On a problem of B. Mityagin^{*}

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1 The problem

In connection with an uncertain principle Boris Mityagin [2] formulated the following problem. For given $0 and <math>d \ge 1$, characterize those non-empty subsets A, B of \mathbf{R}^d for which

$$f(x+a) - f(x) \in L^p(\mathbf{R}^d) \quad \text{for all } a \in A, \tag{1}$$

and

$$f(x)\sin\langle x,b\rangle \in L^p(\mathbf{R}^d) \quad \text{for all } b \in B, \tag{2}$$

imply $f \in L^p(\mathbf{R}^d)$ for any measurable function f on \mathbf{R}^d (here $\langle x, b \rangle$ denotes the inner product of x and b). He showed (for $p \ge 1$) that if

(i) $A = \alpha \mathbf{Z}^d$ and $B = \beta \mathbf{Z}^d$, or

(ii) $A = \{a\}$ and $B = \{b\}$ are singletons,

then (1) and (2) imply $f \in L^p(\mathbf{R}^d)$ if and only if $\alpha\beta$ is not an integer multiple of π in case (i) and $\langle a, b \rangle$ is not an integer multiple of π in case of (ii). He has also conjectured the statement in

Proposition 1 (1) and (2) imply $f \in L^p(\mathbf{R}^d)$ for every measurable function f on \mathbf{R}^d if and only if there are $a \in A$ and $b \in B$ such that $\langle a, b \rangle$ is not an integer multiple of π .

This paper is devoted to the proof of this proposition. A relatively simple modification of the proofs shows that the claim is true also for $L^{\infty}(\mathbf{R}^d)$.

The sufficiency part of Proposition 1 easily follows from the method of [2] (which fact was mentioned in that paper), but we follow a different and shorter path.

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2 Sufficiency in Proposition 1

Let $a \in A$ and $b \in B$ be such that $\langle a, b \rangle$ is not an integer multiple of π . If we multiply the function in (1) by $\sin\langle x, b \rangle$ and add the function in (2), then we obtain $f(x+a) \sin\langle x, b \rangle \in L^p(\mathbf{R}^d)$, i.e. $f(x) \sin\langle x-a, b \rangle \in L^p(\mathbf{R}^d)$, which is the same as $f(x) | \sin\langle x-a, b \rangle | \in L^p(\mathbf{R}^d)$. Thus, $f(x)h(x) \in L^p(\mathbf{R}^d)$, where

$$h(x) = |\sin\langle x, b\rangle| + |\sin\langle x - a, b\rangle|.$$

On the line $\ell = \{tb \mid t \in \mathbf{R}\}$ the function h(tb) is continuous, non-zero (a zero would mean that both $t\langle b, b \rangle$ and $t\langle b, b \rangle - \langle a, b \rangle$ — and hence also $\langle a, b \rangle$ — are integer multiples of π , which is not the case by the assumption) and periodic in $t \in \mathbf{R}$ with period $\pi/\langle b, b \rangle$, hence it has there a positive lower bound: $h(tb) \geq \delta > 0$. Since on the hyperplanes of \mathbf{R}^d that are perpendicular to ℓ the function h is constant, it follows that $h(x) \geq \delta$ for all $x \in \mathbf{R}^d$, and hence $fh \in L^p(\mathbf{R}^d)$ implies $f \in L^p(\mathbf{R}^d)$.

3 Necessity in Proposition 1

Suppose now that

$$\langle a, b \rangle \in \pi \mathbf{Z}$$
 for all $a \in A$ and $b \in B$. (3)

We are going to construct a measurable function $f \notin L^p(\mathbf{R}^d)$ for which (1) and (2) are true.

Let \mathcal{A} be the additive group generated by A with vector addition in \mathbf{R}^d as the group operation. Then

$$\langle a, b \rangle \in \pi \mathbf{Z}$$
 for all $a \in \mathcal{A}$ and $b \in B$, (4)

is also true, hence we may replace A by A. If \overline{A} is the closure of A in the metric of \mathbf{R}^d , then (4) remains true when A is replaced by \overline{A} , so we may assume that Ais a closed subgroup of \mathbf{R}^d . We shall need the following description of A, which is basically known (c.f. [1, M. 4.8], [3, Theorem 4.20]) and fairly easy to prove. Since our formulation is somewhat more precise than what is in the literature, for completeness we give a proof at the and of this note.

