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Logarithmic Concavity of Series in Gamma Ratios

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Abstract—We find two-sided bounds and prove non-negativeness of Taylor coefficients for the Turán determinants of power series with coefficients involving the ratio of gamma-functions. We consider these series as functions of simultaneous shifts of the arguments of the gamma-functions located in the numerator and the denominator. The results are then applied to derive new inequalities for the Gauss hypergeometric function, the incomplete normalized beta-function and the generalized hypergeometric series. This communication continues the research of various authors who investigated logarithmic convexity and concavity of hypergeometric functions in parameters.

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We consider a class of power series

$$g_{a,c}(\mu;x)=\sum_{n=0}^{\infty}g_n\frac{\Gamma(a+\mu+n)}{\Gamma(c+\mu+n)}x^n, \tag{1}$$

where $\{g_n\}_{n=0}^{\infty}$ is certain non-negative number sequence, and $\Gamma(z)$ stands for the Euler gamma-function. The main question concerns restrictions on the sequence $\{g_n\}_{n=0}^{\infty}$ and values a, c such that under these restrictions the difference of products

$$\psi_{a,c}(\mu,\nu;x)=g_{a,c}(\mu;x)g_{a,c}(\nu;x)-g_{a,c}(0;x)g_{a,c}(\mu+\nu;x)=\sum_{m=0}^{\infty}\psi_m x^m \tag{2}$$

has non-negative coefficients ψ_m at all powers of x. Evident consequence of such a non-negativity is logarithmic concavity of the function $\mu\to g_{a,c}(\mu;x)$. Here all series are formal, and we do not consider their convergence. Important examples of series (1) are hypergeometric series and their derivatives in parameters. In addition, results of the present paper will be illustrated by implied new inequalities for hypergeometric functions. Analogous questions for other power series are studied in [1–4]. For the proof of logarithmic concavity of hypergeometric functions with respect to their parameters for negative values of argument one can use integral representations from [5]. The bounds for ratios of gamma-functions are applied in [6] for the proof of inequalities for Bessel functions, and in [7] for bounding of norms for transformation operators in the Lebesgue spaces on a half-axis with power weights.

For formulation of results we need the following conventional definitions.

Definition 1. A non-negative sequence $\{f_k\}_{k=0}^{\infty}$ is called logarithmically concave if its elements satisfy inequalities $f_k^2\geq f_{k-1}f_{k+1}$, $k=1,2,3,\ldots$. We say that it has no inner zeros if relation $f_N=0$ implies $f_k=0$ either for all $0\leq k\leq N$ or for all $k\geq N$.

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Definition 2. A function $f(x)$ is called monotone on the interval (a, b) (maybe, unbounded) if $f^{(k)}(x) \geq 0$ for all $k = 0, 1, \ldots$ and $x \in (a, b)$. It is called completely monotone if for the same values k and x the inequalities are valid

$$(-1)^k f^{(k)}(x) \geq 0.$$

Definition 3. A function $f(x)$ is multiplicative convex on the interval $(0, \infty)$ if it satisfies the condition

$$f(x^\lambda y^{1-\lambda}) \leq f^\lambda(x) f^{1-\lambda}(y)$$

for $\lambda \in [0, 1]$ and $x, y > 0$.

The following theorem is proved in [4].

Theorem 1. *Let one of the following conditions be fulfilled*:

a) $c + 1 \geq a \geq c > 0$ *and* $\{g_n\}_{n=0}^{\infty}$ *is an arbitrary non-negative sequence,*

b) $a > c + 1 > 1$ *and* $\{g_n n!\}_{n=0}^{\infty}$ *is non-negative log-concave sequence without inner zeros.*

Then $\psi_{a,c}(\mu, \nu; x) \geq 0$ *for all* $x, \mu \geq 0$ *and* $\nu \in \mathbb{N}$. *If, in addition,* $\mu \geq \nu - 1$, *then the Taylor coefficients of the function* $\psi_{a,c}(\mu, \nu; x)$ *are non-negative,* $\psi_m \geq 0$ *for all* $m = 0, 1, \ldots,$ *and the function* $x \to \psi_{a,c}(\mu, \nu; x)$ *is absolutely monotone and multiplicative convex on* $(0, \infty)$.

