H will be denoted by $\text{cap}(H)$.

For a continuous nondecreasing function F on $\mathbf{R}$ and for an $x \in \mathbf{R}$ let

$$I_x(F) = \int_0^1 \frac{F(x+u) - F(x-u)}{u} du, \tag{1}$$

and set

$$E(F) = \{x \,|\, I_x(F) = \infty.\}$$

The finiteness of I_x has a close connection with the existence of the conjugate function at the given point, with the convergence of various singular integrals as well as with the convergence of conjugate series (see the "nondecreasing periodic extension" discussion below).

*AMS Classification 42A20, 31A15; Key words Monotone functions, conjugate functions, sets of zero capacity, logarithmic potentials

†Supported by the European Research Council Advanced Grant No. 267055

The following theorem is due to R. Salem and A. Zygmund (cf. Theorem VII in [12]), who proved it in connection with convergence questions of Fourier-series.

Theorem A *Let F be a continuous and nondecreasing function on the real line. Then the set $E(F)$ is of logarithmic capacity zero.*

Since $E(F)$ is the intersection of the open sets

$$\bigcup_{n=1}^{\infty} \left\{ x \,\middle|\, \int_{1/n}^{1} \frac{F(x+t)-F(x-t)}{t}\, dt > N \right\}$$

for $N = 1, 2, \ldots$, it is also a G_δ set (i.e. a countable intersection of open sets). Thus, the condition of being a G_δ set of logarithmic capacity zero is necessary for a set to be the set $E(F)$ for some continuous and nondecreasing function F. The question whether the same condition is sufficient, as well, has remained open. The main purpose of the present paper is to give an affirmative answer to this question.

Theorem 1 *If $E \subset \mathbf{R}$ is a G_δ set of logarithmic capacity zero, then there is a continuous and nondecreasing function F such that $E = E(F)$.*

Remark 2 Salem and Zygmund were working with continuous nondecreasing functions on $[0, 2\pi]$ and with their "nondecreasing periodic extension" defined by $F(x + 2\pi) - F(x) = F(2\pi) - F(0)$ for all x (see the footnote on p. 35 in [12] and also [13], p. 11.). Of course, in that case E is 2π-periodic and of zero logarithmic capacity, and the converse takes the form that if $E \subset \mathbf{R}$ is a 2π-periodic G_δ set of zero logarithmic capacity, then there is a continuous and nondecreasing F on $[0, 2\pi]$ such that for its nondecreasing periodic extension we have $E(F) = E$. In fact, we may assume that $0 \notin E$ (otherwise use an appropriate translation of E), apply Theorem 1 to $E \cap [0, 2\pi]$, and restrict the function guaranteed by Theorem 1 to $[0, 2\pi]$ (note that the finiteness of $I_x(F)$ depends only on the behavior of F around x).

We shall prove Theorem 1 in the next section, but before that we mention some corollaries.

As an immediate corollary we have

Corollary 3 *Let f be a 2π-periodic continuous function on the real line such that f is piecewise monotone on $[0, 2\pi]$. Then the integral*

$$-\frac{1}{\pi} \int_0^{\pi} \frac{f(x+t)-f(x-t)}{2\tan(t/2)} dt, \qquad (2)$$

representing the trigonometric conjugate function $\tilde{f}$, exists and is finite with the exception of a G_δ set of zero logarithmic capacity. Conversely, if $E \subset [0, 2\pi)$ is *a G_δ set of zero logarithmic capacity, then there is a continuous and 2π-periodic function f such that f is piecewise monotone on $[0, 2\pi]$ and the set of points x in $[0, 2\pi)$ for which (2) does not exist coincides with E.*

In the literature the integral (2) is usually taken in principal value sense, and it is classical that in this sense (2) exists almost everywhere for any integrable f. The corollary sharpens this for continuous, piecewise-monotone functions. In this case, except for the points where f changes its monotonicity, (2) exists (as an ordinary Lebesgue-integral) as a $[-\infty, \infty]$–valued function, and the corollary is about the set of points where its value is finite.