Lemma 2 (a) There is a subspace V of \mathbf{R}^d and a discrete subgroup \mathcal{G} in its orthogonal complement V^{\perp} such that $\mathcal{A} = \mathcal{G} + V$.

(b) The discrete subgroups of \mathbf{R}^d are the free groups generated by linearly independent elements.

(a) means that every $a \in \mathcal{A}$ can be uniquely written in the form a = g + vwhere $g \in \mathcal{G}$ and $v \in V$. (b) means for the \mathcal{G} in (a) that there are linearly independent elements $g_1, \ldots, g_m \in \mathcal{G}$ such that every $g \in \mathcal{G}$ can be uniquely written in the form

$$g = \alpha_1(g)g_1 + \dots + \alpha_m(g)g_m,$$

with some integers $\alpha_1(g), \ldots, \alpha_m(g)$. Set

$$\alpha(g) = \max_{1 \le j \le m} |\alpha_j(g)|,$$

and

$$S_k = \{g \in \mathcal{G} \,|\, \alpha(g) = k\}.$$

Since the different elements $kg_1 + \alpha_2g_2 + \cdots \alpha_mg_m$ with $-k \leq \alpha_j \leq k$ all belong to S_k , we have $|S_k| \geq (2k+1)^{m-1}$. On the other hand, every element of S_k belongs to one of the sets $\{g \mid \alpha_j(g) = \pm k, -k \leq \alpha_i(g) \leq k \text{ if } i \neq j\}, 1 \leq j \leq m$. Each of these sets has $2(2k+1)^{m-1}$ elements, hence $|S_k| \leq 2m(2k+1)^{m-1}$. Thus, if $A \sim B$ means that A/B lies in between two positive constants, then we have $|S_k| \sim (k+1)^{m-1}$ for all k. As a consequence we obtain that if M > 0is any number, then for $\varepsilon \geq 0$

$$\sum_{a \in \mathcal{A}} \frac{1}{(\alpha(a) + M)^{m + \varepsilon}} < \infty \quad \Leftrightarrow \quad \varepsilon > 0.$$
(5)

Indeed, this is immediate since

$$\sum_{a \in \mathcal{A}} \frac{1}{(\alpha(a) + M)^{m+\varepsilon}} = \sum_{k=0}^{\infty} \sum_{a \in S_k} \frac{1}{(\alpha(a) + M)^{m+\varepsilon}} = \sum_{k=0}^{\infty} \frac{|S_k|}{(k+M)^{m+\varepsilon}}$$
$$\sim \sum_{k=0}^{\infty} \frac{(k+1)^{m-1}}{(k+M)^{m+\varepsilon}},$$

and it is clear that the last sum diverges (terms are $\sim 1/k$) if $\varepsilon = 0$, and converges (terms are $\sim 1/k^{1+\varepsilon}$) if $\varepsilon > 0$.

In the proof of the necessity we distinguish two cases.

Case I: \mathcal{A} is discrete. Thus, in this case $V = \{0\}$ and $\mathcal{A} = \mathcal{G}$. Since \mathcal{A} is discrete, there is an M such that the distance in between different elements of \mathcal{A} is at least $2/M^{1/d}$ (just note that if there were different elements arbitrarily close to each other, then their difference would be non-zero and arbitrarily close to 0, contradicting the discrete character of \mathcal{A}).

Assume first that the number m in the description of \mathcal{G} is bigger than 0. For $a \in \mathcal{A}$ let \mathcal{B}_a be the (closed) ball of radius $1/(\alpha(a) + M)^{m/d}$ with center at a, and set $f = \chi_{\bigcup_{a \in \mathcal{A}} \mathcal{B}_a}$, where χ_E denotes the characteristic function of the set E. Since the balls \mathcal{B}_a are disjoint by the choice of M, and the d-dimensional volume of a ball of radius r is $\theta_d r^d$ with some number θ_d , it follows that the L^p norm of f is

$$\theta_d \sum_{a \in \mathcal{A}} \frac{1}{(\alpha(a) + M)^m} = \infty$$

by (5), so $f \notin L^p(\mathbf{R}^d)$. On the other hand, below we show that (1) and (2) are true, and that will complete the proof of the necessity in the case when \mathcal{A} is discrete and $m \geq 1$.

