A sharp L^p-Bernstein inequality on finitely many intervals *

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Abstract

An asymptotically sharp Bernstein-type inequality is proven for trigonometric polynomials in integral metric. This extends Zygmund's classical inequality on the L^p norm of the derivatives of trigonometric polynomials to the case when the set consists of several intervals. The result also contains a recent theorem of Nagy and Toókos, who proved a similar statement for algebraic polynomials.

1 Introduction

In a recent paper B. Nagy and F. Toókos [7] proved an asymptotically sharp form of Bernstein's inequality for algebraic polynomials in integral metric on sets consisting of finitely many intervals. In the present paper we propose an analogue of their inequality for trigonometric polynomials, which, using the standard $x = \cos t$ substitution, gives back the Nagy-Toókos inequality.

S. N. Bernstein's famous inequality

$$\|T_n'\|_{\sup} \le n\|T_n\|_{\sup}$$

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for trigonometric polynomials T_n of degree at most $n = 1, 2, \ldots$ was proved in 1912. It was extended by Videnskii [15] in 1960 to intervals less than a whole period: if $0 < \beta < \pi$ then

$$|T_n'(\theta)| \le n \frac{\cos\theta/2}{\sqrt{\sin^2\beta/2 - \sin^2\theta/2}} \|T_n\|_{[-\beta,\beta]}, \qquad \theta \in (-\beta, \beta). \tag{1}$$

Here, and in what follows, $\|\cdot\|_E$ denotes the supremum norm on the set E.

The general form of Videnskii's inequality for an arbitrary system of intervals is due to A. Lukashov [4]. For a set $E \subset (-\pi, \pi]$ let

$$\Gamma_E = \{e^{it} \,|\, t \in E\}$$

be the set that corresponds to E when we identify $(-\pi, \pi]$ with the unit circle C_1, and let ω_{Γ_E} denote the density of the equilibrium measure of Γ_E, where the density is taken with respect to arc measure on C_1. See [1], [3], [9] or [10] for the potential theoretical concepts (such as equilibrium measure, balayage etc.) used in this work. With this notation A. Lukashov's result [4] can be stated as follows. Let $E \subset (-\pi, \pi]$ consist of finitely many intervals. If $e^{i\theta}$ is an inner point of Γ_E, then for any trigonometric polynomial T_n of degree at most $n = 1, 2, \ldots$ we have

$$|T_n'(\theta)| \le n 2\pi \omega_{\Gamma_E}(e^{i\theta}) \|T_n\|_E. \tag{2}$$

The L^p, $1 \le p < \infty$, extension of Bernstein's inequality in the form

$$\|T_n'\|_{L^p} \le n \|T_n\|_{L^p} \tag{3}$$

was given in [14, Ch. X, (3∘16)Theorem] by A. Zygmund (here the L^p norm is taken on the whole period, i.e. $\|\cdot\|_{L^p} \equiv \|\cdot\|_{L^p[-\pi,\pi]}$). The main purpose of this paper is to find a form of this inequality on a finite system of intervals (mod 2π). We state

Theorem 1.1 *Let $1 \le p < \infty$, and assume that $E \subset (0, 2\pi]$ consists of finitely many intervals. Then for trigonometric polynomials T_n of degree at most n we have*

$$\int_E \left| \frac{T_n'(t)}{n 2\pi \omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it})\, \mathrm{d}t \le (1 + o(1)) \int_E |T_n(t)|^p \omega_{\Gamma_E}(e^{it})\, \mathrm{d}t, \tag{4}$$

where $o(1)$ tends to zero uniformly in T_n as n tends to ∞.

If $E = (0, 2\pi]$ then $\omega_{\Gamma_E}(e^{it}) \equiv 1/2\pi$, so we get back Zygmund's inequality (with the factor $(1 + o(1))$).

We also mention that the result is sharp: there are trigonometric polynomials $T_n \not\equiv 0$ of degree $n = 1, 2, \dots$ for which

$$\int_E \left| \frac{T_n'(t)}{n2\pi\omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it})\,\mathrm{d}t \geq (1 - o(1)) \int_E |T_n(t)|^p \omega_{\Gamma_E}(e^{it})\,\mathrm{d}t. \qquad (5)$$

This follows from the use of T-sets below in the same fashion as Theorem 4 follows in [7, Sec. 7] from the use of polynomial inverse images. We do not give details.

Let now $K \subset \mathbf{R}$ be a set consisting of finitely many intervals, which we may assume to lie in $[-1, 1]$. Let ω_K denote the density of the equilibrium measure of K with respect to linear Lebesgue-measure.

Set $E = \{t \in (-\pi, \pi] \mid \cos t \in K\}$, and for an algebraic polynomial P_n of degree at most n consider the trigonometric polynomial $T_n(t) = P_n(\cos t)$. In this case it is known [11, (4.12)] that

$$\omega_{\Gamma_E}(e^{it}) = \frac{1}{2}\omega_K(\cos t)|\sin t|. \qquad (6)$$

If we substitute this into (4) applied to $T_n(t) = P_n(\cos t)$, then we obtain the inequality

$$\int_K \left| \frac{P_n'(x)}{n\pi\omega_K(x)} \right|^p \omega_K(x)\,\mathrm{d}x \leq (1 + o(1)) \int_K |P_n(x)|^p \omega_K(x)\,\mathrm{d}x, \qquad (7)$$

which is the Nagy–Toókos result from [7] mentioned before. As far as we know this latter inequality is the only Bernstein-type inequality with an asymptotically sharp factor that is known on general sets. Although, as we have just shown, (7) is a special case of Theorem 1.1, the present paper was motivated by the inequality of Nagy and Toókos, and the resemblance of (4) with (7) is obvious. Besides that, we shall closely follow the proof of (7) from [7], which was based on the polynomial inverse image method. We shall replace here polynomial inverse images of intervals by their trigonometric analogues, the so called T-sets of F. Peherstorfer and R. Steinbauer [8] and S. Khruschev [5],[6]. We shall be rather brief, for we are not going to repeat the technical steps that are identical with those in [7].

2 Proof of Theorem 1.1

When $E = (-\pi, \pi]$, then the statement in Theorem 1.1 is included in Zygmund's inequality (3), hence we may assume that $E \neq (-\pi, \pi]$, and then, by the periodicity of trigonometric polynomials, that $-\pi, \pi \notin E$, i.e. that E is a closed subset of $(-\pi, \pi)$.