It is not difficult to modify the proof so that the function f in the last claim of the corollary is increasing on $[0, \pi]$ and decreasing on $[\pi, 2\pi]$.

A completely similar statement holds for the set where the Hilbert transform

$$\int_{\mathbf{R}} \frac{f(t)}{x-t} dt$$

exists for a piecewise monotone, and on $\mathbf{R}$ integrable function f.

Let Δ be the open unit disk and let $\mathbf{T}$ be the unit circle. Suppose that f is a univalent function in Δ such that its radial limits (denoted again by $f(\theta)$, $|\theta| = 1$) exist everywhere on $\mathbf{T}$. By a theorem of Beurling [2] the set $\{\theta \in \mathbf{T} \mid f(\theta) = 0\}$ is of logarithmic capacity zero. This set is also a G_δ set, since it is the intersection of the open sets

$$\bigcup_{n=1}^{\infty} \left\{ \theta \in \mathbf{T} \,\middle|\, |f(e^{-1/n}\theta)| + \frac{1}{n} < \frac{1}{N} \right\}$$

for $N = 1, 2, \ldots$. As an application of Theorem 1 we prove the following converse.

Corollary 4 *Let E be a G_δ subset of $\mathbf{T}$ of zero logarithmic capacity. Then there exists a univalent function f in Δ such that its radial limits exist everywhere on $\mathbf{T}$ and they are equal to zero exactly on E.*

The particular case when E is closed is of special interest, and in this case the conclusion can also be strengthened.

Corollary 5 *If $E \subset \mathbf{T}$ is a closed set of zero logarithmic capacity, then there exists a continuous function f on the closed unit disk $\overline{\Delta}$ which is univalent in Δ such that f vanishes precisely at the points of E.*

This corollary is the univalent analogue of Fatou's interpolation theorem, according to which if $E \subset \mathbf{T}$ is a closed set of Lebesgue measure zero, then there exists a continuous function f in the closed unit disk $\overline{\Delta}$ which is analytic in Δ such that f vanishes precisely at the points of E. As Beurling's theorem and Corollary 5 show, under the additional condition of univalency the role of sets of zero measure is taken over by sets of zero capacity. We also mention that an analogue of the Rudin-Carleson interpolation theorem for univalent functions has been proved recently by C. J. Bishop (see [3], p. 608): if $E \subset \mathbf{T}$ is a closed set of zero logarithmic capacity and $h : \mathbf{T} \to \mathbf{T}$ is an orientation-preserving homeomorphism, then there is a conformal map $f : \Delta \to \Omega \subset \Delta$ onto a Jordan domain Ω such that $f\big|_E = h\big|_E$. Since Ω is a Jordan domain, f can be extended to a continuous function on $\overline{\Delta}$. Even then Bishop's theorem and Corollary 5 are independent results since the function h is a homeomorphism, while Corollary 5 concerns the univalent extension of the identically zero function.

Finally, we derive from Theorem 1 a result of S. V. Kolesnikov. Suppose again that f is a univalent function in Δ, and consider the set of points on $\mathbf{T}$ at which the radial limit of f does not exist. By a classical theorem of Beurling [2] this set is of zero logarithmic capacity. The following partial converse is due to S. V. Kolesnikov [8].

Corollary 6 (Kolesnikov) *Let $E \subset \mathbf{T}$ be any G_δ set of zero logarithmic capacity. Then there exists a bounded univalent function f in Δ such that f has radial limit at every point of $\mathbf{T} \setminus E$ and has no radial limit at any point of E.*

The problem whether similar result is true for any $G_{\delta\sigma}$-set of logarithmic capacity zero on $\mathbf{T}$ remains open.

2 Proof of Theorem 1

It is enough to prove the theorem for bounded E. Indeed, if we know the result for bounded sets, then there is an F_n such that $E \cap [n, n+1] = E(F_n)$. We may modify F_n to be constant on $(-\infty, n-1)$ and on $(n+2, \infty)$, and then by multiplication by a small number we may assume $|F_n| \le 2^{-n}$. But then $F = \sum F_n$ is an increasing continuous function for which $E = \cup_n E(F_n) = E(F)$.