The aim of the present paper is to strengthen part b) of Theorem 1 by replacement of $g_n n!$ by g_n, and to apply this strengthening for the proof of new inequalities for hypergeometric functions. The main result is

Theorem 2. *Assume that* $a > c + 1 > 1$ *and* $\{g_n\}_{n=0}^{\infty}$ *is non-negative log-concave sequence without inner zeros. Then* $\psi_{a,c}(\mu, \nu; x) \geq 0$ *for any* $x, \mu \geq 0$ *and* $\nu \in \mathbb{N}$. *If, in addition,* $\mu \geq \nu - 1$, *then the Taylor coefficients of the function* $\psi_{a,c}(\mu, \nu; x)$ *are non-negative,* $\psi_m \geq 0$ *for all* $m = 0, 1, \ldots$ *Therefore, the function* $x \to \psi_{a,c}(\mu, \nu; x)$ *is absolutely monotone and multiplicative convex on* $(0, \infty)$.

Scheme of the proof. According to lemmas 2 and 3 from [4] it suffices to prove the theorem for $\nu = 1$. For this meaning of ν we obtain by immediate calculations

$$\psi_m = \frac{(a - c)\Gamma(a)\Gamma(a + \mu)}{\Gamma(c + 1)\Gamma(c + \mu + 1)} \sum_{k=0}^{[m/2]} g_k g_{m-k} M_k,$$

where

$$M_k = \frac{(a)_k (a + \mu)_{m-k}}{(c + 1)_k (c + 1 + \mu)_{m-k}} (m - 2k + \mu) - \frac{(a)_{m-k} (a + \mu)_k}{(c + 1)_{m-k} (c + 1 + \mu)_k} (m - 2k - \mu)$$

for $k < m/2$, and

$$M_k = \frac{\mu (a)_k (a + \mu)_{m-k}}{(c + 1)_k (c + 1 + \mu)_{m-k}}$$

for $k = m/2$. Here $(a)_k = \Gamma(a + k)/\Gamma(a)$ is shifted factorial, or Pochhammer symbol. According to lemma 6 from [4], for the proof of non-negativity of coefficients ψ_m it suffices to show that

$$\sum_{0 \leq k \leq m/2} M_k \geq 0 \tag{3}$$

and the sequence $M_0, M_1, \ldots, M_{[m/2]}$ changes sign at most once, i.e., it has the structure $(- - \cdots - -00 \cdots 00 + + \cdots + +)$, where zero and minus signs may be missing.

The non-negativity of the sum is asserted by

Lemma. *The inequality*

$$\sum_{k=0}^{m} \frac{(a)_k (a+\mu)_{m-k}}{(b)_k (b+\mu)_{m-k}} (m-2k+\mu) \geq 0 \tag{4}$$

is valid for all integer $m \geq 1$ and all $\mu \geq 0$ if $b \geq a \geq 0$ or $a \geq b \geq 1$.

Scheme of the proof of Lemma 14. We introduce the notation $u_k = (a)_k(a+\mu)_{m-k}/[(b)_k(b+\mu)_{m-k}]$. If $a = b$ or $a = 0$, then the assertion is obvious. For $b > a > 0$ the function $x \to (a+x)/(b+x)$ increases, and, consequently, $u_k > u_{m-k}$ for all $k < m-k$. It remains to note that in this case

$$u_k(m-2k+\mu) + u_{m-k}(2k-m+\mu) = (m-2k)(u_k - u_{m-k}) + \mu(u_k + u_{m-k}) > 0$$

for all $k \leq m-k$. If $a > b \geq 1$, then by means of the Gosper algorithm ([8], [9], Chap. 5) we find anti-differences for values $u_k(m-2k+\mu)$. One can check the result by immediate calculation

$$u_k(m-2k+\mu) = \alpha_{k+1} - \alpha_k, \text{ where } \alpha_k = \frac{(b-1)(b-1+\mu)(a)_k(a+\mu)_{m+1-k}}{(a-b+1)(b-1)_k(b-1+\mu)_{m+1-k}}, \ k = 0, 1, \ldots, m+1.$$

Consequently,

$$\sum_{k=0}^{m} u_k(m-2k+\mu) = \sum_{k=0}^{m} (\alpha_{k+1} - \alpha_k) = \alpha_{m+1} - \alpha_0$$

$$= \frac{(b-1)(b-1+\mu)}{(a-b+1)} \left\{ \frac{(a)_{m+1}}{(b-1)_{m+1}} - \frac{(a+\mu)_{m+1}}{(b-1+\mu)_{m+1}} \right\} \geq 0.$$

The latter inequality is valid, because the function $x \to (a+x)/(b-1+x)$ decreases for $x > 0$ by virtue of the assumption $a > b \geq 1$. □

It remains to note that the left-hand sides of inequalities (3) and (4) coincide, and the proof of unique change of sign by sequence $M_0, M_1, \ldots, M_{[m/2]}$ repeats the corresponding part of the proof of Theorem 1 (see [4], theorem 4). The multiplicative convexity follows from non-negativity of coefficients by virtue of the Hardy, Littlewood and Polya theorem ([10], proposition 2.3.3).

