Sharp constants in asymptotic higher order Markov inequalities*

Vilmos Totik[†] and Yuan Zhou[‡]

November 21, 2016

Abstract

The best asymptotic constant for k-th order Markov inequality on a general compact set is determined.

1 Introduction

Let $\mathcal{P}_n$ denote the set of all (complex) polynomials of degree at most n, and let $\|f\|_E = \sup_{x\in E} |f(x)|$ denote the supremum norm of the function f on the set E. Two of the most classical polynomial inequalities are the Bernstein inequality (see [2], [3, Corollary 4.1.2])

$$|P_n'(x)| \le \frac{n}{\sqrt{1-x^2}}\|P_n\|_{[-1,1]}, \quad x \in (-1,1), \tag{1}$$

and the Markov inequality (see [3, Theorem 4.1.4], [7])

$$\|P_n'\|_{[-1,1]} \le n^2 \|P_n\|_{[-1,1]}, \tag{2}$$

where $P_n \in \mathcal{P}_n$. For higher order derivatives iteration of (2) gives

$$\|P_n^{(k)}\|_{[-1,1]} \le n^{2k} \|P_n\|_{[-1,1]}, \tag{3}$$

but the correct estimate is (see [8] or [9, Theorem 1.2.2, Sec. 6.1.2]),

$$\|P_n^{(k)}\|_{[-1,1]} \le C_{n,k} \|P_n\|_{[-1,1]}, \quad P_n \in \mathcal{P}_n, \tag{4}$$

with

$$C_{n,k} := \frac{n^2(n^2-1)\cdots(n^2-(k-1)^2)}{(2k-1)!!}, \tag{5}$$

*AMS Classification: 26D05, 42A05; Keywords: Markov inequality, asymptotically sharp constants

[†]Supported by ERC Advanced Grant No. 267055

[‡]Supported by NSF grant DMS 1564541

where $(2k-1)!! = 1 \cdot 3 \cdot 5 \cdots (2k-1)$. The equality is attained for the standard Chebyshev polynomial $\mathcal{T}_n(x) := \cos(n \arccos(x))$. If we write (4) in the asymptotic form

$$\|P_n^{(k)}\|_{[-1,1]} \le (1+o(1)) \frac{n^{2k}}{(2k-1)!!} \|P_n\|_{[-1,1]},$$

where $o(1)$ tends to zero (uniformly in P_n) as $n \to \infty$, then we can see that for large n the factor $1/(2k-1)!!$ appears compared to the iterated (3). We shall show that the appearance of this factor is universal, it emerges on other compact sets, as well.

The classical Markov inequality implies that if E consists of finitely many intervals, then

$$\|P_n^{(k)}\|_E \le C n^{2k} \|P_n\|_E \tag{6}$$

with some constant C that depends only on the set E. Therefore, there is a smallest $\mathcal{M}_{E,k}$ such that

$$\|P_n^{(k)}\|_E \le \mathcal{M}_{E,k}(1+o(1)) n^{2k} \|P_n\|_E, \tag{7}$$

where $o(1) \to 0$ (uniformly in P_n) as $n \to \infty$, and in this paper our aim is to determine this $\mathcal{M}_{E,k}$, thereby providing the best possible asymptotic constant in the k-th order Markov inequality. It follows from (1) that

$$|P_n'(x)| \le C_K n \|P_n\|_E, \quad x \in K,$$

with some constant C_K uniformly on compact subsets K of the interior of E, and if we iterate this k times (for some fixed k) on nested intervals, then we obtain that if K is a compact subset of the interior of E, then

$$|P_n^{(k)}(x)| \le C_K^* n^k \|P_n\|_E, \quad x \in K, \tag{8}$$

i.e. inside the set E the k-th order Bernstein-Markov factor is of the order $O(n^k)$. Therefore, the k-th derivative can be of size n^{2k} only around endpoints of E, and the constant in front of this n^{2k} depends on what endpoint we are considering. Thus, let $E = \cup_{j=1}^{l} [a_{2j-1}, a_{2j}]$, and let a_j be one of the endpoints of E. If $\delta > 0$ is so small that $[a_j - \delta, a_j + \delta]$ does not contain any other endpoint of E, then the asymptotic k-th order Markov constant for the endpoint a_j is the smallest number $\mathcal{M}_{a_j,k}$ for which it is true that

$$\|P_n^{(k)}\|_{E \cap [a_j-\delta, a_j+\delta]} \le (1+o(1)) \mathcal{M}_{a_j,k} n^{2k} \|P_n\|_E. \tag{9}$$

(8) shows that this smallest $\mathcal{M}_{a_j,k}$ is independent of $\delta > 0$.

In view of (8) it is clear that the $\mathcal{M}_{E,k}$ in (7) is the maximum of all these $\mathcal{M}_{a_j,k}$, $1 \le j \le 2l$:

$$\mathcal{M}_{E,k} = \max_{1 \le j \le 2l} \mathcal{M}_{a_j,k},$$

so it is sufficient to determine $\mathcal{M}_{a_j,k}$ for each j. To describe it we need some facts from potential theory. For the necessary concepts we refer to [10], [12] or to [15].

Let E be a compact set on the real line. The equilibrium measure ν_E of E minimizes the logarithmic energy

$$\iint \log \frac{1}{|z-t|} d\nu(z) d\nu(t)$$

among all probability measures ν on E. This ν_E is absolutely continuous (with respect to linear Lebesgue measure) in the interior of E, and we denote by ω_E its density (= Radon-Nikodym derivative) with respect to the Lebesgue measure.

