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ON POLYNOMIALS AND RATIONAL FUNCTIONS NORMALIZED ON CIRCULAR ARCS

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Applications of geometric function theory to some inequalities for algebraic polynomials and rational functions normalized on circular arcs are considered. In particular, coefficients estimates and also covering and distortion theorems are proved. The latter theorems supplement recent results of the author and V. N. Dubinin. Bibliography: 18 *titles.*

Introduction

Inequalities for polynomials and rational functions are studied in many publications (e.g., see the references in [1, 2]). Lately, considerable attention has been paid to polynomials with constraints on arcs of the unit circle [3–10]. In particular, in [9] it was demonstrated how new covering and distortion theorems, as well as estimates on the absolute value of the product of the leading coefficient and constant term of an algebraic polynomial with constraints on circular arcs, can be derived from the majorization principles for meromorphic functions [11–13].

The aim of the present paper is to refine and generalize the results in [9]. The paper consists of two parts. The first one presents theorems for rational functions that are immediately obtained by applying the majorization principle (see [11–14]) to a suitable meromorphic function. These theorems depend on the Green's function and inner radius of domains complementary to circular arcs. In the second part, inequalities for polynomials with constraints on a circular arc are established. The latter inequalities supplement the corresponding results in [9]. In this part, the proofs are based on the approach suggested by V. N. Dubinin in [15], which can be outlined as follows: Given a polynomial, an analytic function is constructed; then, to this function methods of geometric function theory are applied. The technical details of the proofs are borrowed from Olesov's paper [3].

Everywhere below, Γ denotes a union of a finite number of disjoint closed nondegenerate arcs of the unit circle $|z| = 1$; $D = \overline{\mathbb{C}}_z \setminus \Gamma$; $g_B(z, \zeta)$ is the Green's function of a domain B; $r(B, z)$ is the inner radius of the domain B with respect to a point z [14];

$$\Gamma_\alpha = \{z = e^{ix} : -\alpha \le x \le \alpha\}, \quad 0 < \alpha < \pi.$$

We consider polynomials with complex coefficients of the form

$$P(z) = c_n z^n + \dots + c_k z^k, \quad n > k, \quad c_n c_k \ne 0, \tag{1}$$

and also rational functions of the form

$$R(z) = \frac{P(z)}{z^{p_0} \prod\limits_{j=1}^{p} (z - a_j)}, \quad p_0 \ge 0, \quad a_j \in \mathbb{C}_z \setminus (\Gamma \cup \{0\}), \tag{2}$$

where P is a polynomial of the form (1). In what follows, we use the notation

$$x_+ = \max\{0, x\},$$

$$m = m(F; \Gamma) = \min_{z \in \Gamma} |F(z)|, \quad M = M(F; \Gamma) = \max_{z \in \Gamma} |F(z)|,$$

where, depending on the context, F is either a polynomial or a rational function. The degree of a rational function is understood as the number of preimages of ∞ lying on the Riemann sphere $\overline{\mathbb{C}}_z$ with account for their multiplicities;

$$\Psi(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

is Zhukovsky's function. The regular branch of the function

$$\widetilde{\Phi}(\omega) = \omega + \sqrt{\omega^2 - 1},$$

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inverse to Zhukovsky's function, is defined on the exterior of the circular arc connecting the points ± 1 and passing through the point $i\tan(\alpha/2)$ by the condition $\widetilde{\Phi}(\infty)=\infty$, and the function

$$\Phi(\omega)=\omega+\sqrt{\omega^2-1}$$

is defined on the exterior of the interval $[-1,1]$ by the condition $\Phi(\infty)=\infty$. Finally, we set

$$\delta(\xi)=\frac{2}{1-\cos\alpha}\Psi(\xi)-\frac{1+\cos\alpha}{1-\cos\alpha}.$$

1. Inequalities for rational functions

Theorem 1. *Let R be an irreducible rational function of the form* (2) *and let $h(z)=R(z)\overline{R(1/\overline{z})}$, $M=M(h,\Gamma)$, $m=m(h,\Gamma)$. Then, for an arbitrary point z, the following inequality holds*:

$$\left|2h(z)-M^2-m^2+2\sqrt{(h(z)-M^2)(h(z)-m^2)}\right|$$
$$\leq (M^2-m^2)\exp((n-k-p)_+(g_D(z,0)+g_D(1/\overline{z},0))+\sum_{j=1}^{p'}(g_D(z,a_j')+g_D(1/\overline{z},a_j')))$$