Corollary 1. Let assumptions of either Theorem 1 a) or Theorem 2 be fulfilled, and $\nu \in \mathbb{N}$, $\mu \geq \nu - 1$. Then the function $y \to \psi_{a,c}(\mu, \nu; 1/y)$ is completely monotone and logarithmically convex on $(0, \infty)$, and, consequently, there exists a non-negative measure τ with a support in $[0, \infty)$ such that

$$\psi_{a,c}(\mu, \nu; x) = \int_{[0,\infty)} e^{-t/x} d\tau(t).$$

Scheme of the proof. According to theorem 3 in [11], the sum of convergent series consisting of completely monotone functions is completely monotone function. Hence, the function $y \to \psi_{a,c}(\mu, \nu; 1/y)$ is completely monotone, because the function $1/y$ is completely monotone, and the coefficients are non-negative by virtue of Theorem 2. The integral representation follows from the classical Bernstein theorem ([12], theorem 1.4), and logarithmic convexity can be obtained, for instance, in accordance with [10] (see exercise 2.1(6)).

Corollary 2. Let assumptions either of Theorem 1 a) or of Theorem 2 be fulfilled. Then for any $\nu \in \mathbb{N}$, $\mu \geq \nu - 1$ and $x \geq 0$ the inequality is valid

$$g_{a,c}(\mu; x) g_{a,c}(\nu; x) - g_{a,c}(0; x) g_{a,c}(\mu+\nu; x) \geq g_0^2 \left\{ \frac{\Gamma(a+\nu)\Gamma(a+\mu)}{\Gamma(c+\nu)\Gamma(c+\mu)} - \frac{\Gamma(a)\Gamma(a+\mu+\nu)}{\Gamma(c)\Gamma(c+\mu+\nu)} \right\}. \tag{5}$$

If $a - c, \mu, \nu \neq 0$, then the equality is attained at the point $x = 0$, only.

Scheme of the proof. The right-hand side of inequality (5) is free term in expansion of the function $\psi_{a,c}(\mu, \nu; x)$ into series in powers of x. Then the inequality follows from non-negativity of coefficients at all positive powers of x.

Corollary 3. Let assumptions either of Theorem 1 a) or of Theorem 2 be fulfilled. Then

$$\frac{(a+\mu)_\nu (c)_\nu}{(c+\mu)_\nu (a)_\nu} \le \frac{g_{a,c}(0;x) g_{a,c}(\mu+\nu;x)}{g_{a,c}(\nu;x) g_{a,c}(\mu;x)} \le 1$$

for all $\nu \in \mathbb{N}$, $\mu \ge 0$ and $x \ge 0$.

Scheme of the proof. The upper bound is equivalent to the inequality $\psi_{a,c}(\mu,\nu;x) \ge 0$, which is a part of assertion of Theorem 2. The lower bound is a part of assertion of theorem 2 from [4] if we apply it to the function $f_{a,c}(\mu;x) = \Gamma(c+\mu) g_{a,c}(\mu;x)/\Gamma(a+\mu)$.

We combine Corollaries 2 and 3, and obtain two-sided bounds for the Turán determinant

$$g_0^2 \frac{\Gamma(a)^2}{\Gamma(c)^2} \left[\frac{(a)_\nu^2}{(c)_\nu^2} - \frac{(a)_{2\nu}}{(c)_{2\nu}} \right] \le g_{a,c}(\nu;x)^2 - g_{a,c}(0;x) g_{a,c}(2\nu;x)$$

$$\le \frac{(c+\nu)_\nu (a)_\nu - (a+\nu)_\nu (c)_\nu}{(a)_\nu (c+\nu)_\nu} g_{a,c}(\nu;x)^2. \quad (6)$$

These inequalities are valid for $\nu \in \mathbb{N}$, $a \ge c > 0$ under assumption that $\{g_n\}_{n\ge 0}$ is non-negative sequence, which is also logarithmically concave and has no inner zeros for $a > c+1$.