Let $E = \cup_{j=1}^{l} [a_{2j-1}, a_{2j}]$ consist of the disjoint intervals $[a_{2j-1}, a_{2j}]$. It is known (see e.g. [13, (2.4)]), that the equilibrium density is of the form

$$\omega_E(x) = \frac{\prod_{i=1}^{l-1} |x-\tau_i|}{\pi \sqrt{\prod_{i=1}^{2l} |x-a_i|}}, \qquad x \in E, \tag{10}$$

where $\tau_i \in (a_{2i}, a_{2i+1}), i = 1, \cdots, l-1$, are the unique numbers satisfying

$$\int_{a_{2j}}^{a_{2j+1}} \frac{\prod_{i=1}^{l-1} (x-\tau_i)}{\pi \sqrt{\prod_{i=1}^{2l} |x-a_i|}} dx = 0$$

for $j = 1, 2, \cdots, l-1$. We define

$$M_{a_j} := 2 \frac{\prod_{i=1}^{l-1} (a_j - \tau_i)^2}{\prod_{i \neq j} |a_j - a_i|}, \qquad j = 1, \cdots, 2l. \tag{11}$$

It was proved in [13, Theorem 4.1] that for $k=1$ we have the equality $\mathcal{M}_{a_j,1} = M_{a_j}$, but, just in the case of $E = [-1,1]$, this cannot be iterated to get the correct result for higher derivative. Indeed, for higher derivative we have

$$\mathcal{M}_{a_j,k} = \frac{M_{a_j}^k}{(2k-1)!!},$$

as is shown by

Theorem 1. *With the above notations, for fixed $k \geq 1$ and for each $1 \leq j \leq 2l$, we have*

$$\|P_n^{(k)}\|_{E \cap [a_j - \delta, a_j + \delta]} \leq (1 + o(1)) \frac{M_{a_j}^k n^{2k}}{(2k-1)!!} \|P_n\|_E, \tag{12}$$

where $o(1)$ tends to 0 uniformly in $P_n \in \mathcal{P}_n$ as $n \to \infty$. Furthermore, this estimate is asymptotically the best possible, for there is a sequence $\{P_n \in \mathcal{P}_n\}_{n=1}^{\infty}$ of nonzero polynomials such that

$$|P_n^{(k)}(a_j)| \geq (1 + o(1)) \frac{M_{a_j}^k n^{2k}}{(2k-1)!!} \|P_n\|_E. \tag{13}$$

A more general result will be proved (with the help of Theorem 1) in Theorem 3.

Let us consider the example $E = [-b, -a] \cup [a, b]$. In this case $l = 2$, $a_1 = -b$, $a_2 = -a$, $a_3 = a$, $a_4 = b$, and, by symmetry, $\tau_1 = 0$. Hence

$$\omega_E(t) = \frac{|t|}{\pi\sqrt{(b^2 - t^2)(t^2 - a^2)}},$$

$$M_{a_1} = M_{a_4} = \frac{2b^2}{(b-a)(b+a)(2b)} = \frac{b}{b^2 - a^2}$$

$$M_{a_2} = M_{a_3} = \frac{2a^2}{(b-a)(b+a)(2b)} = \frac{a}{b^2 - a^2}.$$

Since $M_{a_1} = M_{a_4} > M_{a_2} = M_{a_3}$, we obtain that for fixed k

$$\|P_n^{(k)}\|_{[-b,-a]\cup[a,b]} \leq (1 + o(1))\frac{n^{2k}}{(2k-1)!!}\left(\frac{b}{b^2 - a^2}\right)^k \|P_n\|_{[-b,-a]\cup[a,b]},$$

and this is the (asymptotically) best possible estimate for the k-th derivative of general polynomials P_n of degree $n = 1, 2, \ldots$ in the sense that one cannot write a smaller constant on the right.

2 Proof of Theorem 1

The proof uses the polynomial inverse image method, see [13, 14]. First we are going to prove (12) in a special case when both the base set and the polynomial P_n are related to polynomial mappings. Then we deduce (12) in its full generality from this special case, and at the end we verify (13).

Polynomial inverse images

Suppose that T_N is a real polynomial of degree $N \geq 2$ with real zeros $X_1 < X_2 < \cdots < X_N$. Let $Y_1 < Y_2 < \cdots < Y_{N-1}$ be zeros of T_N', and assume that $|T_N(Y_s)| \geq 1$ for $s = 1, 2, \cdots, N-1$. Then there exists a unique sequence of closed intervals $E_s = [\alpha_s, \beta_s]$ such that $T_N(E_s) = [-1, 1], X_s \in E_s, s = 1, 2, \cdots, N$ and for each $1 \leq s \leq N-1$ the set $E_s \cap E_{s+1}$ contains at most one point, call it θ_s (if the intersection is not empty). We call such polynomials admissible.

For an admissible polynomial the inverse image $T_N^{-1}[-1, 1]$ consists of l disjoint intervals where $1 \leq l \leq N$. At the endpoints of subintervals of $T_N^{-1}[-1, 1]$, as well at the points θ_s, the value of T_N is ± 1. Furthermore, T_N' does not vanish at the endpoints of the subintervals of $T_N^{-1}[-1, 1]$, and it has a simple zero at every θ_s.

Polynomial inverse images under admissible polynomials possess several properties. One of them is the density among all sets consisting of finitely many intervals (see [14, Theorem 3.1] and the references there).

Proposition 2. *Given a set* $\Sigma = \cup_{j=1}^{l}[a_{2j-1}, a_{2j}]$ *of disjoint closed intervals and a positive number* ε*, there is another set* $\Sigma' = \cup_{j=1}^{l}[a'_{2j-1}, a'_{2j}]$ *consisting of the same number of intervals such that* $\Sigma' = T^{-1}[-1,1]$ *for an admissible polynomial* T*, and for each* $1 \le j \le 2l$ *we have*

$$|a_j - a'_j| < \varepsilon.$$

The theorem also implies its strengthened form when we can choose if a given a'_j is smaller or bigger than a_j. In particular, we can require $\Sigma \subset \Sigma'$ or $\Sigma' \subset \Sigma$. The proof of proposition 2 (given for example in [13]) also gives that we can choose $a_{2j-1} = a'_{2j-1}$ for all j. Alternatively we can fix all a_{2j}.