Thus, we may assume E to be bounded, and then by simple scaling that $E \subset [1/3, 2/3]$. Since E is G_δ and of logarithmic capacity 0, by a theorem of Deny [4] there is a finite measure ρ on $\mathbf{C}$ such that

$$U(\rho, z) := \int \log|z - t| d\rho(t) = -\infty$$

precisely if $z \in E$. Unfortunately, in Deny's theorem the measure may not be supported on the real line, so first we reprove Deny's result along the original proof.

In the lemmas that follow "open" is referring to the topology on $\mathbf{R}$.

Lemma 7 *Let $O \subset (0, 1)$ be an open set. Then there is a continuous probability measure ν_O on $\mathbf{R}$ for which*

$$U(\nu_O, x) \le \log \operatorname{cap}(O), \qquad x \in O, \tag{3}$$

and $U(\nu_O, z)$ is finite at all points.

Recall that the continuity of ν_O means that $\nu_O(\{x\}) = 0$ for all x (i.e. ν_O has no point masses).

Proof. Let $O = \cup_{j=1}^{\infty}(a_j, b_j)$ where the (a_j, b_j)'s are disjoint, and set $K_n = \cup_{j=1}^{n}[a_j + 1/n, b_j - 1/n]$ (skip the j-th term if $a_j + 1/n \geq b_j - 1/n$). This is a compact set of logarithmic capacity $< \mathrm{cap}(O)$, and if ν_n is its equilibrium measure then, by Frostman's theorem [9, Theorem 3.3.4],

$$\int \log|x - t| d\nu_n(t) = \log \mathrm{cap}(K_n) \leq \log \mathrm{cap}(O), \qquad x \in K_n. \tag{4}$$

Let ν_O be a weak*-limit of the sequence $\{\nu_n\}$, say $\nu_n \to \nu_O$ as $n \to \infty$, $n \in \mathcal{N}$. ν_n is absolutely continuous on the interior of K_n and its density w_n (with respect to Lebesgue-measure) is a C^∞-function (see e.g. for an explicit representation [11, Lemma 4.4.1]). Let $x \in O$ and choose $\delta > 0$ such that $[x-2\delta, x+2\delta] \subset O$. Then $[x - 2\delta, x + 2\delta] \subset K_n$ for all $n \geq n_0$, and on the set $[x - 2\delta, x + 2\delta]$ (actually on the whole set K_{n_0}) the sequence $\{\nu_n\}_{n \geq n_0}$ is a decreasing sequence of measures ([10, Theorem IV.1.6(e)]). These imply that ν_O is absolutely continuous on $(x - 2\delta, x + 2\delta)$ and on $(x - \delta, x + \delta)$ we have $w_n \searrow w_O(x)$, where w_O is the density of ν_O. Hence, by the monotone convergence theorem, we have as $n \to \infty$, $n \in \mathcal{N}$,

$$\int_{x-\delta}^{x+\delta} \log|x - t| d\nu_n(t) \to \int_{x-\delta}^{x+\delta} \log|x - t| d\nu_O(t).$$

On the other hand, according to what we have just said $\nu_n\Big|_{\mathbf{R} \setminus [x-\delta, x+\delta]} \to \nu_O\Big|_{\mathbf{R} \setminus [x-\delta, x+\delta]}$ in the weak* topology as $n \to \infty$, $n \in \mathcal{N}$, hence

$$\int_{\mathbf{R}\setminus[x-\delta,x+\delta]} \log|x - t| d\nu_n(t) \to \int_{\mathbf{R}\setminus[x-\delta,x+\delta]} \log|x - t| d\nu_O(t).$$

Thus, we have proven that along the sequence $n \in \mathcal{N}$

$$\int \log|x - t| d\nu_n(t) \to \int \log|x - t| d\nu_O(t),$$

and then (3) follows from (4).