It is sufficient to prove (1) for the generators g_j , j = 1, ..., m. Choose such a g_j , and consider the set

$$F_j = \{x \mid f(x+g_j) - f(x) \neq 0\}$$

Since $f(x+g_j)-f(x)$ takes only the values $0, \pm 1$, if we show that $\operatorname{meas}(F_j) < \infty$, then (1) follows. But $x \in F_j$ means that either $x \in \mathcal{B}_{a_0}$ for some $a_0 \in \mathcal{A}$ and $x+g_j \notin \bigcup_{a \in \mathcal{A}} \mathcal{B}_a$, or the other way around (i.e. $x+g_j \in \mathcal{B}_{a_0}$ and $x \notin \bigcup_{a \in \mathcal{A}} \mathcal{B}_a$). These two cases are similar (just replace x by $x+g_j$ and g_j by $-g_j$), so consider the first one. Let $B_r(z)$ denote the (closed) ball about z and of radius r. Since $x+g_j \notin \bigcup_{a \in \mathcal{A}} \mathcal{B}_a$, we have in particular

$$x + g_j \notin \mathcal{B}_{a_0 + g_j} = B_{(\alpha(a_0 + g_j) + M)^{-m/d}}(a_0 + g_j),$$

which is the same as

$$x \notin B_{(\alpha(a_0+g_i)+M)^{-m/d}}(a_0).$$

Therefore, by the assumption

$$x \in B_{(\alpha(a_0)+M)^{-m/d}}(a_0) \setminus B_{(\alpha(a_0+g_i)+M)^{-m/d}}(a_0).$$
(6)

This is possible only if $\alpha(a_0 + g_j) > \alpha(a_0)$. But in any case, the definition of the function α shows that $\alpha(a_0 + g_j) \leq \alpha(a_0) + 1$, so we must have $\alpha(a_0 + g_j) = \alpha(a_0) + 1$. But then from (6) it follows that

$$\max(F_j \cap \mathcal{B}_{a_0}) \leq 2 \max\left(B_{(\alpha(a_0)+M)^{-m/d}}(a_0+g_j) \setminus B_{(\alpha(a_0+g_j)+M)^{-m/d}}(a_0+g_j)\right) \\ = 2\theta_d\left(\frac{1}{(\alpha(a_0)+M)^m} - \frac{1}{(\alpha(a_0)+1+M)^m}\right) \sim \frac{1}{(\alpha(a_0)+M)^{m+1}},$$

and so

$$\operatorname{meas}(F_j) = \sum_{a_0 \in \mathcal{A}} \operatorname{meas}(F_j \cap \mathcal{B}_{a_0}) < \infty$$

in view of (5). This proves (1).

Now consider property (2). Let $b \in B$. Since $\langle a, b \rangle \equiv 0 \pmod{\pi}$ for all $a \in \mathcal{A}$, it follows that if $x \in \mathcal{B}_a$, then (in what follows |x| denotes the Euclidean norm of $x \in \mathbf{R}^d$)

$$|\sin\langle x,b\rangle| = |\sin\langle x-a,b\rangle| \le |x-a||b| \le \frac{|b|}{(\alpha(a)+M)^{m/d}},$$

and so

$$\int_{\mathcal{B}_a} |f(x)\sin\langle x,b\rangle|^p dx \le \left(\frac{|b|}{(\alpha(a)+M)^{m/d}}\right)^p \operatorname{meas}(\mathcal{B}_a) = |b|^p \frac{\theta_d}{(\alpha(a)+M)^{m+mp/d}}.$$

Therefore, (5) implies

$$\int |f(x)\sin\langle x,b\rangle|^p dx = \sum_{a\in\mathcal{A}} \int_{\mathcal{B}_a} |f(x)\sin\langle x,b\rangle|^p dx = \sum_{a\in\mathcal{A}} |b|^p \frac{\theta_d}{(\alpha(a)+M)^{m+mp/d}} < \infty$$

This is property (2), and the proof is complete when \mathcal{A} is discrete and $m \geq 1$.