For definiteness we assume that a_j is a right endpoint of a subinterval of E (left endpoints can be similarly handled).

In the proof of (12) in Theorem 1 first we assume E to be the inverse image of $[-1,1]$ under an admissible polynomial T_N of degree N, and also assume that P_n is of the form $P_n(x) = R_m(T_N(x))$ with some $R_m \in \mathcal{P}_m$, so that $n = mN$.

Taking derivatives we get

$$\begin{aligned}
P'_n(x) &= R'_m(T_N(x))T'_N(x), \\
P''_n(x) &= R''_m(T_N(x))(T'_N(x))^2 + R'_m(T_N(x))T''_N(x), \\
&\vdots \\
P_n^{(k)}(x) &= R_m^{(k)}(T_N(x))(T'_N(x))^k + \frac{k(k-1)}{2}R_m^{(k-1)}(T_N(x))(T'_N(x))^{k-2}T''_N(x) \\
&\qquad + \cdots + R'_m(T_N(x))T_N^{(k)}(x). \qquad (14)
\end{aligned}$$

Here we have used Faà di Bruno's formula to calculate higher order derivatives of composed functions Faà di Bruno's formula [4] (see also [11, pp. 35–37])

$$\frac{d^k}{dx^k}f(g(x)) = \sum \frac{k!}{m_1!m_2!\cdots m_k!} f^{(m_1+\cdots+m_k)}(g(x)) \prod_{i=1}^{k}\left[\frac{g^{(i)}(x)}{i!}\right]^{m_i}, \qquad (15)$$

where the sum is over all k-tuples of nonnegative integers $(m_1, \cdots, m_k)$ satisfying

$$m_1 + 2m_2 + \cdots + km_k = k. \qquad (16)$$

For fixed N and k, the functions $T_N, T'_N, \cdots, T_N^{(k)}$ are all bounded on E. When m is large, the first term in (14) can be of order m^{2k}, all other terms are of smaller order by (6). Therefore, by the classical Markov inequality (4)

$$|P_n^{(k)}(a_j)| \le (1 + o(1))C_{m,k}\|R_m\|_{[-1,1]}|T'_N(a_j)|^k.$$

In view of (4.10) of [13], we have $|T'_N(a_j)| = N^2 M_{a_j}$, and since $n = mN$, we obtain

$$\begin{aligned}
C_{m,k}N^{2k} &= \frac{(mN)^2[(mN)^2 - N^2]\cdots[(mN)^2 - (k-1)^2N^2]}{(2k-1)!!} \\
&\le \frac{(mN)^{2k}}{(2k-1)!!} = \frac{n^{2k}}{(2k-1)!!},
\end{aligned}$$

Therefore,

$$|P_n^{(k)}(a_j)| \le (1+o(1))\frac{M_{a_j}^k n^{2k}}{(2k-1)!!}\|P_n\|_E,$$

where we used that $\|P_n\|_E = \|R_m\|_{[-1,1]}$. This is the desired inequality but only for the endpoint a_j.

The argument for points close to a_j is similar. In fact, let $\varepsilon > 0$ be given. We can select $\eta > 0$ such that $[a_j - 2\eta, a_j] \subset E$ and for $x \in [a_j - \eta, a_j]$ it is true that

$$|T_N'(x)| \le (1+\varepsilon)|T_N'(a_j)| = (1+\varepsilon)M_{a_j}N^2.$$

Then for $x \in [a_j - \eta, a_j]$ we get from (14) and again from the classical Markov inequality (4) that

$$\begin{aligned} |P_n^{(k)}(x)| &\le (1+o(1))(1+\varepsilon)^k \frac{m^{2k}}{(2k-1)!!} N^{2k} M_{a_j}^k \|R_m\|_{[-1,1]} \\ &= (1+o(1))(1+\varepsilon)^k \frac{M_{a_j}^k}{(2k-1)!!} n^{2k} \|P_n\|_E. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, (12) follows (with δ replaced by η) for $P_n = R_m(T_N)$ as $m \to \infty$.

The general case of Theorem 1

We proceed with the proof of (12) in the general case. In view of (6), it is sufficient to prove (12) for large n. So let E be an arbitrary set consisting of a finite number of intervals: $E = \cup_{j=1}^{l}[a_{2j-1}, a_{2j}]$. By Proposition 2 we can choose admissible polynomials T_N such that the inverse image set $E' = T_N^{-1}[-1,1] = \cup_{j=1}^{l}[a'_{2j-1}, a'_{2j}]$ consists of l intervals and it lies arbitrary close to E. For a given j we may choose a_j to be an endpoint of E' (i.e. $a'_j = a_j$), and we may also have $E' \subset E$. For the numbers τ_i in (10) it is clear that they are C^∞-functions of the endpoints a_j. But then, if M'_{a_j} is the quantity (11) for E' and the corresponding τ_i are denoted by τ_i', given $\varepsilon > 0$, we have $M'_{a_j} \le (1+\varepsilon)M_{a_j}$ if E' lies sufficiently close to E.