Since, by Frostman's theorem,

$$\int \log|z - t| d\nu_n(t) \geq \log \mathrm{cap}(K_n)$$

for all z, one can easily deduce that $U(\nu_O, z)$ is finite everywhere. Finally, the finiteness of $U(\nu_O, z)$ at every z implies that ν_O is a continuous measure, i.e. it has not atoms: $\nu_O(\{x\}) = 0$ for all $x \in \mathbf{R}$.

∎

Lemma 8 *Let $E \subset [1/3, 2/3]$ be a Borel set of logarithmic capacity 0, $\varepsilon > 0$, and let $\Omega \subset (1/4, 3/4)$ be an open set containing E such that $\mathrm{cap}(\Omega) \le \varepsilon$. Then there exists a continuous measure $\nu_{E,\Omega}$ on $[1/4, 3/4]$ of total mass ≤ 1 such that*

$$U(\nu_{E,\Omega}, x) \le -\frac{1}{2}\log(1/2\varepsilon) \qquad \text{for all } x \in E \tag{5}$$

and

$$U(\nu_{E,\Omega}, x) \ge -1. \qquad \text{for all } x \notin \Omega, \tag{6}$$

Furthermore, $U(\nu_{E,\Omega}, z)$ is finite for all z.

Proof. Let

$$E_m = \{x \in E \,\big|\, e^{-2^{m+1}} \le \mathrm{dist}(x, \mathbf{R} \setminus \Omega) < e^{-2^m}\}.$$

Since the capacity of a segment of length l is $l/4$, the condition $\mathrm{cap}(\Omega) < \varepsilon$ implies that E_m is empty if $2^{m+1} < \log(1/2\varepsilon)$. By Choquet's capacitability theorem for every m there are open sets $E_m \subset O_m \subset \Omega$ such that $\mathrm{cap}(O_m) < e^{-8^m}$. We may also assume that all points in O_m lie of distance

$$e^{-2^{m+2}} < \cdot < e^{-2^{m-1}}$$

from $\mathbf{R} \setminus \Omega$.

Since O_m lies in $(1/4, 3/4)$, the measures ν_{O_m} from the preceding lemma are supported on $[1/4, 3/4]$. The measure

$$\nu_{E,\Omega} := \sum_{2^{m+1} \ge \log(1/2\varepsilon)}^{\infty} \frac{1}{4^m} \nu_{O_m}$$

is continuous, and, by (3), we have for all $x \in E$, say for $x \in E_m$,

$$\int \log|x-t| d\nu_{E,\Omega}(t) \le \frac{1}{4^m} \int \log|x-t| d\nu_{O_m}(t) \le \frac{1}{4^m}(-8^m) = -2^m \le -\frac{1}{2}\log(1/2\varepsilon),$$

while for $x \notin \Omega$

$$\int \log|x-t| d\nu_{E,\Omega}(t) = \sum_{2^{m+1} \ge \log(1/2\varepsilon)}^{\infty} \int \log|x-t| d\nu_{O_m}(t),$$

and since the support of ν_{O_m} lies in the closure of O_m which is of $\ge e^{-2^{m+2}}$ distance from x, we can continue the preceding line as

$$\int \log|x-t| d\nu_{E,\Omega}(t) \ge \sum_{2^{m+1} \ge \log(1/2\varepsilon)}^{\infty} \left(-2^{m+2}\right) \frac{1}{4^m} \ge -1.$$

That $U(\nu_{E,\Omega}, z)$ is finite for all z is clear if $z \notin \mathbf{R}$, and for $z \in \mathbf{R} \setminus \Omega$ we have just proved it. Finally, for $z \in \Omega$ there is an m_0 such that $\mathrm{dist}(z, O_m) \geq \delta$ with some δ (that depends on z) for all $m \geq m_0$, hence, as before,

$$\sum_{m \geq m_0} \frac{1}{4^m} \int \log |z - t| d\nu_{O_m}(t) \geq \log \delta$$

(recall also that $U(\nu_{O_m}, z)$ is finite for all m).