After Peherstorfer and Steinbauer [8] we call a closed set $E \subset (-\pi, \pi)$ a *T-set of order N* if there is a real trigonometric polynomial U_N of degree N such that $U_N(t)$ runs through $[-1, 1]$ $2N$-times as t runs through E. In this case we shall say that E is associated with U_N. Note that the definition implies that if $U_N'(t_0) = 0$ then $|U_N(t_0)| \geq 1$. An interval $[\zeta_1, \zeta_2]$ is a *"branch"* of E if $|U_N(\zeta_1)| = |U_N(\zeta_2)| = 1$ and U_N runs through $[-1, 1]$ precisely once as t runs through $[\zeta_1, \zeta_2]$. This implies that $U_N(\zeta_1) = -U_N(\zeta_2)$. If, furthermore, $U_N'(\zeta_1) = 0$, then we say that ζ_1 is an *inner extremal point* since in this case ζ_1 is necessarily in the interior of E.

The proof of Theorem 1.1 consists of the following steps.

(a) Verify the statement when E is a T-set associated with the trigonometric polynomial U_N and T_n is a polynomial of U_N.

(b) Verify the statement when E is a T-set, and the trigonometric polynomial T_n is arbitrary.

(c) Verify the statement when $E \subset (-\pi, \pi)$ is an arbitrary set consisting of finitely many closed intervals.

These are precisely the steps Nagy and Toókos used, but they used instead of T-sets polynomial inverse images of intervals under a suitable algebraic polynomial mapping.

First we verify **(a)**. Thus, let E be a T-set of degree N associated with the trigonometric polynomial U_N, and assume that $T_n = P_m(U_N)$, where P_m is an algebraic polynomial of degree m. Then $n = Nm$ and $T_n'(t) = P_m'(U_N(t))U_N'(t)$. It is also known (see [5, (25)], [12, Lemma 3.1]) that

$$\omega_{\Gamma_E}(e^{it}) = \frac{1}{2\pi N} \frac{|U_N'(t)|}{\sqrt{1 - U_N(t)^2}}, \qquad t \in E. \tag{8}$$

Therefore,

$$\int_E \left| \frac{T_n'(t)}{n 2\pi \omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it})\, \mathrm{d}t = \int_E \left| \frac{P_m'(U_N(t))\sqrt{1 - U_N(t)^2}}{m} \right|^p \frac{|U_N'(t)|}{2\pi N \sqrt{1 - U_N(t)^2}}\, \mathrm{d}t.$$

In the last integral while t runs through a "branch" $[\zeta_1, \zeta_2]$, the trigonometric polynomial $U_N(t)$ runs through $[-1,1]$ exactly once, and there are $2N$ such "branches". So with $V_m(t) = P_m(\cos t)$ the last integral is equal to

$$\frac{1}{\pi}\int_{-1}^{1}\left|\frac{P_m'(x)\sqrt{1-x^2}}{m}\right|^p \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{V_m'(t)}{m}\right|^p \mathrm{d}t.$$

By Zygmund's result (3) this last expression is at most

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|V_m(t)|^p\,\mathrm{d}t,$$

which is equal to

$$\int_E |T_n(t)|^p \omega_{\Gamma_E}(e^{it})\,\mathrm{d}t$$

by doing the above substitutions backwards. This proves the case **(a)** of Theorem 1.1. Note that in this case the $(1+o(1))$ in (4) is actually 1.

In **(b)** the set E is still a T-set but T_n is an arbitrary trigonometric polynomial. This case will be discussed in the next section in more details because our proof differs in some points from [7, Sec. 5]. This is the technically most involved part of the proof.

Finally, in proving **(c)** one can follow the proof of [7, Sec. 6], if the subsequent lemmas are used.

Lemma 2.1 ([12, Lemma 3.4]) *Let*

$$E = \bigcup_{l=1}^{m}[v_{2l-1}, v_{2l}]$$

be finite interval-system in $(-\pi,\pi)$ $(v_i < v_{i+1})$. Then for every $\epsilon > 0$ there exist $0 < x_1, y_1, x_2, y_2, \ldots, x_m, y_m < \epsilon$ such that both

$$E^- := \bigcup_{l=1}^{m}[v_{2l-1}, v_{2l} - x_l]$$

and

$$E^+ := \bigcup_{l=1}^{m}[v_{2l-1}, v_{2l} + y_l]$$

are T-sets.

In other words, every finite interval-system can be approximated by T-sets. Note that the lemma tells nothing about the order of the approximating T-set, generally it converges to ∞.

Denote by ω_{Γ_E}, $\omega_{\Gamma_{E^+}}$ and $\omega_{\Gamma_{E^-}}$ the equilibrium measure of Γ_E, Γ_{E^+} and Γ_{E^-} respectively.

Lemma 2.2 *Both $\omega_{\Gamma_{E^+}}(e^{it})$ and $\omega_{\Gamma_{E^-}}(e^{it})$ converge to $\omega_{\Gamma_E}(e^{it})$ pointwise on E as $\epsilon \to 0$, moreover $\omega_{\Gamma_{E^+}}(e^{it})/\omega_{\Gamma_E}(e^{it})$ $(\omega_{\Gamma_{E^-}}(e^{it})/\omega_{\Gamma_E}(e^{it}))$ uniformly converges to 1 on sets of the form $[v_{2l-1}, v_{2l} - \delta_l]$ where $\delta_l > 0$ are arbitrarily fixed.*

We will indicate in Remark 3.10 below how to prove this lemma.

3 Proof of (b)

We shall follow the relevant arguments from [7], but we make some modifications.

First a general remark: whenever in [7] the authors write ω_K, in the trigonometric case one should write ω_{Γ_E}. Also, [7] used frequently the inequality

$$|P_n'(x)| \leq n\pi\omega_K(x)\|P_n\|_K, \qquad x \in \mathrm{Int}(K), \tag{9}$$

valid for algebraic polynomials P_n of degree at most n, and in the trigonometric case this should be replaced everywhere by the inequality (2).

Splitting the set E

This part is the same as [7, Sec. 4], but we shall need it for our discussion, therefore we give details. Suppose that the T-set $E \subset (-\pi, \pi)$ is the union of m disjoint intervals $[v_{2l-1}, v_{2l}]$, $l = 1, 2, \ldots, m$, that is:

$$E = \bigcup_{l=1}^{m} [v_{2l-1}, v_{2l}],$$

where $-\pi < v_{2l-1} < v_{2l} < v_{2l+1} < \pi$. Denote the inner extremal points in $[v_{2l-1}, v_{2l}]$ by $\zeta_{l,1} < \zeta_{l,2} < \cdots < \zeta_{l,r_l-1}$ and use the notation $\zeta_{l,0}$ and ζ_{l,r_l} for v_{2l-1} and v_{2l} respectively, where r_l refers to the number of the "branches" covering $[v_{2l-1}, v_{2l}]$.

Fix a number $\kappa \in (0, 1/8)$.

Split E into closed intervals I_j of length at least $1/2n^\kappa$ but at most $1/n^\kappa$ in such a way that each inner extremal point $\zeta_{l,i}$ is a division point, i.e. each "branch" $[\zeta_{l,i}, \zeta_{l,i+1}]$ of E is split up into the union of some of the I_j's separately. Let J_n be the set of indices for these intervals I_j. We assume that this enumeration is monotone, i.e. if $j < j'$ then I_j lies to the left of $I_{j'}$.