Let $E'_s = [\alpha'_s, \beta'_s]$ be the intervals for E' from the beginning of this section (so that $T_N(E'_s) = [-1,1]$), and assume that $a_j \in E'_{s_0}$. Then a_j is the right endpoint of $[\alpha'_{s_0}, \beta'_{s_0}]$, i.e. $a_j = \beta'_{s_0}$. Assume that $\eta > 0$ is so small that $[a_j - 2\eta, a_j] \subset E'_{s_0}$. By Theorem VI.3.6 of [12], there are polynomials $L_{\sqrt{n}}$ of degree at most[1] $[\sqrt{n}]$ such that with some constants $0 < \beta < 1$ and C we have

$$\begin{aligned} &0 \le L_{\sqrt{n}}(x) \le 1, && \text{for } x \in E', \\ &0 \le 1 - L_{\sqrt{n}}(x) \le C\beta^{\sqrt{n}}, && \text{for } x \in [a_j - \eta, a_j], \\ &0 \le L_{\sqrt{n}}(x) \le C\beta^{\sqrt{n}}, && \text{for } x \in E' \setminus E'_{s_0}. \end{aligned}$$

[1] $[\cdot]$ denotes integral part

For an arbitrary polynomial P_n consider $P_n^*(x) = L_{\sqrt{n}}(x)P_n(x)$, which has degree at most $n+[\sqrt{n}]$ and which satisfies

$$\begin{aligned}
\|P_n^*\|_{E'} &\leq \|P_n\|_{E'}, \\
P_n^*(x) &= (1+O(\beta^{\sqrt{n}}))P_n(x), \qquad \text{for } x \in [a_j-\eta, a_j], \\
P_n^*(x) &= O(\beta^{\sqrt{n}})\|P_n\|_{E'}, \qquad \text{for } x \in E' \setminus E'_{s_0}.
\end{aligned} \tag{17}$$

Now

$$\begin{aligned}
(P_n^*)^{(k)}(x) &= (L_{\sqrt{n}}P_n)^{(k)}(x) \\
&= L_{\sqrt{n}}(x)P_n^{(k)}(x) + \sum_{i=1}^{k} \binom{k}{i} L_{\sqrt{n}}^{(i)}(x)P_n^{(k-i)}(x),
\end{aligned}$$

and so

$$(P_n^*)^{(k)}(x) - P_n^{(k)}(x) = (L_{\sqrt{n}}(x)-1)P_n^{(k)}(x) + \sum_{i=1}^{k} \binom{k}{i} L_{\sqrt{n}}^{(i)}(x)P_n^{(k-i)}(x).$$

In view of (6) there exists a constant C_1 (that may depend on E') such that for all $x \in E'$ and $1 \leq i \leq k$

$$\begin{aligned}
&|L_{\sqrt{n}}^{(i)}(x)| \leq C_1(\sqrt{n})^{2i}\|L_{\sqrt{n}}\|_{E'} = C_1 n^i \|L_{\sqrt{n}}\|_{E'} \leq C_1 n^i, \\
&|P_n^{(i)}(x)| \leq C_1 n^{2i}\|P_n\|_{E'},
\end{aligned}$$

and, in addition, on $E' \setminus E'_{s_0} = E' \setminus [\alpha'_{s_0}, \beta'_{s_0}]$

$$|L_{\sqrt{n}}^{(i)}(x)| \leq C_1(\sqrt{n})^{2i}\|L_{\sqrt{n}}\|_{E' \setminus E'_{s_0}} \leq C_1 n^i \beta^{\sqrt{n}}.$$

These show that we have

$$|(P_n^*)^{(k)}(x) - (P_n)^{(k)}(x)| = O\Big(n^{2k}\beta^{\sqrt{n}} + n^{2k-1}\Big)\|P_n\|_{E'}, \quad x \in [a_j-\eta, a_j], \tag{18}$$

and

$$|(P_n^*)^{(k)}(x)| = O\Big(n^{2k}\beta^{\sqrt{n}}\Big)\|P_n\|_{E'}, \qquad \text{uniformly for } x \in E' \setminus E'_{s_0}. \tag{19}$$

We denote by $T_{N,i}^{-1}$ the branch of T_N^{-1} that maps $[-1,1]$ onto E'_i. If we define

$$S(x) = \sum_{i=1}^{N} P_n^*(T_{N,i}^{-1}(T_N(x))),$$

then $S(x)$ is a polynomial of degree at most $\deg(P_n^*)/N \leq (n+\sqrt{n})/N$ of $T_N(x)$, see [14, Section 5]. Thus, the degree of S is at most $[(n+\sqrt{n})/N]N \leq n+\sqrt{n}$. Let $x \in [a_j-\eta, a_j]$. When $i = s_0$ then

$$P_n^*(T_{N,i}^{-1}(T_N(x))) = P_n^*(x),$$

and for all $i \neq s_0$ the points $T_{N,i}^{-1}(T_N(x))$ belong to the set $E' \setminus E'_{s_0}$. We shall prove in the next subsection that for all sufficiently large n

$$\left| S^{(k)}(x) - (P_n^*)^{(k)}(x) \right| \leq C_2(\sqrt{\beta})^{\sqrt{n}} \|P_n\|_{E'}, \qquad x \in [a_j - \eta, a_j], \tag{20}$$

with a constant C_2 independent of $x \in [a_j - \eta, a_j]$ and n.

By the properties of P_n^* (see (17)) and also by the fact that out of $T_{N,i}^{-1}(T_N(x))$, $1 \leq i \leq N$, only one can belong to $E'_{s_0} = [\alpha'_{s_0}, \beta'_{s_0}]$, we have

$$\|S\|_{E'} \leq (1 + O(\beta^{\sqrt{n}}))\|P_n\|_{E'} \leq (1 + O(\beta^{\sqrt{n}}))\|P_n\|_E \tag{21}$$

(recall that $E' \subseteq E$). Therefore, we get from (20) and (18)

$$\begin{aligned}
\|P_n^{(k)}\|_{[a_j-\eta,a_j]} &\leq \|S^{(k)}\|_{[a_j-\eta,a_j]} + O((\sqrt{\beta})^{\sqrt{n}} + n^{2k-1})\|P_n\|_{E'} \\
&\leq (1+o(1))\frac{(M'_{a_j})^k}{(2k-1)!!}(\deg(S))^{2k}\|S\|_{E'} + O((\sqrt{\beta})^{\sqrt{n}} + n^{2k-1})\|P_n\|_{E'} \\
&\leq (1+o(1))\frac{(M'_{a_j})^k}{(2k-1)!!}n^{2k}\|S\|_{E'} \\
&\leq (1+o(1))(1+\varepsilon)^k\frac{M_{a_j}^k}{(2k-1)!!}n^{2k}\|P_n\|_E,
\end{aligned}$$

where in the second inequality we used the special case of the theorem (applied to E' and to S) that we proved in the first part of this section, in the third inequality that $\deg(S) \leq [(n + \sqrt{n})/N]\, N \leq n + \sqrt{n}$, and in the last inequality we used that $E' \subset E$ and $M'_j \leq (1+\varepsilon)M_{a_j}$. Since $\varepsilon > 0$ is arbitrary, we obtain (12) (with δ replaced by η which is permitted by (8)).