■

Now we are ready to prove Deny's theorem with a small addition on the measure:

Lemma 9 *If $E \subset [1/3, 2/3]$ is a G_δ set of zero logarithmic capacity, then there is a continuous and finite measure μ supported on $[1/4, 3/4]$ such that $U(\mu, x) = -\infty$ for all $x \in E$ and $U(\mu, x) > -\infty$ for all $x \in \mathbf{R} \setminus E$.*

Proof. By Choquet's capacitability theorem for every n there are open sets $E \subset H_n$ of capacity $\mathrm{cap}(H_n) \leq e^{-2^{n+1}}$. Represent E as $E = \cap_n \Omega_n$ where the Ω_n are open. We can replace Ω_n by $\Omega_n \cap H_n$, so we may assume that $\mathrm{cap}(\Omega_n) \leq \frac{1}{2} e^{-2^n}$. With the ν_{E,Ω_n} from the preceding lemma we set

$$\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} \nu_{E,\Omega_n}. \tag{7}$$

This is a finite and continuous measure, and (5) (with $\varepsilon = \frac{1}{2} e^{-2^n}$) shows that

$$U(\mu, x) \leq \sum_n \frac{1}{2^n} (-\frac{1}{2} 2^n) = -\infty$$

if $x \in E$. On the other hand, if $x \notin E$ then there is an n_0 such that $x \notin \Omega_n$ for $n \geq n_0$, and then (6) yields

$$U(\mu, x) \geq \sum_{n < n_0} U(\nu_{E,\Omega_n}, x) + \sum_{n \geq n_0} \frac{1}{2^n} (-1) > -\infty.$$

■

After these preparations we turn now to the proof of Theorem 1. With the μ from the previous lemma set

$$F(x) = \mu((-\infty, x]). \tag{8}$$

This is a continuous nondecreasing function on $\mathbf{R}$, it is 0 on $(-\infty, 1/4)$ and it is constant on $[4/5, \infty)$. We claim that $I_x(F) = \infty$ for all $x \in E$. Indeed, let the measure $d\nu(t) = \frac{dt}{t}$ be defined on $[0, 1]$, and for $x \in E$ consider the triangle V_x with vertices at $(x, 0), (x-1, 1), (x+1, 1)$. Consider the product measure $\mu \times \nu$ on V_x:

$$(\mu \times \nu)(V_x) = \int\int \chi_{V_x}(u, v) d(\mu \times \nu)(u, v) = \int_{\mathbf{R}} \int_0^1 \chi_{V_x}(u, v) d\nu(v) d\mu(u), \quad (9)$$

where χ_{V_x} is the characteristic function of V_x and we have used Fubini's theorem. For a fixed $u \in [x-1, x+1]$ (otherwise the inner integral is 0), we have

$$\int_0^1 \chi_{V_x}(u, v) d\nu(v) = \int_{|x-u|}^1 d\nu(v) = \int_{|x-u|}^1 \frac{dv}{v} = -\log|x-u|,$$

so $U(\mu, x) = -\infty$ gives that the double integral in (9) is ∞. But that integral can also be written as

$$\int_0^1 \int_{\mathbf{R}} \chi_{V_x}(u, v) d\mu(u) d\nu(v),$$

and for a fixed $v \in [0, 1]$ we have

$$\int_{\mathbf{R}} \chi_{V_x}(u, v) d\mu(u) = \mu((x-v, x+v]) = F(x+v) - F(x-v),$$

that is

$$\infty = (\mu \times \nu)(V_x) = \int_0^1 \frac{F(x+v) - F(x-v)}{v} dv,$$

which proves the claim $x \in E(F)$.

Similar reasoning shows that if $x \notin E$ then $x \notin E(F)$ (since in this case $(\mu \times \nu)(V_x)$ is finite).

■

Remark 10 It is also clear from the connection between μ and F why the Salem-Zygmund result is true: if F is continuous and nondecreasing, and, say, is constant on $(-\infty, 0)$ and on $(1, \infty)$ then F generates a measure μ of compact support for which $U(\mu, x) = -\infty$ for all $x \in E(F)$. Now the Salem-Zygmund theorem follows from the fact that a potential can be $-\infty$ only on a set of zero logarithmic capacity. (If $\operatorname{cap}(E(F)) > 0$ then there is a compact set $K \subseteq E(F)$ of positive capacity. If ν is the equilibrium measure of K, then, by Frostman's theorem, $U(\nu, x) \geq \log \operatorname{cap}(K)$, so

$$-\infty < \int U(\nu, x) d\mu(x) = \int U(\mu, t) d\nu(t) = \int_K (-\infty) d\nu = -\infty,$$

a contradiction.)