Remark. Earlier we proved the theorem (see [4], theorem 1) on properties of the series

$$\sum_{n=0}^{\infty} f_n \frac{(a+\mu)_n}{(c+\mu)_n} \frac{x^n}{n!},$$

which for $c \ge a$ are analogous to the properties of the series $g_{a,c}(\mu;x)$ from (1). In this connection the following question arises: Whether it is possible to remove factorial in the series keeping validity of its properties (and, consequently, strengthening theorem 1 from [4])? The answer is negative. One can verify immediately that

$$\sum_{k=0}^{2} \left(\frac{(a+1)_k (a+\mu)_{2-k}}{(c+1)_k (c+\mu)_{2-k}} - \frac{(a)_k (a+\mu+1)_{2-k}}{(c)_k (c+\mu+1)_{2-k}} \right) < 0$$

for $a = 1$, $\mu = 1/2$, $c = 20$. Consequently, the coefficient at x^2 in expansion of difference of products, which is analogous to (2), is negative in the present case. The coefficients at higher powers of x are negative, too.

Example 1. If we put in (1) $g_n = (b)_n/n!$, then

$$g_{a,c}(\mu;x) = \sum_{n=0}^{\infty} \frac{(b)_n}{n!} \frac{\Gamma(a+\mu+n)}{\Gamma(c+\mu+n)} x^n = \frac{\Gamma(a+\mu)}{\Gamma(c+\mu)} {}_2F_1(a+\mu, b; c+\mu; x),$$

where ${}_2F_1$ is the Gauss hypergeometric function ([13], Chap. 2). One can verify easily that the sequence $\{(b)_n/n!\}$ is logarithmically concave if and only if $b \ge 1$ (the abscence of inner zeros is evident for all b). Consequently, if either $c+1 \ge a \ge c > 0$ and $b > 0$ or $a > c+1 > 1$ and $b \ge 1$, then the function $g_{a,c}(\mu;x)$ satisfies inequalities from Corollaries 2 and 3, and also inequality (6). In particular, for $\nu = 1$ the latter inequality turns into

$$\frac{a}{c}({}_2F_1(a+1,b;c+1;x))^2 - \frac{a+1}{c+1}\, {}_2F_1(a,b;c;x) {}_2F_1(a+2,b;c+2;x) \ge \frac{a-c}{c(c+1)} \ge 0, \quad 0 \le x < 1. \quad (7)$$

Note that for $c \ge a$ and $b > 0$ the function $g_{a,c}(\mu;x)$ satisfies theorem 3 from [4] and all its corollaries.

Example 2. Normalized incomplete beta-function is defined by the formula

$$I_x(a,b) = \frac{1}{B(a,b)} \int_0^x t^{a-1}(1-t)^{b-1} dt,$$

and is a function of distribution of random variable satisfying the law of beta-distribution. The beta-function in denominator equals $\Gamma(a)\Gamma(b)/\Gamma(a+b)$. We perform the change of variable $t = ux$, and obtain the relation

$$I_x(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^a \int_0^1 u^{a-1}(1-ux)^{b-1}du = \frac{\Gamma(a+b)x^a}{\Gamma(a+1)\Gamma(b)} {}_2F_1(1-b, a; a+1; x),$$

where the Euler representation ([13], theorem 2.2.1) is applied. Furthermore, we apply the Euler transformation ([13], theorem 2.2.5)

$${}_2F_1(\alpha, \beta; \gamma; x) = (1-x)^{\gamma-\alpha-\beta} {}_2F_1(\gamma-\alpha, \gamma-\beta; \gamma; x),$$

and obtain the representation

$$I_x(a,b) = \frac{\Gamma(a+b)x^a}{\Gamma(a+1)\Gamma(b)} (1-x)^b {}_2F_1(a+b, 1; a+1; x) = \frac{x^a(1-x)^b}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+b+n)}{\Gamma(a+1+n)} x^n.$$

Since the factor at the sum is logarithmically neutral in parameter a, theorem 3 from [4] and Theorems 1 and 2 from the present paper imply the following proposition: If $0 < b \leq 1$, then the function $a \to I_x(a,b)$ is logarithmically convex on $(0,\infty)$ for arbitrarily fixed $0 < x < 1$; if $b > 1$, then the function $a \to I_x(a,b)$ satisfies inequalities from Corollaries 2 and 3, and also inequality (6) for arbitrarily fixed $0 < x < 1$. In another way we can show that in the second case the function $a \to I_x(a,b)$ is logarithmically convex on $(0,\infty)$. Moreover, the function $b \to I_x(a,b)$ is also logarithmically convex on $(0,\infty)$ for any $a > 0$ and $0 < x < 1$. Proof of these facts will be given in other publication.