(*for any choice of the value of the root*), *where a_j' are those points a_j which are poles of the function h, whereas p' is their number with account for the multiplicities. The equality at a point $z\neq 0,\infty$ and $z\notin\Gamma$ for a certain value of the root is attained if and only if the function h satisfies the condition $h(D)=\overline{\mathbb{C}}_w\setminus[m^2,M^2]$ and h is a complete N-fold covering of the domain $\overline{\mathbb{C}}_w\setminus[m^2,M^2]$ by the domain D, where N is the degree of h.*

Proof. The domains D and $G=\overline{\mathbb{C}}_w\setminus[m^2,M^2]$ have classical Green's functions. The function h is meromorphic in D, and in D it has poles at the points a_j' and $1/\overline{a_j'}$, $j=1\dots p'$, and also poles of order $(n-k-p)_+$ at the points $z=0$ and $z=\infty$. In addition, as z tends to the set Γ, all the limit boundary values of h lie on the interval $[m^2,M^2]$. By Theorem 1 in [13], at points of the domain D it holds that

$$g_G(f(z),\infty)\leq(n-k-p)_+(g_D(z,0)+g_D(1/\overline{z},0))+\sum_{j=1}^{p'}(g_D(z,a_j')+g_D(z,1/\overline{a_j'})),$$

and the equality at a point $z\neq 0,\ \infty$ occurs if and only if $G=h(D)$ and the function h is a complete N-fold covering of the domain $\overline{\mathbb{C}}_w\setminus[m^2,M^2]$ by the domain D. In view of the symmetry of the domain D with respect to the circle $|z|=1$, we have

$$g_D(z,\zeta)=g_D(1/\overline{z},1/\overline{\zeta}),\quad z,\zeta\in D.$$

It remains to observe that

$$g_G(w,\infty)=\log\left|\frac{2w-M^2-m^2}{M^2-m^2}+\sqrt{\left(\frac{2w-M^2-m^2}{M^2-m^2}\right)^2-1}\right|.$$

This completes the proof. $\square$

Remark. The extremal rational function is defined up to multiplication by an integer power of z and by the Blaschke product.

Theorem 2. *In the notation of Theorem* 1, *the coefficients of an irreducible rational function R of the form* (2) *for $n-k>p$ satisfy the inequality*

$$\frac{|c_nc_k|}{\prod_{j=1}^{p}|a_j|}\leq\frac{1}{4}(M^2-m^2)r^{2(n-k-p)}(D,0)\times\exp\left(\sum_{j=1}^{p'}(g_D(\infty,a_j')+g_D(\infty,1/\overline{a_j'}))\right).\tag{3}$$

Equality in (3) *is attained if and only if for the function $h(z)=R(z)\overline{R(1/\overline{z})}$ we have $h(D)=\overline{\mathbb{C}}_w\setminus[m^2,M^2]$ and h is a complete N-fold covering of the domain $\overline{\mathbb{C}}_w\setminus[m^2,M^2]$ by the domain D, where N is the degree of h.*

Proof. From Mityuk's inequality [14] (also see [11, Corollary 1]) it follows that

$$\frac{|c_n c_k|}{\prod_{j=1}^{p}|a_j|} \le \frac{r^{(n-k-p)}(D,\infty)}{r(G,\infty)} \exp\left\{(n-k-p)g_D(0,\infty) + \left(\sum_{j=1}^{p'}(g_D(\infty,a'_j)+g_D(\infty,1/\overline{a'_j}))\right)\right\},$$

and equality occurs if and only if the corresponding conditions of the theorem are fulfilled. As is readily seen,

$$\log|z| + g_D(z,0) \equiv g_D(z,\infty).$$

Consequently,

$$\log r(D,0) = g_D(0,\infty).$$

Finally, straightforward computations yield

$$r(\overline{\mathbb{C}}_w \setminus [m^2,M^2],\infty)\frac{1}{\operatorname{cap}([m^2,M^2])} = \frac{4}{M^2-m^2},$$

which completes the proof. □

By $\omega(z,E,\Omega)$ denote the harmonic measure of a set $E\subset\partial\Omega$ at a point z relative to a domain Ω. In the case where $\Omega = D$, the density of the harmonic measure is defined as follows:

$$\varpi(\zeta,e^{ix}) = \frac{\partial}{\partial x}\omega\big(\zeta,\Gamma\cap\{e^{i\theta}: 0\le\theta\le x\},D\big),\quad \zeta\in D,\quad e^{ix}\in\Gamma.$$

Theorem 3. *In the notation of Theorem* 1, *for an irreducible rational function of the form* (2) *it holds that*

$$|(|R(z)|^2)'_x| \le 2\pi((n-k-p)_+\varpi(\infty,z) + \sum_{j=1}^{p'}\varpi(a'_j,z))\sqrt{(M^2-|R(z)|^2)(|R(z)|^2-m^2)}, \tag{4}$$

where $z = e^{ix}\in\Gamma$.