3 Proof of Corollaries 3–6

In the following lemma we use the definition of $I_x(F)$ from (1) not only for continuous and nondecreasing, but for any continuous function F.

Lemma 11 *Let $E \subset [0, 2\pi]$ be a G_δ set of logarithmic capacity zero such that $0, \pi$, and 2π do not belong to E. Then there exists a continuous and 2π-periodic function $F \geq 0$ with the properties:*

1) *F is nondecreasing on $[0, \pi]$ and the set $\{x \in (0, \pi) \,|\, I_x(F) = \infty\}$ coincides with $(0, \pi) \cap E$,*

2) *F is nonincreasing on $[\pi, 2\pi]$ and $\{x \in (\pi, 2\pi) \,|\, I_x(F) = -\infty\}$ coincides with $(\pi, 2\pi) \cap E$,*

3) *The numbers $I_0(F)$, $I_\pi(F)$, $I_{2\pi}(F)$ exist and they are finite.*

Note that in 1) and 2) the existence of $I_x(F)$ is automatic by the assumed monotonicity.

Proof. In the following reasoning we use the following simple fact: if g is increasing and h is continuously differentiable in a neighborhood of a point x and $h'(x) > 0$, then $I_x(g)$ is finite precisely if $I_x(gh)$ is finite.

By Theorem 1 there exists a continuous nondecreasing function φ (on $\mathbf{R}$) such that $E = E(\varphi)$. Denote by φ_1 the restriction of φ to $[0, 2\pi]$. Since $E \subset (0, 2\pi)$, we have $E = E(\varphi_1)$. By shifting the function φ_1 vertically (by adding a constant), without loss of generality, we may assume $\varphi_1(\pi) = 0$.

Also, without loss of generality, we may assume $\varphi_1(0) = -\varphi_1(2\pi)$. Indeed, if both $\varphi_1(0)$ and $\varphi_1(2\pi)$ are different from zero, this can be done for example by replacing the values of the function φ_1 on the subinterval $[0, \pi]$ by the function $c\, \varphi_1|_{[0,\pi]}$ where $c = -\varphi_1(2\pi)/\varphi_1(0)$. The case when both $\varphi_1(0)$ and $\varphi_1(2\pi)$ are zero is trivial, since then φ_1 is identically zero and E is empty. Finally, if for example $\varphi_1(0) = 0$ (thus, φ_1 is identically zero on $[0, \pi]$), but $\varphi_1(2\pi) \neq 0$ (then in fact $\varphi_1(2\pi) > 0$), we redefine φ_1 on $[0, \pi]$ as a linear function which takes the values $-\varphi_1(2\pi)$ and 0 at the points 0 and π, respectively.

Thus we may assume that we have a continuous nondecreasing function φ_1 on $[0, 2\pi]$, such that $E = E(\varphi_1)$, $\varphi_1(\pi) = 0$, and $\varphi_1(0) = -\varphi_1(2\pi)$. Consider $\varphi_2 = -|\varphi_1|$, the 2π-periodic extension of which has all properties of F listed in the lemma except the nonnegativity condition and possibly property 3).

The function $\varphi_3(x) = \varphi_2(x)(x - \pi)^2$ has all properties of φ_2, and in addition it is differentiable at π, which implies that $I_\pi(\varphi_3)$ is finite.

Next, $\varphi_4(x) = \varphi_3(x) + \varphi_1(2\pi)\pi^2$ has all properties of φ_3 but φ_4 is also nonnegative and $\varphi_4(0) = \varphi_4(2\pi) = 0$.

Finally, the 2π-periodic extension of $F(x) = \varphi_4(x)x^2(x-2\pi)^2$, $x \in [0, 2\pi]$, has all the required properties (note that F is differentiable at $0, \pi$ and 2π, which implies property 3).