Example 3. This example complements example 2 from [4]. Consider the Gauss fraction ([13], § 2.5)

$$r(x) = \frac{{}_2F_1(a+1, b; c+1; x)}{{}_2F_1(a, b; c; x)}.$$

We apply adjacency relation ([13], (2.5.3))

$$\frac{a+1}{c+1} {}_2F_1(a+2, b; c+2; x) = \frac{c+(a-b+1)x}{(c-b+1)x} {}_2F_1(a+1, b; c+1; x) - \frac{c}{(c-b+1)x} {}_2F_1(a, b; c; x).$$

We substitute this relation in inequality (7) and obtain

$$\frac{a}{c}({}_2F_1(a+1, b; c+1; x))^2 \geq \frac{c+(a-b+1)x}{(c-b+1)x} {}_2F_1(a, b; c; x) {}_2F_1(a+1, b; c+1; x) - \frac{c({}_2F_1(a, b; c; x))^2}{(c-b+1)x}$$

or, after division by $({}_2F_1(a, b; c; x))^2$,

$$\frac{a}{c} r(x)^2 - \frac{c+(a-b+1)x}{(c-b+1)x} r(x) + \frac{c}{(c-b+1)x} \geq 0.$$

We resolve this inequatily separately for $c - b + 1 < 0$ and $c - b + 1 > 0$, combine the result with inequalities from example 2 from [4], and obtain the table

Table.

	$c+1<b$	$c+1>b$
$c+1 \geq a \geq c > 0, b > 0$ or $a > c+1 > 1, b > 1$	$r(x) \geq \Lambda_{a,b,c}(x)$	$r(x) \leq \Lambda_{a,b,c}(x)$
$c \geq a > 0, b > 0$	$r(x) \leq \Lambda_{a,b,c}(x)$	$r(x) \geq \Lambda_{a,b,c}(x)$

where $\Lambda_{a,b,c}(x) = \dfrac{c+(a-b+1)x - \sqrt{(c+(a-b+1)x)^2 - 4a(c-b+1)x}}{2(a/c)(c-b+1)x}$.

Example 4. Intrinsic generalization of Example 1 is the function

$$g(\mu;x)=\frac{\Gamma(a_1+\mu)}{\Gamma(c_1+\mu)}{}_{q+1}F_q(a_1+\mu,a_2,\ldots,a_{q+1};c_1+\mu,c_2,\ldots,c_q;x),\tag{8}$$

where ${}_{q+1}F_q$ is generalized hypergeometric series ([13], (2.1.2)). Application of lemma 9 from [4] implies the following proposition: Let either $c_1+1\ge a_1\ge c_1>0$ and $a_2,\ldots,a_{q+1},c_2,\ldots,c_q$ be any positive numbers, or $a>c+1>1$ and the inequalities are fulfilled

$$\frac{e_q(c_2,\ldots,c_q,1)}{e_q(a_2,\ldots,a_{q+1})}\le\frac{e_{q-1}(c_2,\ldots,c_q,1)}{e_{q-1}(a_2,\ldots,a_{q+1})}\le\cdots\le\frac{e_1(c_2,\ldots,c_q,1)}{e_1(a_2,\ldots,a_{q+1})}\le 1.$$

Then $g(\mu;x)$ from (8) and constructed for it by formula (2) function $\psi(\mu,\nu;x)$ satisfy assumptions of Theorem 2, Corollaries 1, 2, 3 and inequality (6). Here $e_k(x_1,\ldots,x_q)$ is kth elementary symmetric polynomial, i.e.,

$$e_0(x_1,\ldots,x_q)=1,\;\; e_k(x_1,\ldots,x_q)=\sum_{1\le j_1<j_2<\cdots<j_k\le q}x_{j_1}x_{j_2}\cdots x_{j_k},\;\; k\ge 1.$$

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