Equality in (4) *occurs if and only if for the function* $h(z) = R(z)\overline{R(1/\overline{z})}$ *we have* $h(D) = \overline{\mathbb{C}}_w\setminus[m^2,M^2]$ *and* h *is a complete* N*-fold covering of the domain* $\overline{\mathbb{C}}_w\setminus[m^2,M^2]$ *by the domain* D*, where* N *is the degree of* h.

Proof. The function h, defined in the domain D, satisfies the assumptions of Corollary 2 in [12], provided that as G one takes the domain $\overline{\mathbb{C}}_w\setminus[m^2,M^2]$. Therefore,

$$\frac{|R'(z)\overline{R(1/\overline{z})} + R(z)\overline{R'(1/\overline{z})}(-1/z^2)|}{\sqrt{(M^2-|R(z)|^2)(|R(z)|^2-m^2)}} = \frac{|zR'(z)\overline{R(z)} - \overline{z}\overline{R'(z)}R(z)|}{\sqrt{(M^2-|R(z)|^2)(|R(z)|^2-m^2)}} = \frac{2|\Im zR'(z)\overline{R(z)}|}{\sqrt{(M^2-|R(z)|^2)(|R(z)|^2-m^2)}}$$
$$\le (n-k-p)_+\left(\frac{\partial g_D(z,\infty)}{\partial n^\pm} + \frac{\partial g_D(z,0)}{\partial n^\pm}\right) + \sum_{j=1}^{p'}\left(\frac{\partial g_D(z,a'_j)}{\partial n^\pm}\frac{\partial g_D(z,1/\overline{a'_j})}{\partial n^\pm}\right),$$

where $z\in\Gamma$, $\frac{\partial}{\partial n^+}$ denotes differentiation along the outward normal to Γ, and $\frac{\partial}{\partial n^-}$ denotes differentiation in the opposite direction. From the relation

$$\varpi(a,z) = \frac{1}{2\pi}\left[\frac{\partial g_D(z,a)}{\partial n^+} + \frac{\partial g_D(z,a)}{\partial n^-}\right],\quad z\in\operatorname{int}\Gamma,$$

it follows that

$$2|\Im zR'(z)\overline{R(z)}| \le \pi((n-k-p)_+(\varpi(\infty,z)+\varpi(0,z)) + \sum_{j=1}^{p'}(\varpi(a'_j,z)+\varpi(1/\overline{a'_j},z)))\sqrt{(M^2-|R(z)|^2)(|R(z)|^2-m^2)}.$$

Here, $\operatorname{int}\Gamma$ means the set Γ from which the endpoints of the arcs composing it are excluded. In view of the symmetry of D, we have

$$\varpi(a,z) = \varpi(1/\overline{a},z),\quad a\in\overline{\mathbb{C}}_z\setminus\Gamma,\quad z\in\operatorname{int}\Gamma.$$

Now, in order to complete the proof of the inequality, it only remains to note that

$$\left|\Im zR'(z)\overline{R(z)}\right| = \left|\Im\frac{zR'(z)}{R(z)}R(z)\overline{R(z)}\right| = \frac{1}{2}\left|\left(|R(z)|^2\right)'_x\right|.$$

The assertion concerning the equality case stems from the respective assertion of Theorem 2 in [12] or from Corollary 1 in [13].

This completes the proof. $\square$

Consider a method for finding the extremal rational function in the case of a unique arc $\Gamma = \Gamma_\alpha$. To this end, use the arguments from [9] and [16, pp. 106–107]. Assume that $M = 1$ and $m = 0$. Equalities in Theorems 1–3 occur if and only if for the function $h(z) = R(z)\overline{R(1/\overline{z})}$ we have $h(D) = \overline{\mathbb{C}}_w \setminus [0,1]$ and h is an N-fold covering of the domain $\overline{\mathbb{C}}_z \setminus [0,1]$ by the domain D, where N is the degree of h. Construct the function h as the superposition of the elementary functions

$$u(z) = \Phi\left[i\frac{z-1}{z+1}\operatorname{cotan}\frac{\alpha}{2}\right] \quad \text{and} \quad \widetilde{u}(z) = \Phi\left(\delta(z)\right), \quad z \in \mathbb{C} \setminus \Gamma_\alpha,$$

which define a one-sheeted and a two-sheeted mappings, respectively, of the exterior of the arc Γ_α onto the exterior of the unit disk. Let

$$B(u) = \prod_{\substack{j=1\\ a_j=-1}}^{p} u^2 \prod_{\substack{j=1\\ a_j\neq -1}}^{p} \frac{(1-\overline{u(a_j)}u)(1-\overline{u(1/\overline{a_j})}u)}{(u-u(a_j))(u-u(1/\overline{a_j}))}.$$

The functions $u(z)$ and $\widetilde{u}(z)$ take pairwise conjugate values at points symmetric about the unit circle. This fact and the symmetry of the points a_j and $1/\overline{a_j}$ imply that the function $B(u)$ is real. In addition, $B(|u| > 1) = \{|u| > 1\}$.