If $J \subset J_n$ is a subset of J_n then set

$$H(J) := \bigcup_{j \in J} I_j.$$

We shall consider these sets only for the case when $H = H(J)$ is an interval, in which case the "boundary" H_b of H be the union of the two intervals I_j attached to H. If, say, there is no I_j attached to H from the left (i.e. if H contains one of the left-endpoints v_{2l-1}), then as H_b we take the union of $[v_{2l-1} - 1/n^\kappa, v_{2l-1}]$ with the interval I_j attached to H from the right, and we use a similar procedure if H has no I_j attached to it from the right.

Now we enlist some properties, labelled by roman numbers, which $H = H(J)$ can possess and which will be important for us: H is strictly inside a "branch", that is

$$H \cup (H_b \cap E) \subset [\zeta_{l,i}, \zeta_{l,i+1}] \tag{I}$$

for some $l \in \{1, 2, \dots, m\}$ and $i \in \{0, 1, \dots, r_l - 1\}$ (recall that $\zeta_{l,0} = v_{2l-1}$ and $\zeta_{l,r_l} = v_{2l}$).

Before defining further properties we set up some notations:

$$A(T_n, X) := \int_X \left| \frac{T_n'(t)}{n 2\pi \omega_{\Gamma_E}(e^{it})} \right|^p \omega_{\Gamma_E}(e^{it})\, \mathrm{d}t, \tag{10}$$

$$B(T_n, X) := \int_X |T_n(t)|^p \omega_{\Gamma_E}(e^{it})\, \mathrm{d}t,$$

$$a(T_n, X) := \frac{A(T_n, X)}{A(T_n, E)},$$

and

$$b(T_n, X) := \frac{B(T_n, X)}{B(T_n, E)}. \tag{11}$$

With these quantities we need to prove that

$$A(T_n, E) \le (1 + o(1)) B(T_n, E).$$

Fix a

$$0 < \gamma < \frac{\kappa}{2}. \tag{12}$$

The next properties are

$$a(T, H_b) \le n^{-\gamma}, \tag{II-a}$$

and

$$b(T, H_b) \le n^{-\gamma}. \tag{II-b}$$

Note that, since

$$\sum_{j\in J_n} a(T_n, I_j) = \sum_{j\in J_n} b(T_n, I_j) = 1,$$

there are at most $2\lceil n^\gamma \rceil$ indices $j \in J_n$ such that $a(T_n, I_j) \ge n^{-\gamma}$ or $b(T_n, I_j) \ge n^{-\gamma}$. This number is small if we compare it to the number of the rest of the indices which is $\sim n^\kappa$. Therefore, if

$$J'_n := \left\{ j \in J_n \,\middle|\, \max\left(a(T_n, I_j), b(T_n, I_j)\right) < n^{-\gamma} \right\}, \tag{13}$$

then

$$|J_n \setminus J'_n| \le 4n^\gamma.$$

This implies that for large n every interval $[\zeta_{l,i-1}, \zeta_{l,i}]$ contains at least two intervals I_j with $j \in J'_n$. Furthermore, if $J \subset (J_n \setminus J'_n)$, then

$$|H(J)| \le 4n^{\gamma-\kappa} = o(1), \tag{III}$$

where $|H(J)|$ is the Lebesgue measure of the set $H(J)$.

We emphasize that $E, v_l, \zeta_{l,j}$ are fixed, they are independent of n and T_n. The collection of the intervals I_j that E (and each of its "branch") is divided into, and hence also the index set J_n, depends on n (the degree of T_n), but it is independent of T_n itself. Finally, the set J'_n depends on the polynomial T_n in question.

Let χ_H denote the characteristic function of H. For its approximation by trigonometric polynomials we need the following analogue of [7, Lemma 6].

Lemma 3.1 *Assume that $H = H(J)$ ($J \subset J_n$) is an interval with characteristic function $\chi_H(t)$. Fix $1/2 > \theta > 4\kappa$. Then there is a constant C (independent of H and E) and a trigonometric polynomial $q = q(H, n; t)$ of degree $O\big(n^{2\theta}\big)$ which satisfies*

$$0 \le q(t) \le 1 \tag{14}$$

on $[-\pi, \pi]$*, furthermore,*

$$|q(t) - \chi_H(t)| \le O\left(e^{-Cn^\theta}\right), \tag{15}$$

$$|q'(t)| \le O\left(e^{-Cn^\theta}\right), \tag{16}$$

whenever $t \in [-\pi, \pi] \setminus H_b$*.*

Proof. The lemma follows from [7, Lemma 6]. Let t_0 be the midpoint of H and take the sets $\hat{H} := \{\cos(t - t_0 + \pi) \,|\, t \in H\}$ and $\hat{H}_b := \{\cos(t - t_0 + \pi) \,|\, t \in H_b\}$. $\hat{H}$ is an interval in $[-1, 1)$ with left-endpoint -1. [7, Lemma 6] implies the existence of a constant $\hat{C}$ and an algebraic polynomial $p(x) = p(\hat{H}, n; x)$ of degree at most $n^{2\theta}$ such that $0 \le p(x) \le 1$ on $[-2, 2]$ as well as

$$|p(x) - \chi_{\hat{H}}(x)| \le O\left(e^{-\hat{C}n^\theta}\right)$$

$$|p'(x)| \le O\left(e^{-\hat{C}n^\theta}\right)$$

whenever $x \in [-2, 2] \setminus \left(\hat{H}_b \cup [-1 - |\hat{H}_b|, -1]\right)$. (We should be cautious a bit because if $|H|$ is small ($\sim n^{-\kappa}$), then $\hat{H}_b$ has a length of about $|H_b|^2 = n^{-2\kappa}$, therefore we have to apply [7, Lemma 6] as if we ought to apply it after a split (similar to the one discussed above) but of magnitude $n^{-2\kappa}$.) Then, as it can easily be checked, $q(t) := p(\cos(t - t_0 + \pi))$ is a suitable trigonometric polynomial, that is $q(t) \in [0, 1]$ ($t \in [-\pi, \pi]$) and satisfies both (15) and (16) with $C = \hat{C}$.

■

Three types of subintervals

In order to estimate the analogue of $A(T_n, E)$ from (10) Nagy and Toókos divided K from (7) into special intervals which were the unions of some I_j's, then they separately gave estimates on these intervals, and finally they summed up the estimates obtained. We are also going to do so. Recall that E is a T-set associated with the trigonometric polynomial $U_N(t)$, thus E has the following form:

$$E = \bigcup_{l=1}^{m} \bigcup_{i=1}^{r_l} [\zeta_{l,i-1}, \zeta_{l,i}],$$

where $[\zeta_{l,i-1}, \zeta_{l,i}]$, $l = 1, 2, \ldots, m$, $i = i_l = 1, 2, \ldots, r_l$, are the "branches" of E, i.e. $U_N(t)$ runs through $[-1, 1]$ precisely once as t runs through $[\zeta_{l,i-1}, \zeta_{l,i}]$. Note that we have $2N = \sum_{l=1}^{m} r_l$. As we have already remarked, if n is large enough then, for every l and i, there are at least two $j \in J_n'$ (for the definition of J_n' see (13)) such that $I_j \subset [\zeta_{l,i-1}, \zeta_{l,i}]$.