■

Proof of Corollary 3. Note that (2) is finite at x if and only if $I_x(f)$ is finite. Now the first statement in the corollary is an immediate consequence of the Salem-Zygmund result Theorem A if we apply it to the finitely many subintervals of $[0, 2\pi]$ where f is monotone (the finitely many points where f changes monotonicity does not play a role in the "G_δ, zero logarithmic capacity" property of the set in question).

In the proof of the second statement we may assume that $0, \pi, 2\pi$ do not belong to E (apply a translation if this is not the case), and then the second claim in the corollary follows from Lemma 11 (set $f = F$) using again the equiconvergence of (2) and $I_x(f)$.

■

Proof of Corollary 4. Since E is of logarithmic capacity zero, one can find two diametrically opposite points on $\mathbf{T}$ which do not belong to E. Without loss of generality we may assume them to be 1 and -1.

We identify the set E with its "image set" on $[0, 2\pi]$ defined as $\{x \in [0, 2\pi] | e^{ix} \in E\}$. Then E on $[0, 2\pi]$ satisfies the conditions of Lemma 11. Let F be the function from Lemma 11. By a result of W. Kaplan [7, Theorem 3], the function

$$G(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} F(\theta)\ d\theta$$

is univalent in Δ. If $G(z) = u(z) + iv(z)$, then u, being the Poisson integral of the continuous and periodic function F, is continuous on the closed unit disk.

Next, note that the function $I_x(F)$ behaves as the boundary function

$$\tilde{F}(x) = -\frac{1}{\pi} \int_0^{\pi} \frac{F(x+t) - F(x-t)}{2 \tan \frac{1}{2} t}\ dt$$

of the conjugate harmonic function $v(z)$, where $G(z) = u(z) + iv(z)$ as above: $I_x(F) + \pi\tilde{F}(x)$ is a continuous function of x. This is so because $1/t - 1/2\tan(t/2)$ is a continuous function around the origin. In particular, $I_x(F)$ and $-\tilde{F}(x)$ are ∞ or $-\infty$ at exactly the same points, and hence $v(z) \to \infty$ or $v(z) \to -\infty$ as $z \in \Delta$ tends radially to a point of E.

The continuity of F and its other properties stated in Lemma 11 imply that the radial limits of v exist and are finite everywhere on the set $\mathbf{T} \setminus E$ (see for example [13], p. 103, Theorem 7.20). The same properties also imply that on the portion of E located on the upper semicircle the radial limits of v are equal to ∞, while on the portion of E located on the lower semicircle they are equal to $-\infty$ (see again [13], Theorem 7.20). From these facts it is clear that the univalent function $f(z) = \frac{1}{G(z)+1}$ satisfies the claim in the corollary.

$\blacksquare$

Proof of Corollary 5. Since E is closed, in addition to the conclusions of Lemma 11, one can also achieve (see the argument below) that for the function F in Lemma 11 the quantity $I_x(F)$ (thus also $\tilde{F}(x)$) is continuous at every point x that belongs to the complement of E, and it is also continuous in the extended sense at each point of E, where the values of the function are $\pm\infty$. Hence, again, $f(z) = \frac{1}{G(z)+1}$ has the claimed properties.

To see that we mention first of all, that, if in Lemma 7 the set O consists of finitely many intervals, then there exists a continuous probability measure $\tilde{\nu}_O$ on $\mathbf{R}$ for which

$$U(\tilde{\nu}_O, x) \le \frac{1}{2} \log \operatorname{cap}(O), \qquad x \in O, \tag{10}$$

$U(\tilde{\nu}_O, z)$ is finite at all points, and, in addition, the density of $\tilde{\nu}_O$ with respect to Lebesgue measure is continuously differentiable everywhere. Indeed, consider the equilibrium measure $\nu_{\overline{O}}$ of the closure $\overline{O}$ of O. For it we have (see the reasoning in Lemma 7)

$$U(\nu_{\overline{O}}, x) = \log \operatorname{cap}(\overline{O}) = \log \operatorname{cap}(O), \qquad x \in \overline{O}. \tag{11}$$