Consider the function

$$\Omega(z) = \widetilde{u}^{(n-p)_+}(z)B[u(z)].$$

From the symmetry principle it follows that the function $\Psi[\Omega(z)]$ is regular on the entire sphere $\overline{\mathbb{C}}_z$, except for the poles a_j and $1/\overline{a_j}$, $j = 1\dots p$, and also the poles of order $(n-p)_+$ at the points 0 and ∞. Therefore,

$$\Psi[\Omega(z)] = \frac{\widetilde{P}(z)}{z^{(n-p)_+}\prod\limits_{j=1}^{p}(z-a_j)(1-\overline{a_j}z)},$$

where $\widetilde{P}$ is an algebraic polynomial of degree $2p + 2(n-p)_+$. Then

$$h(z) = \frac{1}{2}(\Psi[\Omega(z)] + 1).$$

On the arc Γ_α, we have $h(z) \equiv |R(z)|^2$, and the zeros of the rational function can be found from the equation $\Psi[\Omega(z)] = -1$.

In the case of a polynomial, the above argument reduces to that in [9]. As a result, up to a constant multiplier, we arrive at the Vidensky polynomial

$$P_\alpha(z) = \begin{cases} \prod\limits_{k=1}^{n/2}(z^2 - 2a_k z + 1) & \text{for } n \text{ even},\\ (z-1)\prod\limits_{k=1}^{(n-1)/2}(z^2 - 2a_k z + 1) & \text{for } n \text{ odd}, \end{cases}$$

where $a_k = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\cos\frac{\pi(2k-1)}{n}$.

Earlier, Maergoiz and Rybakova [5] proved that P_α is the polynomial least deviating from zero on the circular arc among all unital polynomials having zeros in this arc (also see [6]).

Remark. The extremal polynomial can also be represented in the form

$$P_\alpha(z) = 2\varepsilon\sin^n(\alpha/2)\sqrt{z^n}T_n\left(\frac{\sqrt{z}-1/\sqrt{z}}{2i\sin(\alpha/2)}\right),$$

where $T_n(z)$ is the Chebyshev polynomial of the first kind of degree n, and ε is an arbitrary number such that $|\varepsilon| = 1$ (see [7, 9]).

The function

$$z = \varphi(w) = w\frac{1 + w\sin(\alpha/2)}{w + \sin(\alpha/2)}$$

maps the domain $|w| > 1$ conformally and univalently onto the exterior of the arc Γ_α, and, in addition, the point at infinity is taken into the point at infinity [3]. The function $w = \psi(z)$ inverse to the function $z = \varphi(w)$ can be represented as

$$\psi(z) = -i\cos\left(\frac{\alpha}{2}\right)\widetilde{\Phi}\left(i\frac{z - \cos\alpha}{\sin\alpha}\right) - \sin\left(\frac{\alpha}{2}\right), \quad z \in \overline{\mathbb{C}}_z \setminus \Gamma_\alpha.$$

Given a polynomial P of the form (1), consider the associated function

$$\rho(z) = \frac{2P(z)\overline{P(1/\overline{z})} - M^2 - m^2}{M^2 - m^2}.$$

On the set $G = \{w : \ |w| > 1,\ \rho(\varphi(w)) \notin [-1, 1]\}$, define the meromorphic function $\zeta = F(w)$ by setting

$$F(w) = w\frac{\Phi[\rho(\varphi(w))]}{\Phi^{n-k}[\delta(\varphi(w))]}$$

at the points w at which $\varphi(w) \neq 0$.

Let $\mathcal{D}$ be the collection of domains composing the set $G \setminus \{w : \ |F(w)| = 1\}$. By the maximum modulus principle for a regular function, we have

$$\left|\frac{w}{\Phi[\delta(\varphi(w))]}\right| < 1, \quad |w| > 1.$$

This fact, along with the boundary properties of the functions $\Phi[\rho(\varphi(w))]$ and $\Phi[\delta(\varphi(w))]$, implies that as the point w approaches the boundary of each of the domains in $\mathcal{D}$, all the limit values of $|F(w)|$ become less than or equal to unity. Moreover, $0 < |F'(\infty)| < \infty$. Repeating the proof of Lemma 2.2 in [15] for the function $1/F(1/w)$, we see that for any domain $D \in \mathcal{D}$, either $F(D) \cap \{\zeta : |\zeta| > 1\} = \varnothing$ or $F(D) = \{\zeta : |\zeta| > 1\}$. In the latter case, there exists a function $w = f(\zeta)$ inverse to $F(w)$, which univalently maps the domain $|\zeta| > 1$ onto D.