Let

$$k_{l,i}^{\text{left}} := \min\{j \in J_n' \,\big|\, I_j \subset [\zeta_{l,i-1}, \zeta_{l,i}]\},$$

and

$$k_{l,i}^{\text{right}} := \max\{j \in J_n' \,\big|\, I_j \subset [\zeta_{l,i-1}, \zeta_{l,i}]\}.$$

We say that $H = H(J)$ $(J \subset J_n')$ is an interval

- of the first type if $J = [k_{l,i}^{\text{left}} + 1, k_{l,i}^{\text{right}} - 1] \cap \mathbf{N}$, that is

  $$H = \bigcup_{j=k_{l,i}^{\text{left}}+1}^{k_{l,i}^{\text{right}}-1} I_j$$

  for some $l \in \{1, 2, \ldots, m\}$ and $i = i_l \in \{1, 2, \ldots, r_l\}$.

- of the second type if $J = [k_{l,i}^{\text{right}} + 1, k_{l,i+1}^{\text{left}} - 1] \cap \mathbf{N}$, that is

  $$H = \bigcup_{j=k_{l,i}^{\text{right}}+1}^{k_{l,i+1}^{\text{left}}-1} I_j$$

  for some $l \in \{1, 2, \ldots, m\}$ and $i = i_l \in \{1, 2, \ldots, r_l - 1\}$ $(i \neq r_l!)$.

- of the third type if H contains a v_{2l-1} and all the subsequent I_j with $j < k_{l,1}^{\text{left}}$ or H contains a v_{2l} and all preceding I_j with $j > k_{l,r_l}^{\text{right}}$.

See Figure 1.

We treat the intervals of the first and third type together. The case of the intervals of the second type is more complicated and we are going to deal with it in more details. Note that the union of these intervals covers the T-set E except for the $4N$ intervals $I_{k_{l,i}^{\text{left}}}$ and $I_{k_{l,i}^{\text{right}}}$.

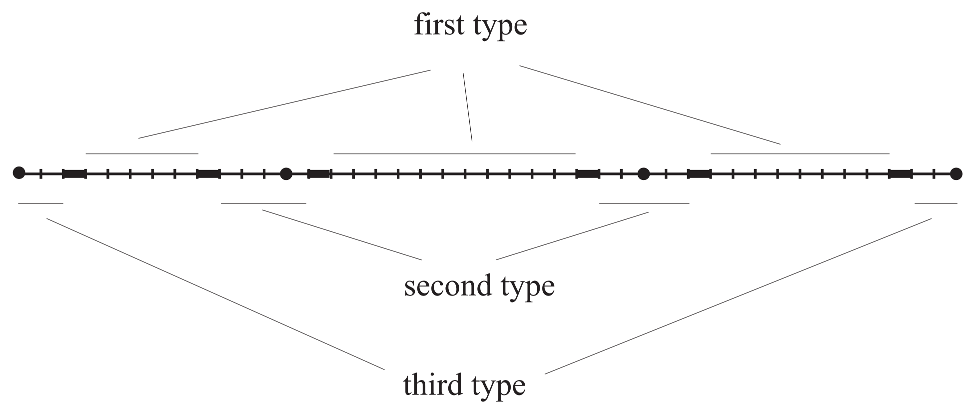

Figure 1: One component of E and the various types of intervals H. The dots represent the points where $|U_N| = 1$, in between two such points there is a "branch", and the thicker-drawn intervals are the intervals $I_{k_{l,i}^{\text{left}}}$ and $I_{k_{l,i}^{\text{right}}}$ in each "branch".

Intervals of the first and third types

From the definition of the intervals H of the first and third type it easily follows that such intervals possess the properties (I), (II-a) and (II-b).

We claim that, if the interval $H = H(J)$ has these properties, in particular, if it is of the first or third type then (see the definitions (10)–(11))

$$A(T_n, H) \leq B(T_n, H) + o(1)A(T_n, E) + o(1)B(T_n, E), \tag{17}$$

where $o(1) \to 0$ as $n \to \infty$ uniformly in T_n. The verification follows [7, Sec. 5.1 and 5.3] almost word for word, one only has to use the following trigonometric analogues of the lemmas there.

Lemma 3.2 ([12, Lemma 3.2]) *Let E be a T-set associated with the trigonometric polynomial U_N of degree N, and for a $t \in E$ with $U_N(t) \in (-1, 1)$ let $t_1, t_2, \ldots, t_{2N}$ be those points in E which satisfy $U_N(t_k) = U_N(t)$. Then, if V_n is a trigonometric polynomial of degree at most n, there is an algebraic polynomial $S_{n/N}$ of degree at most n/N such that*

$$\sum_{k=1}^{2N} V_n(t_k) = S_{[n/N]}(U_N(t)).$$

With this lemma at hand we can take the trigonometric analogue of [7, (19)]. If $H \subset E$ is an interval with property (I), and $q(t) = q(H, n; t)$ is the polynomial by Lemma 3.1, then, by Lemma 3.2,

$$T_n^*(t) := \sum_{k=1}^{2N} T_n(t_k) q(t_k)$$

is a polynomial of the trigonometric polynomial U_N, so we can apply part **(a)** from Section 2 to this T_n^*. This leads to the following lemma which is the analogue of [7, Lemma 7 and 8] and which is verified exactly as Lemmas 7 and 8 were proved in [7] .

Lemma 3.3 *Let E, U_N, T_n be the same as in Lemma 3.2 and let $H = H(J)$ be an interval with property (I). Then, if $n^* = n + \deg q (= (1 + o(1))n)$, we have*

$$\left| \left(\frac{n^*}{n} \right)^p A(T_n^*, E) - (2N) A(T_n, H) \right|$$
$$\leq \big(o(1) + c_1\, a(T_n, H_b) \big) A(T_n, E) + o(1) B(T_n, E),$$

and

$$|B(T_n^*, E) - (2N) B(T_n, H)| \leq \big(o(1) + c_2\, b(T_n, H_b) \big) B(T_n, E),$$

where $o(1) \to 0$ as $n \to \infty$. Furthermore, the $o(1)$ and the constants c_1, c_2 are independent of T_n.