In order to prove (13), we select a polynomial inverse image set $E' = T_N^{-1}[-1,1]$, $E \subseteq E'$, consisting of l intervals that lies close to E for which a_j is an endpoint, and for which M'_{a_j} is close to M_{a_j}, say $M'_{a_j} \geq M_{a_j}(1-\varepsilon)$ for some given $\varepsilon > 0$. Let $\mathcal{T}_m = \cos(m \arccos x)$ be the classical Chebyshev polynomials and set $P_n := \mathcal{T}_m(T_N)$. Since $|\mathcal{T}_m^{(k)}(\pm 1)| = C_{m,k}$ (see (5)) and $|T'_N(a_j)| = M'_{a_j}N^2$, we get for $n = mN$ as before

$$|P_n^{(k)}(a_j)| = |(\mathcal{T}_m(T_N))^{(k)}(a_j)| = (1+o(1))C_{m,k}N^{2k}(M'_j)^k,$$

and here

$$C_{m,k}N^{2k}(M'_j)^k \geq (1+o(1))\frac{m^{2k}}{(2k-1)!!}N^{2k}M_{a_j}^k(1-\varepsilon)^k.$$

Since $E \subset E'$ we have

$$\|P_n\|_E \leq \|P_n\|_{E'} = \|\mathcal{T}_m\|_{[-1,1]} = 1,$$

and so from $n = mN$ we get

$$|P_n^{(k)}(a_j)| \geq (1+o(1))(1-\varepsilon)^k\frac{n^{2k}}{(2k-1)!!}M_{a_j}^k\|P_n\|_E.$$

This is only for integers n of the form $n = mN$. For others just use $P_{N[n/N]}$ as P_n, where $[\cdot]$ denotes integral part. Since here $\varepsilon = \varepsilon_N > 0$ is arbitrary, (13) follows if we let N tend to ∞ slowly (and at the same time $T_N^{-1}[-1,1]$ close to E) as $n \to \infty$ (in which case we have $\varepsilon_N \to 0$).

Proof of (20)

The preceding proof used (20), and now we proceed with its proof. We keep the notations used before.

Let $x \in (a_j - \eta, a_j)$, and for an $i \neq s_0$ let $T_{N,i}^{-1}(T_N(a_j)) = \gamma_s$ (it is one of the endpoints of a subinterval of E_s', $s \neq s_0$). Since a_j is an endpoint of a subinterval of E', we have $T_N'(a_j) \neq 0$, hence close to a_j

$$|T_N(x) - T_N(a_j)| \sim |x - a_j|,$$

where $T_N(a_j) = \pm 1$ and $A \sim B$ means that the ratio A/B remains in between two positive constants. In a similar manner, if γ_s is an endpoint of a subinterval of E' then

$$|T_N(y) - T_N(\gamma_s)| \sim |y - \gamma_s|,$$

for y lying close to γ_s. However, if γ_s is an interior point of E', then T_N' has a simple zero at γ_s, therefore

$$|T_N(y) - T_N(\gamma_s)| \sim |y - \gamma_s|^2,$$

for y lying close to γ_s. These imply that in $[a_j - \eta, a_j]$

$$|T_{N,i}^{-1}(T_N(x)) - \gamma_s| \sim \begin{cases} |x - a_j|^{1/2} & \text{if } \gamma_s = T_{N,i}^{-1}(T_N(a_j)) \text{ is not an endpoint of } E' \\ |x - a_j| & \text{otherwise} \end{cases} \tag{22}$$

Note also that T_N' has a simple zero or no zero at γ_s depending on if γ_s is not an endpoint of E' or it is.

Differentiation gives

$$\frac{d}{dx}\left(T_{N,i}^{-1}(T_N(x))\right) = \frac{T_N'(x)}{T_N'(T_{N,i}^{-1}(T_N(x)))},$$

$$\frac{d^2}{dx^2}\left(T_{N,i}^{-1}(T_N(x))\right) = \frac{-(T_N'(x))^2}{(T_N'(T_{N,i}^{-1}(T_N(x))))^3} + \frac{T_N''(x)}{T_N'(T_{N,i}^{-1}(T_N(x)))},$$

and in general we obtain that

$$\frac{d^m}{dx^m}\left(T_{N,i}^{-1}(T_N(x))\right) = \frac{Q_{N,m}(x)}{(T_N'(T_{N,i}^{-1}(T_N(x))))^{2\nu-1}}$$

with some $Q_{N,m}$ built up from $T_N^{(\nu)}(x)$ and $T_N^{(\nu)}(T_{N,i}^{-1}(T_N(x)))$, $1 \leq \nu \leq m$ using multiplication and addition. Hence, in view of (22) and of what we said about the derivative of T_N' at the point $\gamma_s = T_{N,i}^{-1}(T_N(a_j))$, it follows that

$$\left|\frac{d^m}{dx^m}\left(T_{N,i}^{-1}(T_N(x))\right)\right| \leq \frac{C}{|x - a_j|^{(2m-1)/2}} \leq \frac{C}{|x - a_j|^m} \tag{23}$$