Theorem 4. *Let P be a polynomial of the form* (1) *and let $h(z) = P(z)\overline{P(1/\overline{z})}$. Then, for all points z, the following inequality holds*:

$$\left|2h(z) - M^2 - m^2 + 2\sqrt{(h(z) - M^2)(h(z) - m^2)}\right| \le (M^2 - m^2)\frac{\beta_{\lambda,r(z)}}{r(z)}\left|\Phi\left[\frac{2}{1 - \cos\alpha}\Psi(z) - \frac{1 + \cos\alpha}{1 - \cos\alpha}\right]\right|^{n-k}. \quad (5)$$

Furthermore, if $|\psi(z)| > r_\lambda$, then

$$\left|2h(z) - M^2 - m^2 + 2\sqrt{(h(z) - M^2)(h(z) - m^2)}\right| \ge (M^2 - m^2)\frac{\alpha_{\lambda,r(z)}}{r(z)}\left|\Phi\left[\frac{2}{1 - \cos\alpha}\Psi(z) - \frac{1 + \cos\alpha}{1 - \cos\alpha}\right]\right|^{n-k}, \quad (6)$$

where

$$\lambda = \frac{4|c_n c_k|\sin^{2(n-k)}(\alpha/2)}{M^2 - m^2}, \qquad r(z) = |\psi(z)|,$$

whereas $\alpha_{\lambda,r(z)}$ and $\beta_{\lambda,r(z)}$ are the roots of the equations

$$\lambda(r(z) + 1)^2 x = r(z)(x + 1)^2 \quad \text{and} \quad \lambda(r(z) - 1)^2 x = r(z)(x - 1)^2,$$

respectively, lying in the interval $(1, r(z)]$;

$$r_\lambda = 2\lambda^{-1} - 1 + 2\sqrt{\lambda^{-1}(\lambda^{-1} - 1)}.$$

For a suitable choice of the root, equalities in (5) *and* (6) *occur, for instance, for the polynomial P_α.*

Proof. Let $w = f(\zeta)$ be the function defined above. Straightforward computations yield

$$\frac{1}{f'(\infty)} = \lim_{w\to\infty}\frac{F(w)}{w} = \frac{4c_n\overline{c_k}\sin^{2(n-k)}(\alpha/2)}{M^2 - m^2},$$

and, by Schwarz's lemma, we have $\lambda = |f'(\infty)|^{-1} \le 1$ (also see [9]).

The function $f_1(\zeta) = f'(\infty)/f(1/\zeta)$ is univalent in the disk $|\zeta| < 1$, smaller than λ^{-1} in the absolute value, and can be represented as the power series

$$f_1(\zeta) = \zeta + \alpha_2\zeta^2 + \alpha_3\zeta^3 + \cdots .$$

In the class of such functions, the following sharp two-sided estimates hold:

$$\left(\frac{1 + |\lambda f_1(\zeta)|}{1 + |\zeta|}\right)^2 \le \left|\frac{f_1(\zeta)}{\zeta}\right| \le \left(\frac{1 - |\lambda f_1(\zeta)|}{1 - |\zeta|}\right)^2, \quad 0 < |\zeta| < 1. \tag{7}$$

An equality on the left-hand or right-hand side of (7) is attained at at least one point if and only if

$$\frac{f_1(\zeta)}{(1 + e^{i\beta}\lambda f_1(\zeta))^2} \equiv \frac{\zeta}{(1 + e^{i\beta}\zeta)^2},$$

where β is a real number (e.g., see [15, 17]).

Let w, with $|w| = r$ and $\varphi(w) \ne 0$, be a point of the set $f(|\zeta| > 1)$. The right-hand-side inequality in (7) yields

$$\frac{(|F(w)| - 1)^2}{|F(w)|} \le \lambda \frac{(r-1)^2}{r}.$$

Since the function $y = (x-1)^2/x$ is strictly increasing on the ray $x > 1$, there exists a unique root $\beta_{\lambda,r}$ of the equation $\lambda(r-1)^2 x = r(x-1)^2$ that is located in the interval $(1, r]$. This also implies that $|F(w)| \le \beta_{\lambda,r}$, i.e.,

$$\bigl|\Phi[\rho(\varphi(w))]\bigr| \le \frac{\beta_{\lambda,r}}{r}\bigl|\Phi^{n-k}[\delta(\varphi(w))]\bigr|. \tag{8}$$

If $w \notin f(|\zeta| > 1)$, then the inequality $|F(w)| \le 1$ holds, i.e.,

$$|\Phi[\rho(\varphi(w))]| \le \frac{1}{r}\Bigl|\Phi^{n-k}[\delta(\varphi(w))]\Bigr| < \frac{\beta_{\lambda,r}}{r}\bigl|\Phi^{n-k}[\delta(\varphi(w))]\bigr|.$$

Thus, (8) holds for any w such that $|w| > 1$ and $\varphi(w) \ne 0$. Using the change of variable $\varphi(w) = z$ and the explicit representation of the function $\delta(\xi)$, we arrive at inequality (5).