Remark 3.4: Note that if H has the property (II-a) then $a(T_n, H_b) = o(1) \to 0$ as $n \to \infty$ and, similarly, if it has the property (II-b) then $b(T_n, H_b) = o(1) \to 0$ as $n \to \infty$.

As we have already mentioned, the proof is the same as those of [7, Lemma 7 and 8], one should only replace "$\omega_K(t)$" by "$\omega_{\Gamma_E}(e^{it})$" and "$P'(t)/\pi(\deg P)$" by "$T_n'(t)/(2\pi n)$" there.

From Lemma 3.3 one can easily deduce (17) as was done in [7, Sec. 5.3].

Intervals of the second type

In this case $H = H(J)$ is an interval of the second type, so it has the following properties:

- H contains an inner extremal point ζ_{l_0,i_0};
- $\max\big(a(T_n, H_b), b(T_n, H_b)\big) < 1/n^\gamma$.

Our aim is to reduce this case to the case of the intervals of the first or third types, and to prove that

$$A(T_n, H) \leq B(T_n, H) + o(1)A(T_n, E) + o(1)B(T_n, E). \tag{18}$$

The idea is the following: We approximate the set E by a sequence $\{E_k\}$ of T-sets of order N (which is the order of E) from the inside: $E_k \subset E$. Every one of these T-sets E_k has an inner extremal point corresponding to ζ_{l_0,i_0} and these extremal points form a strictly increasing sequence converging to ζ_{l_0,i_0}. We take an appropriate T-set from the sequence for which the point corresponding to ζ_{l_0,i_0} is outside of H, so with respect to this T-set H behaves as if it was of the first or third type. Then we only have to show that the estimates on H with regard to E hardly differ from those with regard to the chosen T-set. In this process we use potential theoretic tools. The subsequent proposition replaces [7, Propositon 9 and 10].

Proposition 3.5 *Let E be the union of the disjoint intervals $[v_{2l-1}, v_{2l}] \subset (-\pi, \pi)$, where $l = 1, 2, \ldots, m$ and $v_{2l-1} < v_{2l} < v_{2l+1}$. If E is a T-set of order N then there is a sequence E_k of T-sets of order N such that*

(i)

$$E_k = \bigcup_{l=1}^{m} [v_{2l-1}, v_{2l}(k)],$$

where each $v_{2l}(k)$ strictly increases in k and converges to v_{2l} for every $l \in \{1, 2, \ldots, m\}$;

(ii) if E has the inner extremal points $\zeta_{l,1} < \zeta_{l,2} < \cdots < \zeta_{l,r_l-1}$ in its l-th component $[v_{2l-1}, v_{2l}]$ then E_k also has $r_l - 1$ inner extremal points $\zeta_{l,1}(k) < \zeta_{l,2}(k) < \cdots < \zeta_{l,r_l-1}(k)$ in $[v_{2l-1}, v_{2l}(k)]$ such that each $\zeta_{l,i}(k)$ strictly increases in k and converges to $\zeta_{l,i}$ ($i \in \{1, 2, \ldots, r_l - 1\}$);

(iii) if ω_{Γ_E}, $\omega_{\Gamma_{E_k}}$ denote the corresponding equilibrium densities of Γ_E and Γ_{E_k} then there is a sequence $D_k = D(E_k) \to 1$ for which the estimates

$$1 \leq \frac{\omega_{\Gamma_{E_k}}(e^{it})}{\omega_{\Gamma_E}(e^{it})} \leq D_k \tag{19}$$

are valid for every

$$t \in \bigcup_{l=1}^{m} \left[\frac{v_{2l-1} + \zeta_{l,1}}{2}, \frac{\zeta_{l,r_l-1} + v_{2l}}{2} \right].$$

and for sufficiently large $k \in \mathbf{N}$.

We need some lemmas for the proof of this proposition. The first is a standard characterization of T-sets. Denote by $[e^{i\xi_1}, e^{i\xi_2}]$ the arc $\{e^{it} \mid t \in [\xi_1, \xi_2]\}$, where $-\pi < \xi_1 < \xi_2 < \pi$.

Lemma 3.6 ([12, Lemma 3.2]) *Let* E *be the union of the disjoint intervals* $[v_{2l-1}, v_{2l}] \subset (-\pi, \pi)$, *where* $l = 1, 2, \ldots, m$ *and* $v_{2l-1} < v_{2l} < v_{2l+1}$. *Then the followings are equivalent:*

(a) E *is a* T*-set of order* N.

(b) For every $l = 1, 2, \ldots, m$ *the measure* $\mu_{\Gamma_E}\big([e^{iv_{2l-1}}, e^{iv_{2l}}]\big)$ *is of the form* $r_l/2N$ *with some integer* r_l.

Furthermore, in this case each subinterval $[v_{2l-1}, v_{2l}]$ *contains precisely* $r_l - 1$ *inner extremal points for the trigonometric polynomial* U_N *which* E *is associated with. If* $[a, b]$ *is a "branch" of* E, *then* $\mu_{\Gamma_E}([e^{ia}, e^{ib}]) = 1/2N$.

The second lemma describes how the equilibrium measure of a subset can be derived from the equilibrium measure of the full set.

Lemma 3.7 ([10, Ch. IV. Theorem 1.6 (e)]) *Let* K *be a compact subset of the complex plane and let* $S \subset K$ *be a closed set of positive capacity. Let* μ_K *and* μ_S *denote the equilibrium measures of* K *and* S *respectively. Then*

$$\mu_S = \mathrm{Bal}(\mu_K) = \mu_K\big|_S + \mathrm{Bal}\left(\mu_K\big|_{K \setminus S}\right).$$

where $\mathrm{Bal}(.)$ *denotes the balayage onto* S.

For the concept of balayage see [10].

Next, we state

Lemma 3.8 ([13, Theorem 9]) *Let* $g_1, g_2, \ldots, g_m$ *be functions with the following properties:*

(A) *each g_j is a continuous function on the cube $[0,a]^m$, where a is some positive number,*

(B) *each $g_j = g_j(x_1, x_2, \ldots, x_m)$ is strictly monotone increasing in x_j and strictly monotone decreasing in every x_i with $i \neq j$, and*

(C) $\sum_{j=1}^m g_j(x_1, x_2, \ldots, x_m) = 1$.

Then there is an $\alpha > 0$ with property that for every $x_m \in (0, \alpha)$ there exist $x_1 = x_1(x_m), x_2 = x_2(x_m), \ldots, x_{m-1} = x_{m-1}(x_m) \in (0, a)$ such that each $g_j(x_1, x_2, \ldots, x_m)$ equals $g_j(0, 0, \ldots, 0)$. Furthermore, these $x_j = x_j(x_m)$ are monotone increasing functions of x_m and $x_j(x_m) \to 0$ as $x_m \to 0$.

The last lemma describes the equilibrium density of an arc-system on the unit circle, it is due to Peherstorfer and Steinbauer.