with a C (that may depend on T_N and m). By the Faà di Bruno formula (15) the k-th derivative of $P_n^*(T_{N,i}^{-1}(T_N(x)))$ is a combination of terms of the form

$$(P_n^*)^{(m_1+\cdots+m_k)}(T_{N,i}^{-1}(T_N(x)))\prod_{\nu=1}^{k}\frac{d^{m_\nu}}{dx^{m_\nu}}\left(T_{N,i}^{-1}(T_N(x))\right)$$

with $m_1+2m_2+\cdots+km_k\le k$. Therefore, we obtain from (19) (apply it not just for the k-th, but also to lower order derivatives of P_n^*) and (23) that for $i\ne s_0$

$$\left|\frac{d^k}{dx^k}P_n^*(T_{N,i}^{-1}(T_N(x)))\right|\le\frac{C_1n^{2k}\beta^{\sqrt{n}}}{|x-a_j|^k}\|P_n\|_{E'}.$$

Let now $\theta<1$ be such that $\theta^k>\sqrt{\beta}$. The preceding estimate gives for $x\in[a_j-\eta,a_j-\theta^n]$ (provided $\theta^n<\eta$)

$$\left|\frac{d^k}{dx^k}P_n^*(T_{N,i}^{-1}(T_N(x)))\right|\le C_1n^{2k}\beta^{\sqrt{n}}\theta^{-kn}\|P_n\|_{E'}\le C_1(\sqrt{\beta})^{\sqrt{n}}\|P_n\|_{E'}.$$

What we have obtained is that

$$|S_n^{(k)}(x)-(P_n^*)^{(k)}(x)|=\left|\sum_{i\ne s_0}\frac{d^k}{dx^k}P_n^*(T_{N,i}^{-1}(T_N(x)))\right|\le NC_1(\sqrt{\beta})^{\sqrt{n}}\|P_n\|_{E'} \tag{24}$$

on the interval $[a_j-\eta,a_j-\theta^n]$, where C_1 may depend on T_N and k. We want to conclude that

$$\|S_n^{(k)}-(P_n^*)^{(k)}\|_{[a_j-\eta,a_j]}\le 2NC_1(\sqrt{\beta})^{\sqrt{n}}\|P_n\|_{E'}. \tag{25}$$

To do that we recall Remez' inequality (see [5, Lemma 7.3]): if R_n is a polynomial of degree at most n and $m(R_n)$ is the measure of those $x\in[-1,1]$ where $|R_n(x)|\le 1$, then

$$\|R_n\|_{[-1,1]}\le\mathcal{T}_n\left(\frac{4}{m(R_n)}-1\right), \tag{26}$$

where $\mathcal{T}_n(t)=\cos(n\arccos t)$ are the classical Chebyshev polynomials. In view of

$$\mathcal{T}_n\left(y\right)=\frac{1}{2}\left((y+\sqrt{y^2-1})^n+(y-\sqrt{y^2-1})^n\right),$$

a transformation of (26) yields that there is a $c_0>0$ such that for any polynomial R_n of degree at most n and for any interval I the inequality

$$\|R_n\|_I\le 2\|R_n\|_{I\setminus J}$$

is true provided the linear measure of $J\subseteq I$ is $\le c_0|I|/n^2$. Thus, for large n the inequality (25) is, indeed, a consequence of (24) (which is true uniformly in n on $[a_j-\eta,a_j-\theta^n]$), and (25) is nothing else than (20).

■

3 General compact sets on R

In this section, we will consider a compact set $E \subset \mathbf{R}$. We say that $a \in E$ is a right endpoint of E if there is a ρ such that $[a-2\rho, a] \subset E$, but $(a, a+2\rho) \cap E = \emptyset$.

As before, for a given $k \geq 1$ the asymptotic Markov factor of order k for E at such an endpoint a is the smallest number $\mathcal{M}_{a,k}$ such that

$$\|P_n^{(k)}\|_{[a-\rho,a]} \leq \mathcal{M}_{a,k} n^{2k} n^{2k} \|P_n\|_E \tag{27}$$

is satisfied for all $P_n \in \mathcal{P}_n$. In this section we determine this $\mathcal{M}_{a,k}$. To do that recall that the equilibrium measure of E is absolutely continuous on $[a-2\rho, a]$ and its density ω_E is defined there. This ω_E has a $1/\sqrt{t}$ type behavior at a, and we define

$$M_a = M_a(E) := 2\pi^2 \lim_{t \to a-0} \omega_E^2(t)|t-a|.$$

This quantity exists (see [6, Lemma 2.1]), and has already been used in the paper [6]. It is immediate from (10) that if E consists of finitely many intervals $[a_{2j-1}, a_{2j}]$ and $a = a_{2j}$, then this M_a is the $M_{a_{2j}}$ defined in (11). Therefore, the following theorem is an extension of Theorem 1.

Theorem 3. *If E is a compact subset of $\mathbf{R}$ and a is a right endpoint of E, then for fixed $k \geq 1$ and $P_n \in \mathcal{P}_n$, we have (for any small fixed $\rho > 0$)*

$$\|P_n^{(k)}\|_{[a-\rho,a]} \leq (1+o(1)) \frac{M_a^k n^{2k}}{(2k-1)!!} \|P_n\|_E, \tag{28}$$

where $o(1)$ tends to 0 as $n \to \infty$. Furthermore, this is asymptotically the best estimate, for there is a sequence $\{P_n \in \mathcal{P}_n\}_{n=1}^{\infty}$ of nonzero polynomials such that

$$|P_n^{(k)}(a)| \geq (1+o(1)) \frac{M_a^k n^{2k}}{(2k-1)!!} \|P_n\|_E. \tag{29}$$

Thus, for the best asymptotic Markov factor $\mathcal{M}_{a,k}$ in (27) we have

$$\mathcal{M}_{a,k} = \frac{M_a^{2k}}{(2k-1)!!}.$$

Proof. First we prove (28), and in doing so first we assume that E is regular with respect to the Dirichlet problem in $\overline{\mathbf{C}} \setminus E$.