Now prove inequality (6). Let w, $|w| = r > r_\lambda$, $\varphi(w) \ne 0$, be a point of the set $f(|\zeta| > 1)$. The left-hand-side inequality in (7) and the fact that the function $y = (x+1)^2/x$ is strictly increasing on the ray $x > 1$ imply that

$$\frac{(|F(w)| + 1)^2}{|F(w)|} \ge \lambda\frac{(r+1)^2}{r} > \lambda\frac{(r_\lambda + 1)^2}{r_\lambda} = 4.$$

Therefore, there exists a unique root $\alpha_{\lambda,r}$ of the equation $\lambda(r+1)^2 x = r(x+1)^2$ lying in the interval $(1, r]$, and

$$|F(w)| \ge \alpha_{\lambda,r}. \tag{9}$$

It follows that

$$\bigl|\Phi[\rho(\varphi(w))]\bigr| \ge \frac{\alpha_{\lambda,r}}{r}\bigl|\Phi^{n-k}[\delta(\varphi(w))]\bigr|. \tag{10}$$

Now we demonstrate that any point w, $|w| > r_\lambda$, belongs to the image $f(|\zeta| > 1)$. Suppose the contrary, i.e., let

$$r_\lambda < r^* = \inf\Bigl\{r : r > 1,\ \bigl|F(w)\bigr| > 1 \quad \text{for all} \quad w,\ |w| = r\Bigr\}.$$

On the circle $|w| = r^*$ there is a point w^* such that

$$|F(w^*)| = 1. \tag{11}$$

On the other hand, for any sequence w_k, $|w_k| > r^*$, $k = 1, 2, \ldots$, converging to w^* from (9) we obtain

$$|F(w_k)| \ge \alpha_{\lambda,r^*}, \quad k = 1, 2, \ldots,$$

which contradicts (11). Thus, inequality (10) holds for all w, $|w| = r > r_\lambda$, $\varphi(w) \ne 0$. Performing the change of variable $\varphi(w) = z$, we see that (6) holds whenever $r(z) = |w|$.

The assertion concerning the equality case stems from the identity $F(w) \equiv w$, which holds for the polynomial indicated.

This completes the proof of the theorem. □

Theorem 5. *For a polynomial P of the form* (1) *it holds that*

$$|(|P(z)|^2)'_x| \leq \frac{\cos(x/2)\sqrt{(M^2-|P(z)|^2)(|P(z)|^2-m^2)}}{\sqrt{\sin^2(\alpha/2)-\sin^2(x/2)}} \times \left[n-k-\frac{\Lambda(\alpha,z)\cos(\alpha/2)(1-2\sin^{n-k}(\alpha/2)\sqrt{|c_nc_k|/(M^2-m^2)})}{2\cos(x/2)}\right], \quad (12)$$

where $z=e^{ix}\in\Gamma_\alpha$ and

$$\Lambda(\alpha,z)=\left|\Phi\left(i\frac{z-\cos\alpha}{\sin\alpha}\right)\right|.$$

Equality in (12) *is attained, for instance, for the polynomial P_α.*

Proof. Let $w=f(\zeta)$ be the function defined above. If a point ζ, $|\zeta|=1$, is a regular point of the function $f(\zeta)$ and if $|f(\zeta)|=1$, then (see [15, p. 21])

$$|f'(\zeta)| \geq \frac{1}{2\sin^{n-k}(\alpha/2)}\sqrt{\frac{M^2-m^2}{|c_nc_k|}}. \quad (13)$$

If a point w, $|w|=1$, is a regular point of the function $|F(w)|$ and if, simultaneously, it lies on the boundary of a domain $D\in\mathcal{D}$ such that $F(D)\cap\{\zeta:\ |\zeta|>1\}=\varnothing$, then, at this point,

$$\frac{\partial|F|}{\partial|w|}\leq 0.$$

If $F(D)=\{\zeta:|\zeta|>1\}$, then, at this point, by applying inequality (13), we obtain

$$\frac{\partial|F|}{\partial|w|}=|f'(\zeta)|^{-1}\leq 2\sin^{n-k}(\alpha/2)\sqrt{\frac{|c_nc_k|}{M^2-m^2}}. \quad (14)$$

Thus, inequality (14) holds at all points of the unit circle except, possibly, for a finite number of such points.