Lemma 3.9 ([8, Lemma 4.1]) *Let $E = \cup_{l=1}^m [v_{2l-1}, v_{2l}] \subset (-\pi, \pi)$. There are points $e^{i\beta_j}$, $j = 1, 2, \ldots, m$, on the complementary arcs (with respect to the unit circle) to Γ_E with which*

$$\omega_{\Gamma_E}(e^{it}) = \frac{1}{2\pi} \frac{\prod_{j=0}^{m-1} |e^{it} - e^{i\beta_j}|}{\sqrt{\prod_{j=1}^{2m} |e^{it} - e^{iv_j}|}}, \qquad t \in E. \tag{20}$$

The $e^{i\beta_j}$ are the unique points on the unit circle for which

$$\int_{v_{2k}}^{v_{2k+1}} \frac{\prod_{j=0}^{m} (e^{it} - e^{i\beta_j})}{\sqrt{\prod_{j=1}^{2m} (e^{it} - e^{iv_j})}} \, \mathrm{d}t = 0, \qquad k = 0, 1, \ldots, m-1, \; v_0 = v_{2m} \tag{21}$$

holds, with appropriate definition of the square root in the denominator.

Proof of Proposition 3.5.

The proof consists of some observations resting on the previous lemmas.
Observation 1 By the assumption E is a T-set of order N, so by Lemma 3.6 $\mu_{\Gamma_E}\big([e^{iv_{2l-1}}, e^{iv_{2l}}]\big) = r_l/2N$ for every l where r_l is a positive integer.
Observation 2 Let

$$E(x_1, x_2, \ldots, x_m) := \bigcup_{l=1}^m [v_{2l-1}, v_{2l} - x_l].$$

Then, as can be easily verified (cf. [13, (2)]),

$$g_l(x_1, x_2, \dots, x_m) := \frac{1 + \frac{1}{m} - \mu_{\Gamma_{E(x_1,x_2,\dots,x_m)}}\left([e^{iv_{2l-1}}, e^{i(v_{2l}-x_l)}]\right)}{m}$$

have the properties (A), (B) and (C) in Lemma 3.8. From this the existence of the sequence E_k in (i) is immediate, since $g_l(x_1, \dots, x_m) = g_l(0, \dots, 0)$ for all l means that $\mu_{\Gamma_{E(x_1,x_2,\dots,x_m)}}([e^{iv_{2l-1}}, e^{i(v_{2l}-x_l)}]) = r_l/2N$ for all $l = 1, \dots, m$, and apply Lemma 3.6.

Observation 3 Accordingly, by Lemma 3.6, E_k is a T-set of order N, associated with some trigonometric polynomial $U_{N,k}$, and $U_{N,k}$ has precisely $r_l - 1$ inner extremal points on the intervals $[v_{2l-1}, v_{2l}(k)]$, $l = 1, 2, \dots, m$. In other formulation, each $[v_{2l-1}, v_{2l}(k)]$ consists of r_l "branches" of E_k.

Observation 4 Recall that $\zeta_{l,1}(k) < \zeta_{l,2}(k) < \cdots < \zeta_{l,r_l-1}(k)$ ($\zeta_{l,1} < \zeta_{l,2} < \cdots < \zeta_{l,r_l-1}$) denote the inner extremal points of the l-th component $[v_{2l-1}, v_{2l}(k)]$ ($[v_{2l-1}, v_{2l}]$) of E_k (E). Set also $\zeta_{l,0}(k) = v_{2l-1}$, $\zeta_{l,r_l}(k) = v_{2l}(k)$. By Lemma 3.7 we have $\mu_{\Gamma_{E_k}} > \mu_{\Gamma_{E_{k+1}}} > \mu_{\Gamma_E}$ on E_k. Also, by Lemma 3.6 we have

$$\mu_{\Gamma_{E_k}}\left(\left[e^{iv_{2l-1}}, e^{i\zeta_{l,j}(k)}\right]\right) = \frac{j}{2N},$$

from which it can be easily inferred that each $\zeta_{l,i}(k)$ strictly increases in k, as well as $\zeta_{l,i}(k) \to \zeta_{l,i}$ ($i \in \{1, 2, \dots, r_l\}$) since $\mu_{\Gamma_{E_k}} \to \mu_{\Gamma_E}$ in weak*-sense.

Observation 5 Apply Lemma 3.9 to E and E_k and denote by $\beta_l(E)$ and $\beta_l(E_k)$ the points with which we get ω_{Γ_E} and $\omega_{\Gamma_{E_k}}$, respectively in the form (20). It can be easily shown (see e.g. the proof of Lemma 3.5 in [12]) that $\beta_l(E_k) \to \beta_l(E)$ as $k \to \infty$. This and the form of ω_{Γ_E} in Lemma 3.9 shows that $\omega_{\Gamma_{E_k}}(e^{it}) \to \omega_{\Gamma_E}(e^{it})$ pointwise on E and also uniformly on

$$\bigcup_{l=1}^{m} \left[\frac{v_{2l-1} + \zeta_{l,1}}{2}, \frac{\zeta_{l,r_l-1} + v_{2l}}{2}\right].$$

This verifies (iii) since $\omega_{\Gamma_{E_k}}(e^{it}) > \omega_{\Gamma_E}(e^{it})$ on Γ_{E_k} by Lemma 3.7.

■

Remark 3.10: A similar argument as in Observation 5 also shows that both $\omega_{\Gamma_{E^+}}(e^{it})$ and $\omega_{\Gamma_{E^-}}(e^{it})$ in Lemma 2.1 converge to $\omega_{\Gamma_E}(e^{it})$ as $\epsilon \to 0$ pointwise on E, moreover, by Lemma 3.9, $\omega_{\Gamma_{E^+}}(e^{it})/\omega_{\Gamma_E}(e^{it})$ ($\omega_{\Gamma_{E^-}}(e^{it})/\omega_{\Gamma_E}(e^{it})$)

uniformly converges to 1 on a set of the form $[v_{2l-1}, v_{2l} - \delta_l]$ where $\delta_l > 0$ is arbitrarily fixed.

Now, by Proposition 3.5, we have a sequence $\{E_k\}$ of T-sets of order N approximating the T-set E. Fix one of the E_k's. According to the property (III) for intervals H of the second type the length of $H \cup H_b$ is at most $4n^{\gamma-\kappa} + 2n^{-\kappa} \leq 6n^{\gamma-\kappa}$, where γ is the number fixed in (12). For large fixed k and for large n (how large depending on k) we have

$$\min_{(l,i)} \big(\zeta_{l,i} - \zeta_{l,i}(k)\big) > 12n^{\gamma-\kappa}$$

and

$$\min_{(l,i)} \big(\zeta_{l,i+1}(k) - \zeta_{l,i}\big) > \min_{(l,i)} \frac{\zeta_{l,i+1} - \zeta_{l,i}}{2} > 12n^{\gamma-\kappa},$$

where the minimums are taken for every $l \in \{1, 2, \ldots, m\}$ and $i = i_l \in \{1, 2, \ldots, r_l - 1\}$. Note that then for each interval H of the second type that contains, say, the inner extremal point ζ_{l_0,i_0}, the set $H \cup H_b$ lies strictly inside the "branch" $[\zeta_{l_0,i_0}(k), \zeta_{l_0,i_0+1}(k)]$ of E_k.