Fix $\varepsilon > 0$, and let $J := [\min E, \max E]$ be the smallest interval that contains E. There exist a $0 < \tau < 1$ and for each large n polynomials $Q_{n\varepsilon}$ of degree not larger than $[n\varepsilon]$ such that

a) $1 - e^{-n\tau} \leq Q_{n\varepsilon} \leq 1$ if $x \in [a-\rho, a+\rho]$,

b) $0 \leq Q_{n\varepsilon}(x) \leq 1$ if $x \in [a-3\rho/2, a-\rho] \cup [a+\rho, a+3\rho/2]$,

c) $0 \leq Q_{n\varepsilon}(x) \leq e^{-n\tau}$ if $x \in J \setminus [a-3\rho/2, a+3\rho/2]$

(see for example, [12, Corollary VI.3.6]). We may assume that E is not a finite union of intervals, for in that case we can apply Theorem 1. Since $\mathbf{R} \setminus E$ is an open set, we have $\mathbf{R} \setminus E = \cup_{j=1}^{\infty} I_j$, where I_j are disjoint open intervals. We assume that I_0 and I_1 are the unbounded subintervals of $\mathbf{R} \setminus E$. For $m > 0$ consider the set

$$E_m := \mathbf{R} \setminus (\cup_{j=0}^{m} I_j). \tag{30}$$

This set contains E and is of the form

$$E_m = \cup_{j=1}^{m} [a_{j,m}, b_{j,m}]$$

with $a_{1,m} < b_{1,m} < a_{2,m} < \cdots < a_{m,m} < b_{m,m}$. For sufficiently large m the point a is a right endpoint of E_m, and by Proposition 2.3 of [6] we have

$$\lim_{m \to \infty} M_a(E_m) = M_a(E). \tag{31}$$

Let g_E denote the Green's function of $\overline{\mathbf{C}} \setminus E$ with pole at infinity. The regularity of E guarantees that g_E is continuous and vanishes on E. Therefore, there exists $0 < \theta < 1$, $\theta = \theta(\tau)$, such that

$$\text{if } \ x \in \mathbf{R}, \ \operatorname{dist}(x, E) \le \theta, \ \text{ then } \ g_E(x) \le \tau^2.$$

Choose m sufficient large such that $\operatorname{dist}(x, E) < \theta$ for all $x \in E_m$. Let P_n be an arbitrary polynomial of degree at most n. We apply Theorem 1 for the polynomial $P_n Q_{n\varepsilon}$ on E_m. If $x \in E$, then, by the properties of $Q_{n\varepsilon}$, we have $|P_n(x) Q_{n\varepsilon}(x)| \le \|P_n\|_E$. On the other hand, if $x \in E_m \setminus E$, then, by the Bernstein-Walsh lemma ([16, p. 77]) and by property c) of $Q_{n\varepsilon}$,

$$\begin{aligned} |P_n(x) Q_{n\varepsilon}(x)| &\le \|P_n\|_E \exp(n g_E(x)) \exp(-n\tau) \\ &\le \|P_n\|_E \exp(n\tau^2) \exp(-n\tau) < \|P_n\|_E. \end{aligned}$$

Therefore

$$\|P_n Q_{n\varepsilon}\|_{E_m} \le \|P_n\|_E. \tag{32}$$

For $x \in [a - \rho, a]$

$$|(P_n Q_{n\varepsilon})^{(k)}(x)| \ge |P_n^{(k)}(x) Q_{n\varepsilon}(x)| - \sum_{j=1}^{k} \binom{k}{j} |P_n^{(k-j)}(x) Q_{n\varepsilon}^{(j)}(x)|.$$

Here $1 - e^{-n\tau} \le Q_{n\varepsilon}(x)| \le 1$, and by (6)

$$\|Q_{n\varepsilon}^{(j)}\|_E \le C(n\varepsilon)^{2j}, \qquad \|P_n^{(j)}\|_E \le C n^{2j} \|P_n\|_E$$

with some constant C for all $j = 1, 2, \cdots, k$. Hence, when $x \in [a - \rho, a]$, we get from Theorem 1 when applied to the polynomial $P_nQ_{\varepsilon n}$ and to the set E_m

$$\begin{aligned}
|P_n^{(k)}(x)|(1-e^{-n\tau}) &\le |(P_nQ_{n\varepsilon})^{(k)}(x)| + \sum_{j=1}^{k}\binom{k}{j}C^2\|P_n\|_E n^{2(k-j)}(n\varepsilon)^{2j} \\
&\le (1+o(1))\frac{((1+\varepsilon)n)^{2k}}{(2k-1)!!}M_a(E_m)^k\|P_nQ_{n\varepsilon}\|_{E_m} \\
&\qquad +\|P_n\|_E C_1\varepsilon^2 n^{2k} \\
&\le \frac{n^{2k}}{(2k-1)!!}\|P_n\|_E\Big((1+o(1))(1+\varepsilon)^2 M_a(E_m)^k + C_1\varepsilon^2\Big)
\end{aligned}$$

Since $\varepsilon > 0$ and m are arbitrary, the inequality (28) follows if we apply (31).

The preceding proof used the regularity of E. In order to remove that, we use a theorem of Ancona [1]. Let $E \subset \mathbf{R}$ be a compact set of positive logarithmic capacity. For each l, there exists a regular compact set $E_l^- \subset E$ such that

$$\operatorname{cap}(E) \le \operatorname{cap}(E_l^-) + \frac{1}{l}.$$

Because the union of two regular compact sets is regular, we may assume that $[a-2\rho, a] \subseteq E_m^-$.