This $\omega_{\overline{O}}$ has C^∞ density ω everywhere, except at the endpoints of the components/subintervals of $\overline{O}$, where it has a $1/\sqrt{t}$-type singularity. Since there are only finitely many endpoints, one can easily choose an increasing sequence $\{h_n\}$ of nonnegative continuously differentiable functions that converge pointwise to ω. By Lebesgue's monotone convergence theorem then we have

$$\int \log|x-t| h_n(t)dt \searrow U(x) = \log \operatorname{cap}(O), \qquad x \in \overline{O},$$

hence this convergence is uniform on $\overline{O}$ by Dini's theorem. Therefore, for large n we have

$$\int \log|x-t| h_n(t)dt < \frac{1}{2} \log \operatorname{cap}(O), \qquad x \in \overline{O},$$

and $d\tilde{\nu}_O(x) = h_n(x)dx$ is an appropriate measure for (10).

Next, note that if E is closed in Lemma 8, then the sets E_m in the proof of that lemma are empty for all large m, furthermore, the Ω_m in that lemma can be taken to consist of finitely many intervals. Hence, the proof gives (use now (10) instead of (3)) a $\nu_{E,\Omega}$ as in the lemma for which the conclusions of Lemma 8 are true with (5) replaced by

$$U(\nu_{E,\Omega}, x) \le -\frac{1}{4} \log(1/2\varepsilon) \qquad \text{for all } x \in E, \tag{12}$$

and, in addition, $\nu_{E,\Omega}$ has continuously differentiable density outside E.

Finally, if we use in the proof of Lemma 9 the just given strengthening of Lemma 8, and we choose in that proof Ω_n as some d_n-neighborhood of the compact set E, then we can conclude Lemma 8 for some μ with the additional property that μ has continuously differentiable density outside E (indeed, one only has to remark that in this case on every closed interval that is disjoint from E all but finitely many terms in the sum (7) are zero, so the density of the measure μ in (7) on every such interval is a finite sum of continuously differentiable functions).

Therefore, if we define again $F(x) = \mu((-\infty, x])$ as in (8), then this F has all the properties as before, and, in addition, it is continuously differentiable outside E. This then implies that the integrals $I_x(F)$ converge locally uniformly in the complement of E, and, as a consequence, the function $I_x(F)$ is a continuous function in the complement of E. But this function is also continuous at every point of E in the extended sense, i.e. $I_y(F) \to \infty$ if $y \to x \in E$ (recall that $I_x(F) = \infty$ if $x \in E$). Indeed, the integrals

$$\int_{1/n}^{1} \frac{F(x+t) - F(x-t)}{t}$$

are continuous functions of x and for $n \to \infty$ they converge monotone increasingly to $I_x(F)$. Hence, $I_x(F)$ is lower semi-continuous in x, therefore it is continuous in the extended sense wherever it is ∞.

Now if we follow the proof of Corollary 4 with this modification, we can conclude that the function f constructed there is continuous everywhere on $\mathbf{T}$.

■

Proof of Corollary 6. Consider the proof of Corollary 4 and the functions F, G constructed there. That proof shows that the range of G is part of the vertical strip S bounded by the imaginary axis and the vertical line through the point $F(\pi) > 0$ of the real axis.

In the complex plane consider a bounded simply connected (spiral shaped) domain U called the "outer snake" which winds infinitely many times around the unit circle. Let $\zeta = H(w)$ be a conformal map of the right half plane onto U such that the boundary point of the right half plane at infinity corresponds to the prime end of U having the unit circle as its impression. Then, as we approach any point of E from Δ radially, the values of the function G approach ∞ (while staying inside the strip S which is part of the right-half plane), and so the values $H(G(z))$ wind indefinitely around the unit circle. It is easy to see that at other points the radial limit of $H(G(z))$ exist, hence $H(G(z))$ has all the properties stated in Corollary 6.

As a final note we mention that conformal mappings onto spiral domains were earlier used by Kolesnikov [8].

■

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