The next lemma compares integrals on H with regard to E with those with regard to E_k. It is the analogue of [7, Lemma 11]. Following the definition of $A(T_n, X)$ and $B(T_n, X)$ from (10)–(11) let us introduce the notations $A_k(T_n, X)$ and $B_k(T_n, X)$ for

$$\int_X \left| \frac{T_n'(t)}{n2\pi\omega_{\Gamma_{E_k}}(e^{it})} \right|^p \omega_{\Gamma_{E_k}}(e^{it})\,\mathrm{d}t,$$

and

$$\int_X |T_n(t)|^p \omega_{\Gamma_{E_k}}(e^{it})\,\mathrm{d}t$$

respectively.

Lemma 3.11 *Let $q = q(H, n, t)$ be the polynomial from Lemma 3.1 and let X be an arbitrary subset of E. Then the following five estimates hold:*

- $$|A(T_n q, H) - A(T_n, H)| \leq o(1)A(T_n, E) + o(1)B(T_n, E), \tag{22}$$

- $$A(T_n q, X) \leq A(T_n, X) + o(1)A(T_n, E) + o(1)B(T_n, E), \tag{22'}$$

- $$B(T_n q, X) \le B(T_n, X). \tag{23}$$

- $$|A_k(T_n q, E_k) - A_k(T_n q, H)| \le o(1)A(T_n, E) + o(1)B(T_n, E), \tag{24}$$

- $$|B_k(T_n q, E_k) - B_k(T_n q, H)| \le o(1)B(T_n, E). \tag{25}$$

(23) is an immediate consequence of (14) while (22) is verified as follows (cf. the proof of [7, Lemma 11 (43)]):

$$\begin{aligned} A(T_n q, X)^{\frac{1}{\alpha}} &= \left(\int_X \left| \frac{\big(T_n(t)q(t)\big)'}{\deg(T_n q) 2\pi \omega_{\Gamma_E}(e^{it})} \right|^\alpha \omega_{\Gamma_E}(e^{it})\,\mathrm{d}t \right)^{\frac{1}{\alpha}} \\ &\le \left(\int_X \left| \frac{T_n'(t)q(t)}{\deg(T_n q) 2\pi \omega_{\Gamma_E}(e^{it})} \right|^\alpha \omega_{\Gamma_E}(e^{it})\,\mathrm{d}t \right)^{\frac{1}{\alpha}} \\ &+ \left(\int_X \left| \frac{T_n(t)q'(t)}{\deg(T_n q) 2\pi \omega_{\Gamma_E}(e^{it})} \right|^\alpha \omega_{\Gamma_E}(e^{it})\,\mathrm{d}t \right)^{\frac{1}{\alpha}}. \end{aligned} \tag{26}$$

To the first integral on the right-hand side we apply (14) and get that

$$\left(\int_X \left| \frac{T_n'(t)q(t)}{\deg(T_n q) 2\pi \omega_{\Gamma_E}(e^{it})} \right|^\alpha \omega_{\Gamma_E}(e^{it})\,\mathrm{d}t \right)^{\frac{1}{\alpha}} \le \frac{\deg(T_n)}{\deg(T_n q)} A(T_n, X)^{\frac{1}{\alpha}}. \tag{27}$$

For the second integral on the right-hand side of (26) the inequality (2) gives that

$$\left(\int_X \left| \frac{T_n(t)q'(t)}{\deg(T_n q) 2\pi \omega_{\Gamma_E}(e^{it})} \right|^\alpha \omega_{\Gamma_E}(e^{it})\,\mathrm{d}t \right)^{\frac{1}{\alpha}} \le \frac{\deg(q)}{\deg(T_n q)} B(T_n, X)^{\frac{1}{\alpha}}. \tag{28}$$

(26), (27) and (28) show that

$$A(T_n q, X)^{\frac{1}{\alpha}} \le \big(1 - o(1)\big) A(T_n, X)^{\frac{1}{\alpha}} + o(1) B(T_n, X)^{\frac{1}{\alpha}}.$$

From this inequality, by the generalized weighted mean inequality, we infer that

$$A(T_n q, X) \le \big(1 - o(1) + o(1)\big)^\alpha \left(\frac{\big(1 - o(1)\big) A(T_n, X)^{\frac{1}{\alpha}} + o(1) B(T_n, X)^{\frac{1}{\alpha}}}{1 - o(1) + o(1)} \right)^\alpha$$

$$\leq \big(1 - o(1) + o(1)\big)^{\alpha - 1}\Big(\big(1 - o(1)\big)A(T_n, X) + o(1)B(T_n, X)\Big) \tag{29}$$
$$\leq A(T_n, X) + o(1)A(T_n, E) + o(1)B(T_n, E).$$

As regards (22), (24) and (25), their proof is much the same as the proof of [7, Lemma 11 (43), (45) and (46)]. We remark only one thing, namely, during the verification of (24) and (25) one needs the following lemma:

Lemma 3.12 *Let I be a fixed subinterval of E and let τ be an arbitrary trigonometric polynomial with the property* $\sup_{t \in I} |\tau(t)| = 1$. *Then there exits a constant c independent of τ such that*

$$\int_I |\tau(t)|^p \omega_{\Gamma_E}(e^{it})\, \mathrm{d}t \geq \frac{c}{(\deg \tau)^2}.$$

Nagy and Toókos derived the algebraic analogue of this from Nikolskii's inequality, but we have no knowledge of a Nikolskii-type inequality for trigonometric polynomial on a proper subinterval of $(-\pi, \pi)$.