If \mathcal{A} is discrete but m = 0, then $\mathcal{A} = A = \{0\}$, so (1) is automatic for all f, and to get the necessity just set $f(x) = |x|^{-d/p}(1+|x|)^{-2}$ which is not in $L^p(\mathbf{R}^d)$, but $f(x)|x| \in L^p(\mathbf{R}^d)$ implying (2).

Case II: \mathcal{A} is not discrete. In this case, $V \neq \{0\}$. Let $l \geq 1$ be the dimension of V, and assume first again that $\mathcal{G} \neq \{0\}$, i.e. $m \geq 1$. Since \mathcal{G} is discrete, there is an M > 0 such that different elements of \mathcal{G} are of distance $> 2/M^{(m+l)/(d-l)}$. This implies that any two elements of g + V and g' + V are of distance $> 2/M^{(m+l)/(d-l)}$. $2/M^{(m+l)/(d-l)}$ if $g, g' \in \mathcal{G}$ are different (note that \mathcal{G} lies in V^{\perp}).

Let \mathcal{D} be the (closed) unit ball in V^{\perp} . It is of dimension d-l > 0 (note that V cannot be the whole \mathbf{R}^d because $m \ge 1$), and for a $y \in V$ and $g \in \mathcal{G}$ let

$$\mathcal{D}_{y,g} = y + g + \mathcal{D} \cdot (|y| + \alpha(g) + M)^{-(m+l)/(d-l)},$$

which is a d-l dimensional ball about g+y of radius $(|y|+\alpha(g)+M)^{-(m+l)/(d-l)}$. Set

$$E_g = \cup_{y \in V} \mathcal{D}_{y,g}$$

and $f = \chi_{\cup_{g \in \mathcal{G}} E_g}$. According to what we have just said, the different E_g 's are disjoint (since any element of E_g is of distance $\leq 1/(\alpha(g) + M)^{(m+l)/(d-l)} \leq 1/M^{(m+l)/(d-l)}$ from g + V). It is easy to see that each E_g is closed, so f is measurable. Using Fubini's theorem we obtain that

$$meas(E_g) = \int_V \theta_{d-l} \frac{1}{(|y| + \alpha(g) + M)^{m+l}} dy \sim \frac{1}{(\alpha(g) + M)^m},$$
(7)

where we used that for $\tau \geq 0$

$$\int_{V} \frac{1}{(|y|+L)^{m+l+\tau}} dy \sim \frac{1}{L^{m+\tau}}$$
(8)

uniformly in L > 0. Indeed, this is immediate if we make the substitution y = Ly' in the integral.

In view of (7) and (5)

meas
$$(\cup_{g \in \mathcal{G}} E_g) \sim \sum_{g \in \mathcal{G}} \frac{1}{(\alpha(g) + M)^m} = \infty,$$

and hence $f \notin L^p(\mathbf{R}^d)$. To complete the proof we shall show that, on the other hand, f satisfies both (1) and (2).

It is enough to prove (1) for all $a = v, v \in V$ and for all generators $a = g_j$ of \mathcal{G} . This second one is similar to what we did in the discrete case. Indeed, let again

$$F_j = \{x \mid f(x+g_j) - f(x) \neq 0\}$$

and it is sufficient to show that $\operatorname{meas}(F_j) < \infty$. Now $x \in F_j$ means that either $x \in \mathcal{D}_{y,a_0}$ for some $y \in V$ and $a_0 \in \mathcal{G}$ and $x + g_j \notin \bigcup_{g \in \mathcal{G}} E_g$, or the other way around, and we may consider the first case. Then

$$x \in y + a_0 + \mathcal{D} \cdot (|y| + \alpha(a_0) + M)^{-(m+l)/(d-l)}$$

but

$$x + g_j \notin y + a_0 + g_j + \mathcal{D} \cdot (|y| + \alpha(a_0 + g_j) + M)^{-(m+l)/(d-l)}$$

i.e.

$$x \notin y + a_0 + \mathcal{D} \cdot (|y| + \alpha(a_0 + g_j) + M)^{-(m+l)/(d-l)}$$

and so

$$x \in y + a_0 + \left(\frac{\mathcal{D}}{(|y| + \alpha(a_0) + M)^{(m+l)/(d-l)}} \setminus \frac{\mathcal{D}}{(|y| + \alpha(a_0 + g_j) + M)^{(m+l)/(d-l)}}\right)$$
(9)