Below, by the values of the function $w=\psi(z)$ at points of an arc Γ_α we understand the values obtained as a result of a regular extension of this function from the domain $|z|>1$. At the points $w\in\psi(\Gamma_\alpha)$ at which the values of the functions $\Phi[\rho(\varphi(w))]$ and $\Phi[\delta(\varphi(w))]$ are defined as described above we have

$$\frac{\partial|F|}{\partial|w|}=1+\left|\frac{\partial}{\partial w}\Phi[\rho(\varphi(w))]\right|-\left|\frac{\partial}{\partial w}\Phi^{n-k}[\delta(\varphi(w))]\right|.$$

Setting $\varphi(w)=z=e^{ix}$ and taking into account (14), we arrive at the inequality

$$|\Phi'[\rho(z)]\rho'(z)\varphi'(\psi(z))|\leq(n-k)|\Phi'[\delta(z)]\delta'(z)\varphi'(\psi(z))|-\left[1-2\sin^{n-k}(\alpha/2)\sqrt{|c_nc_k|/(M^2-m^2)}\right],$$

implying that

$$\frac{|\rho'(z)|}{\sqrt{1-\rho^2(z)}}|\leq(n-k)\frac{\cos(x/2)}{\sqrt{\sin^2(\alpha/2)-\sin^2(x/2)}}-\frac{1-2\sin^{n-k}(\alpha/2)\sqrt{|c_nc_k|/(M^2-m^2)}}{|\varphi'(\psi(z))|}. \quad (15)$$

Since

$$\varphi'(\psi(z))=\sin\left(\frac{\alpha}{2}\right)\left[1-\widetilde{\Phi}^{-2}\left(i\frac{z-\cos\alpha}{\sin\alpha}\right)\right],$$

we have

$$|\varphi'(\psi(z))|=2\left|\widetilde{\Phi}\left(i\frac{z-\cos\alpha}{\sin\alpha}\right)\right|^{-1}\frac{\sqrt{\sin^2(\alpha/2)-\sin^2(x/2)}}{\cos(\alpha/2)}.$$

Now, in order to prove inequality (12), it remains to observe that for points w on the circle $|w|=1$ chosen in this way it holds that

$$\left|\widetilde{\Phi}\left(i\frac{z-\cos\alpha}{\sin\alpha}\right)\right|=\Lambda(\alpha,z),$$

and, in addition (see the proof of Theorem 3),

$$|\rho'(z)| = |P'(z)\overline{P(1/\overline{z})} + P(z)\overline{P'(1/\overline{z})}(-1/z^2)| = |zP'(z)\overline{P(z)} - \overline{z}\overline{P'(z)}P(z)| = 2|\Im zP'(z)\overline{P(z)}|$$
$$= 2\left|\Im \frac{zP'(z)}{P(z)}P(z)\overline{P(z)}\right| = |(|P(z)|^2)'_x|.$$

The assertion concerning the equality case stems from the identity $F(w) \equiv w$, which holds for the polynomial in question. □

Letting α tend to π for $k = 0$, we arrive at the inequality first obtained by Dubinin, see [18, Theorem 2].

Theorem 6. *For $n - k \geq 3$, the coefficients of a polynomial P of the form* (1) *satisfy the inequality*

$$\frac{4|c_n c_k| \sin^{2(n-k)}(\alpha/2)}{M^2 - m^2}\left(1 + \frac{1}{\sin(\alpha/2)}\left|\left(\frac{c_{n-1}}{2c_n} + \frac{\overline{c_{k+1}}}{2\overline{c_k}}\right) + (n-k)\cos^2(\alpha/2)\right|\right) \leq 1. \tag{16}$$

Equality in (16) *is attained, for instance, for the polynomial P_α.*

Proof. Following Olesov's paper [3], at a punctured neighborhood of the point $w=0$, consider the function

$$\widetilde{F}(w) := \frac{1}{F(1/w)} \equiv w\frac{\Phi^{n-k}[\delta(\varphi(1/w))]}{\Phi[\rho(\varphi(1/w))]}.$$

Set $\Delta(w) = w\widetilde{F}'(w)/\widetilde{F}(w)$. For the latter function we have

$$\Delta(w) = 1 + \frac{\varphi'(1/w)}{w}\left[\frac{\Phi'[\rho(\xi)]\rho'(\xi)}{\Phi[\rho(\xi)]} - (n-k)\frac{\Phi'[\delta(\xi)]\delta'(\xi)}{\Phi[\delta(\xi)]}\right], \quad \xi = \varphi(1/w).$$