According to what we have proven,

$$\begin{aligned}
\|P_n^{(k)}\|_{[a-\rho,a]} &\le (1+o(1))n^{2k}\frac{M_a(E_l^-)^k}{(2k-1)!!}\|P_n\|_{E_l^-} \\
&\le (1+o(1))n^{2k}\frac{M_a(E_l^-)^k}{(2k-1)!!}\|P_n\|_E,
\end{aligned}$$

Since $M_a(E_l^-)$ can be made arbitrarily close to $M_a(E)$ by choosing l large enough (see [6, Proposition 2.3] and its proof), the inequality (28) follows.

Finally, we prove (29). We are going to select a sequence of polynomials $\{P_n\}_{n=1}^\infty$ with $\deg(P_n) = n$, such that

$$\lim_{n\to\infty}\frac{|P_n^{(k)}(a)|(2k-1)!!}{n^{2k}\|P_n\|_E} = M_a(E)^k. \tag{33}$$

Consider the set E_m from (30) for such a large m, for which a is already a right endpoint of E_m. This E_m is the union of finitely many closed intervals some of them may be a singleton. Replace each such point in E_m by an interval of length less than $1/m$, and denote the resulting set again by E_m, which consists of non-degenerated intervals. We can use the result in the previous section for this E_m: there exists a sequence $\{P_{m,n}\}_{n=1}^\infty$, $\deg(P_{m,n}) \le n$, of nonzero polynomials such that

$$|P_{m,n}^{(k)}(a)| \ge (1-o_{E_m}(1))M_a(E_m)^k\frac{n^{2k}}{(2k-1)!!}\|P_{m,n}\|_{E_m},$$

where $o_{E_m}(1)$ depends on E_m and it tends 0 as $n \to \infty$ for any fixed m. Since $E \subset E_m$, we have $\|P_{m,n}\|_{E_m} \geq \|P_{m,n}\|_E$, and hence

$$|P_{m,n}^{(k)}(a)| \geq (1 - o_{E_m}(1)) M_a(E_m)^k \frac{n^{2k}}{(2k-1)!!} \|P_{m,n}\|_E.$$

By choosing m sufficiently large, $M_a(E_m)$ can be made arbitrarily close to $M_a(E)$ (see Proposition 2.3 and its proof in [6]), and then (33) follows for $P_n := P_{m_n,n}$ if m_n goes slowly to infinity as $n \to \infty$.

■

References

[1] A. Ancona, Sur une conjecture concernant la capacité et l'effilement, in: In théorie du potentiel, Lecture notes in Math., **1096**, Springer-Verlag, Berlin, 1984, 34–68.

[2] S. N. Bernstein, On the best approximation of continuos functions by polynomials of given degree, (O nailuchshem problizhenii nepreryvnykh funktsii posredstrvom mnogochlenov dannoi stepeni), *Sobraniye sochinenii*, Vol. I(1912), 11–104, *Izd. Akad. Nauk SSSR*, Vol. I(1952), Vol. II(1954).

[3] R. A. DeVore and G. G. Lorentz, *Constructive approximation*, Grundlehren der mathematischen Wissenschaften, **303**, Springer-Verlag, Berlin, Heidelberg, New York 1993.

[4] C. F. Faà di Bruno, Note sur une nouvelle formule de calcul differentiel, *Quarterly J. Pure Appl. Math.*, **1**(1857), 359–360.

[5] G. Freud, *Orthogonal polynomials*, Akadémiai Kiadó, Budapest 1971.

[6] S. Kalmykov, B. Nagy and V. Totik, Asymptotically sharp Markov and Schur inequalities on general sets, *Complex Anal. Oper. Theory*, **9**(2015), 1287–1302.

[7] A. A. Markov, Sur une question posée par D.I. Mendeleieff, *Izv. Akad. Nauk St. Petersburg*, **62**(1889), 1–24.

[8] V.A. Markov, On functions of least deviation from zero in a given interval, (1892), (Russian). Appeared in German with a foreword by Sergei Bernstein as "V. A. Markov, Über Polynome, die in einem gegebenen Intervalle möglichst wenig von Null abweichen", *Math. Ann.*, **77**(1916), 213–258.

[9] G. V. Milovanovic, D. S. Mitrinovic and Th. M. Rassias, *Topics in polynomials: extremal problems, inequalities, zeros*, World Scientific Publishing Co., Inc., River Edge, NJ, 1994.

[10] T. Ransford, *Potential theory in the complex plane*, London Mathematical Society Student Texts, **28**, Cambridge University Press, Cambridge, 1995.

[11] J. Riordan, *An introduction to combinatorial analysis*, Wiley, New York, 1958.

[12] E. B. Staff and V. Totik, *Logarithmic potentials with external fields*, Grundlehren Math. Wiss., **316**, Springer-Verlag, Berlin, 1997.

[13] V. Totik, Polynomial inverse images and polynomial inequalities, *Acta Math.*, **187**(2001), 139–160.

[14] V. Totik, The polynomial inverse image method, in: M. Neamtu, L. Schumaker (Eds.), Approximation Theory XIII: San Antonio 2010, Springer Proceedings in Mathematics, Vol. 13, Springer, 2012, 345–367.

[15] M. Tsuji, *Potential theory in modern function theory*, Maruzen, Tokyo, 1959.

[16] J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain*, third edition, Amer. Math. Soc. Coll. Publ., **XX**, Amer. Math. Soc., Providence, 1960.

Vilmos Totik
MTA-SZTE Analysis and Stochastics Research Group, Bolyai Institute, University of Szeged, Szeged, Aradi v. tere 1, 6720, Hungary
and
Department of Mathematics and Statistics, University of South Florida, 4202 E. Fowler Ave. CMC342, Tampa, FL 33620
email address: `totik@mail.usf.edu`

Yuan Zhou
Department of Mathematics and Statistics, University of South Florida, 4202 E. Fowler Ave. CMC342, Tampa, FL 33620
email address: `zhouy@mail.usf.edu`