As in the discrete case this is possible only if $\alpha(a_0 + g_j) = \alpha(a_0) + 1$, and then it follows that the (d - l)-dimensional measure of $F_j \cap (E_{a_0} \cap (y + V^{\perp}))$ is at most twice the difference

$$\frac{\theta_{d-l}}{(|y|+\alpha(a_0)+M)^{m+l}} - \frac{\theta_{d-l}}{(|y|+\alpha(a_0)+1+M)^{m+l}} \sim \frac{1}{(|y|+\alpha(a_0)+M)^{m+l+1}}.$$

If we integrate this with respect to $y \in V$, then we obtain from (8) that the measure of $F_j \cap E_{a_0}$ is at most a constant times $(\alpha(a_0) + M)^{-(m+1)}$, and hence

$$\operatorname{meas}(F_j) = \sum_{a_0 \in \mathcal{G}} \operatorname{meas}(F_j \cap E_{a_0}) \le C \sum_{a_0 \in \mathcal{G}} \frac{1}{(\alpha(a_0) + M)^{m+1}} < \infty,$$

where we used again (5).

Consider now (1) for $a = v \in V$. This time set

$$F_v^* = \{ x \, | \, f(x+v) - f(x) \neq 0 \}.$$

Now $x \in F_v^*$ means that either $x \in \mathcal{D}_{y,a_0}$ for some $y \in V$ and $a_0 \in \mathcal{G}$ and $x + v \notin \bigcup_{g \in \mathcal{G}} E_g$, or the other way around, and consider again the first case. Then

$$x \in y + a_0 + \mathcal{D} \cdot (|y| + \alpha(a_0) + M)^{-(m+l)/(d-l)}$$

$$x + v \notin y + v + a_0 + \mathcal{D} \cdot (|y + v| + \alpha(a_0) + M)^{-(m+l)/(d-l)},$$

but

i.e.

$$x \notin y + a_0 + \mathcal{D} \cdot (|y + v| + \alpha(a_0) + M)^{-(m+l)/(d-l)}.$$

Hence

$$x \in y + a_0 + \left(\frac{\mathcal{D}}{(|y| + \alpha(a_0) + M)^{(m+l)/(d-l)}} \setminus \frac{\mathcal{D}}{(|y| + |v| + \alpha(a_0) + M)^{(m+l)/(d-l)}}\right).$$

It follows that the (d-l)-dimensional measure of $F_v^* \cap (E_{a_0} \cap (y+V^{\perp}))$ is at most twice the difference

$$\frac{\theta_{d-l}}{(|y|+\alpha(a_0)+M)^{m+l}} - \frac{\theta_{d-l}}{(|y|+|v|+\alpha(a_0)+M)^{m+l}} \sim \frac{1}{(|y|+\alpha(a_0)+M)^{m+l+1}}$$

(in this very last step the ~ depends on |v| but not on y or a_0). If we integrate this with respect to $y \in V$, then we obtain from (8) that the measure of $F_v^* \cap E_{a_0}$ is at most a constant times $(\alpha(a_0) + M)^{-(m+1)}$, and hence

$$\operatorname{meas}(F_v^*) = \sum_{a_0 \in \mathcal{G}} \operatorname{meas}(F_v^* \cap E_{a_0}) \le C \sum_{a_0 \in \mathcal{G}} \frac{1}{(\alpha(a_0) + M)^{m+1}} < \infty,$$

because of (5). This finishes the proof of (1).

Next, consider property (2). Let $b \in B$. Since $\langle a, b \rangle \equiv 0 \pmod{\pi}$ for all $a \in \mathcal{A}$, it follows that if $x \in \mathcal{B}_{y,g}$ then

$$|\sin\langle x,b
angle| = |\sin\langle x-g-y,b
angle| \le |x-g-y||b| \le \frac{|b|}{(|y|+\alpha(a)+M)^{(m+l)/(d-l)}},$$

and so

$$\begin{split} \int_{\mathcal{B}_{y,g}} |f(x)\sin\langle x,b\rangle|^p dx &\leq \left(\frac{|b|}{(|y|+\alpha(a)+M)^{(m+l)/(d-l)}}\right)^p \theta_{d-l} (\text{radius of } \mathcal{B}_{y,g})^{d-l} \\ &= |b|^p \theta_{d-l} \frac{1}{(|y|+\alpha(a)+M)^{m+l+(m+l)p/(d-l)}}. \end{split}$$