Proof. The lemma is a simple consequence of a Markov-type inequality by Videnskii (see e.g [2, Sec. 5.1 E/13c] or [15]), which says that if $\alpha \leq \pi$ and Q_m is a trigonometric polynomial of degree at most m, then

$$||Q_m'||_{[-\alpha,\alpha]} \leq \big(1 + o(1)\big)2m^2 \mathrm{cotan}\left(\frac{\alpha}{2}\right)||Q_m||_{[-\alpha,\alpha]}, \tag{30}$$

where $o(1) \to 0$ as $m \to \infty$. A simple transformation shows that it can be assumed that the center of I is 0. Let $t_0 \in I$ such that $|\tau(t_0)| = 1$ and e.g. $[t_0, t_0 + 1/(2d(\deg \tau)^2)] \subset I$ with $d := 4\mathrm{cotan}(|I|/4)$ (at least one of the two sides of t_0 belongs to I for large degrees). Hence, in view of (30), for all $t \in [t_0, t_0 + 1/(2d(\deg \tau)^2)]$ we have

$$|\tau(t)| \geq 1 - d(\deg \tau)^2 \frac{1}{2d(\deg \tau)^2} \geq \frac{1}{2}.$$

Thus, if ρ denotes the minimum of $\omega_{\Gamma_E}(e^{it})$ on E, then we get

$$\int_I |\tau(t)|^p \omega_{\Gamma_E}(e^{it})\, \mathrm{d}t \geq \int_{t_0}^{t_0 + \frac{1}{2d(\deg \tau)^2}} \left(\frac{1}{2}\right)^p \rho\, \mathrm{d}t = \frac{\rho}{2^p} \frac{1}{2d(\deg \tau)^2},$$

which proves the Lemma 3.12 with $c := \rho/(2^{p+1}d)$.

■

Using (22) and (19) we obtain

$$A(T_n,H) \le A(T_nq,H) + o(1)A(T_n,E) + o(1)B(T_n,E) \tag{31}$$

$$\le (D_k)^{p-1}A_k(T_nq,H) + o(1)A(T_n,E) + o(1)B(T_n,E)$$

For large n the interval H possesses the property (I) with respect to E_k, thus we can apply Lemma 3.3 to H with respect to E_k and this yields, similarly to (17), that

$$A_k(T_nq,H) \le B_k(T_nq,H) + o(1)A_k(T_nq,E_k) + o(1)B_k(T_nq,E_k) \tag{32}$$

$$+c_1a_k(T_nq,H_b)A_k(T_nq,E_k) + c_2b_k(T_nq,H_b)B_k(T_nq,E_k).$$

We have deliberately written the error terms in the form that include $a_k(T_nq,H_b)$ and $b_k(T_nq,H_b)$ separately, because we do not know wether H possesses the properties (II-a) and (II-b) with respect to E_k. Recall that $||q||_{[-\pi,\pi]} \le 1$. For $A_k(T_nq,E_k)$ we get by (24), (19) and (22') that

$$A_k(T_nq,E_k) \le A_k(T_nq,H) + o(1)A(T_n,E) + o(1)B(T_n,E) \tag{33}$$

$$\le A(T_nq,H) + o(1)A(T_n,E) + o(1)B(T_n,E)$$

$$\le A(T_n,H) + o(1)A(T_n,E) + o(1)B(T_n,E).$$

For $B(T_nq,E_k)$ we use (25), (19) and (23) to conclude

$$B_k(T_nq,E_k) \le B_k(T_nq,H) + o(1)B(T_n,E) \tag{34}$$

$$\le D_k\, B(T_nq,H) + o(1)B(T_n,E)$$

$$\le D_k\, B(T_n,H) + o(1)B(T_n,E).$$

As regards $a_k(T_nq,H_b)A_k(T_nq,E_k)$, we apply (19) and (22') which give

$$a_k(T_nq,H_b)A_k(T_nq,E_k) = A_k(T_nq,H_b) \le A(T_nq,H_b) \tag{35}$$

$$\le a(T_n,H_b)A(T_n,E) + o(1)A(T_n,E) + o(1)B(T_n,E),$$

and for $b_k(T_nq,H_b)B_k(T_nq,E_k)$ by (19) and (23) we similarly get

$$b_k(T_nq,H_b)B_k(T_nq,E_k) = B_k(T_nq,H_b) \le D_k\, B(T_n,H_b) \tag{36}$$

$$= D_k\, b(T_n,H_b)B(T_n,E).$$

By the assumption that H has the property (II-a) and (II-b) with respect to E we know that both $a(T_n, H_b)$ and $b(T_n, H_b)$ are $< n^{-\gamma} = o(1)$. Hence, by (33), (34), (35) and (36), we can continue the estimate (32) for $A_k(T_n q, H)$ as follows

$$A_k(T_n q, H) \le B_k(T_n q, H) + o(1)A(T_n, E) + o(1)B(T_n, E).$$

First employing the previous equation then (19) and (23) we can now continue (31) as

$$\begin{aligned}
&(D_k)^{p-1}A_k(T_n q, H) + o(1)A(T_n, E) + o(1)B(T_n, E) \\
&\le (D_k)^{p-1}B_k(T_n q, H) + \big((D_k)^{p-1}+1\big)o(1)A(T_n, E) + \big((D_k)^{p-1}+1\big)o(1)B(T_n, E) \\
&\le (D_k)^p B(T_n, H) + \big((D_k)^{p-1} + 1\big)o(1)A(T_n, E) \\
&\quad + \big((D_k)^{p-1} + 1\big)o(1)B(T_n, E).
\end{aligned}$$

Since $D_k \to 1$ as $k \to \infty$ we can finally conclude (by selecting a large k and then a large n)

$$A(T_n, H) \le (1 + o(1))B(T_n, H) + o(1)A(T_n, E) + o(1)B(T_n, E),$$

which is just (18) considering that $B(T_n, H) \le B(T_n, E)$.

Synthesis of the case when E is a T-set and T_n is an arbitrary trigonometric polynomial

Recall that E is a T-set associated with a trigonometric polynomial U_N and it is the union of the m disjoint intervals $[v_{2l-1}, v_{2l}]$ ($l \in \{1, 2, \ldots, m\}$), the l-th one of which consists of r_l "branches". Denote by S_j the union of all intervals which are of the j-th type ($j = 1, 2, 3$). Then each S_j is the union of at most $4N$ intervals H of the same type. Let S_4 be the union of the remaining intervals I_j, that is

$$S_4 := \bigcup_{l=1}^{m} \bigcup_{i=1}^{r_l} \left(I_{k_{l,i}^{\text{left}}} \cup I_{k_{l,i}^{\text{right}}} \right).$$

From (17), (18) and from the fact that $k_{l,i}^{\text{left}}, k_{l,i}^{\text{right}} \in J'_n$ we obtain:

$$A(T_n, E) = \big(A(T_n, S_1) + A(T_n, S_2) + A(T_n, S_3)\big) + A(T_n, S_4) \le$$

$$B(T_n, S_1 \cup S_2 \cup S_3) + 3 \cdot 4N\big(o(1)A(T_n, E) + o(1)B(T_n, E)\big) + 4N\frac{1}{n^\gamma}A(T_n, E) \leq$$

$$B(T_n, E) + o(1)A(T_n, E) + o(1)B(T_n, E)$$

Comparing the leftmost side to the rightmost one we conclude

$$A(T_n, E) \leq \frac{1 + o(1)}{1 - o(1)}B(T_n, E) = \big(1 + o(1)\big)B(T_n, E).$$

This verifies Theorem 1.1 for the case when E is a T-set and T_n is an arbitrary trigonometric polynomial, i.e. **(b)** from Section 2 has been proven.

■

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