Observe that $w\varphi(1/w) \to \varphi'(\infty) = \sin(\alpha/2)$ as $w \to 0$. Therefore,

$$\lim_{w\to 0}\frac{\Delta(w) - 1}{w} = \lim_{\xi\to\infty}\frac{\xi^2}{\sin(\alpha/2)}\left[\frac{\rho'(\xi)}{\sqrt{\rho^2(\xi) - 1}} - (n-k)\frac{\delta'(\xi)}{\sqrt{\delta^2(\xi) - 1}}\right]$$
$$= \lim_{\xi\to\infty}\frac{(M^2 - m^2)\sin(\alpha/2)}{c_n\overline{c_k}\xi^{n-k}}\left[\frac{2}{M^2 - m^2}\left(P'(\xi)\overline{P(1/\overline{\xi})}\xi - \frac{P(\xi)\overline{P'(1/\overline{\xi})}}{\xi}\right)\delta(\xi) - \frac{(n-k)(\xi - 1/\xi)\rho(\xi)}{2\sin^2(\alpha/2)}\right]$$
$$= \lim_{\xi\to\infty}\frac{1}{c_n\overline{c_k}\sin(\alpha/2)\xi^{n-k}}\left[\left((nc_n\xi^n + (n-1)c_{n-1}\xi^{n-1} + \cdots + kc_k\xi^k)\left(\frac{\overline{c_n}}{\xi^n} + \cdots + \frac{\overline{c_{k+1}}}{\xi^{k+1}} + \frac{\overline{c_k}}{\xi^k}\right)\right.\right.$$
$$\left. - (c_n\xi^n + c_{n-1}\xi^{n-1} + \cdots + c_k\xi^k)\left(\frac{n\overline{c_n}}{\xi^n} + \cdots + \frac{(k+1)\overline{c_{k+1}}}{\xi^{k+1}} + \frac{k\overline{c_k}}{\xi^k}\right)\right)$$
$$\left.\times\big(\xi - 2\cos^2(\alpha/2)\big) - (n-k)(c_n\xi^{n+1} + c_{n-1}\xi^n + \cdots + c_k\xi^k)\left(\frac{\overline{c_n}}{\xi^n} + \cdots + \frac{\overline{c_{k+1}}}{\xi^{k+1}} + \frac{\overline{c_k}}{\xi^k}\right)\right]$$
$$= \frac{-c_n\overline{c_{k+1}} - c_{n-1}\overline{c_k} - 2(n-k)c_n\overline{c_k}\cos^2(\alpha/2)}{c_n\overline{c_k}\sin(\alpha/2)}.$$

On the other hand, by l'Hôpital's rule, we find

$$\lim_{w\to 0}\frac{\Delta(w) - 1}{w} = \frac{\widetilde{F}''(0)}{2\widetilde{F}'(0)},$$

implying that

$$\widetilde{F}''(0) = 2\widetilde{F}'(0)\frac{-c_n\overline{c_{k+1}} - c_{n-1}\overline{c_k} - 2(n-k)c_n\overline{c_k}\cos^2(\alpha/2)}{c_n\overline{c_k}\sin(\alpha/2)}.$$

By $\widetilde{f}(\zeta)$ denote the function univalent in the unit disk $|\zeta| < 1$ that is inverse to $\widetilde{F}(w)$. For this function, we have

$$\widetilde{f}''(0) = -\widetilde{F}''(0)(\widetilde{f}'(0))^3 2\frac{c_n\overline{c_{k+1}} + c_{n-1}\overline{c_k} + 2(n-k)c_n\overline{c_k}\cos^2(\alpha/2))c_n\overline{c_k}\sin^{4(n-k)-1}(\alpha/2)}{(M^2 - m^2)^2}.$$

Then, for $\lambda = \dfrac{4|c_n c_k| \sin^{2(n-k)}(\alpha/2)}{M^2 - m^2}$, the function $f^*(0) = \widetilde{f}(\zeta)/\widetilde{f}'(0)$ in the unit disk $|\zeta| < 1$ is univalent, smaller than λ^{-1} in the absolute value, and it can be represented by the power series

$$f^*(\zeta) = \zeta + \alpha_2 \zeta^2 + \alpha_3 \zeta^3 + \cdots ,$$

where

$$\alpha_2 = 4\frac{(c_n \overline{c_{k+1}} + c_{n-1}\overline{c_k} + 2(n-k)c_n \overline{c_k} \cos^2(\alpha/2)) \sin^{2(n-k)-1}(\alpha/2)}{(M^2 - m^2)}.$$

In this case, in accordance with [17, p. 94],

$$|\alpha_2| \le 2(1 - \lambda), \tag{17}$$

which implies (16). The assertion concerning the equality case follows from the identity $F(w) \equiv w$, valid for the polynomial in question.

This completes the proof. □

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