If we integrate this for $y \in V$ then (8) implies

$$\begin{split} \int_{E_g} |f(x)\sin\langle x,b\rangle|^p dx &\leq \int_V |b|^p \theta_{d-l} \frac{1}{(|y|+\alpha(a)+M)^{m+l+(m+l)p/(d-l)}} dy \\ &\sim \frac{1}{(|y|+\alpha(a)+M)^{m+(m+l)p/(d-l)}}. \end{split}$$

Therefore, we obtain from (5)

$$\int |f(x)\sin\langle x,b\rangle|^p dx = \sum_{g\in\mathcal{G}} \int_{E_g} |f(x)\sin\langle x,b\rangle|^p dx \sim \sum_{g\in\mathcal{G}} \frac{1}{(\alpha(a)+M)^{m+(m+l)p/(d-l)}} < \infty$$

and the proof of the necessity is complete when $m \ge 1$.

If m = 0 (i.e. $\mathcal{G} = \{0\}$) but $V \neq \mathbf{R}^d$, then do the preceding proof with m = 0 with the modification that now instead of (8) we use

$$\int_V \frac{1}{(|y|+L)^l} dy = \infty.$$

However, in the m = 0 case it is now possible that $V = \mathbf{R}^d$. In that case necessarily $B = \{0\}$, so (2) is automatic, and to have the necessity just pick a function f on \mathbf{R}^d which is not in L^p but for which (1) holds for all $a \in \mathbf{R}^d$ (for example, set $f(x) = (|x|+1)^{-d/p}$.)

4 Proof of Lemma 2

For part (b) see [3, Theorem 4.20]. To prove part (a), let $\mathcal{A} \subset \mathbf{R}^d$ be the closed group in question. Let $V \subset \mathbf{R}^d$ be the largest subspace of \mathbf{R}^d that lies in \mathcal{A} (since the sum of two subspaces lying in \mathcal{A} also lies in \mathcal{A} , there is such a largest subspace), and let V^{\perp} be the orthogonal complement of V. We claim that there is a $\delta > 0$ such that all $a \in \mathcal{A} \setminus V$ lies of distance $\geq \delta$ from V. Indeed, if this is not the case, then for every n there are $a_n \in \mathcal{A}$ that lie outside V such that their distance from V is < 1/n. Let $v_n \in V$ be the closest element of V to a_n . Then $a_n - v_n \in V^{\perp}$. By compactness, the sequence $\{(a_n - v_n)/|a_n - v_n|\}$ has a convergent subsequence, and we may assume that $(a_n - v_n)/|a_n - v_n| \to u$. Then u is a unit vector lying in V^{\perp} . If $\lambda > 0$, then $(a_n - v_n)[\lambda/|a_n - v_n|] \to \lambda u$, where [·] denotes integral part, and since each $(a_n - v_n)[\lambda/|a_n - v_n|]$ belongs to \mathcal{A} , we obtain that $\lambda u \in \mathcal{A}$ for all $\lambda > 0$, and hence for all $\lambda \in \mathbf{R}$. But this means that all vectors $v + \lambda u$, $v \in V$, $\lambda \in \mathbf{R}$, lie in \mathcal{A} , which is impossible by the maximality of V. As a corollary it follows that $\mathcal{G} := \mathcal{A} \cap V^{\perp}$ is a discrete group (if we had different elements $a, a' \in \mathcal{A} \cap V^{\perp}$ arbitrarily close to each other, then their difference a - a' would be in V^{\perp} and hence would lie outside V, but would lie close to zero, and hence to V, which is not possible).

Every $a \in \mathcal{A}$ has a unique representation $a = a_{V^{\perp}} + a_V$ with $a_{V^{\perp}} \in V^{\perp}$ and $a_V \in V$. Since $a_V \in V \subset \mathcal{A}$, it follows that $a_{V^{\perp}} \in \mathcal{A}$. Therefore, $\mathcal{G} := \{a_{V^{\perp}}\} = \mathcal{A} \cap V^{\perp}$, so this is a subgroup, and part (a) follows.

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