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CHRISTOFFEL FUNCTIONS FOR DOUBLING MEASURES ON QUASISMOOTH CURVES AND ARCS

T. VARGA*

Bolyai Institute, University of Szeged, Aradi v. tere 1, H-6720 Szeged, Hungary
e-mail: vargata@math.u-szeged.hu

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Abstract. Matching two-sided estimates are given for Christoffel functions associated with a doubling measure ν over a quasismooth curve or arc. The size of the the n-th Christoffel function at a point z is given by the ν-measure of the largest disk about z which lies within the $1/n$-level line of the Green's function. The main theorem contains as special case all previously known weak asymptotics for Christoffel functions, and it also gives their size in explicit form about smooth corners. Applications are given for estimating the size of orthonormal polynomials and for Nikolskii-type inequalities.

1. Introduction

Let ν be a measure with support in the complex plane. The n-th Christoffel function of ν with parameter p is defined as

$$\lambda_n(\nu, p, z) := \inf_{\substack{p_n \in P_n \\ p_n(z)=1}} \int \left| p_n(t) \right|^p d\nu(t),$$

where n is a non-negative integer, $p \in [1, \infty)$ and P_n is the set of all polynomials of degree at most n.

The Christoffel function plays an important role in the theory of orthogonal polynomials. In the classical case $p = 2$ it is well known that

$$\lambda_n(\nu, 2, z) = \frac{1}{\sum_{k=0}^{n} \left| \pi_k(z) \right|^2},$$

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where π_k is the k-th orthonormal polynomial with respect to ν. This shows that we can estimate the values of orthonormal polynomials at a point with the help of the Christoffel functions.

When the measure is concentrated on the real line, the Markov–Stieltjes inequalities show that the distribution of the measure ν can be well estimated via its Christoffel functions [6, § I.5]. This made it possible to use Christoffel functions for estimating the distance between two adjacent zeros of orthogonal polynomials [7,9,13,15,17]. Mastroianni and Totik gave a general estimate for Christoffel functions expanding the classical unweighted case. They worked with doubling measures with support $[-1,1]$: the measure ν with support $[-1,1]$ has the doubling property if there is a constant c_ν such that for every $x \in [-1,1]$ and $\delta > 0$

$$c_\nu \, \nu\big([x-\delta, x+\delta]\big) \geqq \nu\big([x-2\delta, x+2\delta]\big)$$

holds.

THEOREM 1.1 ([8, (7.14)]). *Let ν be a doubling measure on $[-1,1]$. Then for any $p \in [0,\infty)$ there is a constant c depending only on the doubling constant c_ν and p such that for all $x \in [-1,1]$ and $n \in \mathbb{N}$*

$$\frac{1}{c}\nu\Big(\big[x-\Delta_n(x), x+\Delta_n(x)\big]\Big) \leqq \lambda_n(\nu,p,x) \leqq c\,\nu\Big(\big[x-\Delta_n(x), x+\Delta_n(x)\big]\Big), \tag{1.1}$$

where $\Delta_n(x) = \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}$.

When the doubling measure is defined on the unit circle $\mathbb{T}$, then a similar result holds, but then $\big[x-\Delta_n(x), x+\Delta_n(x)\big]$ must be replaced by the arc of length $1/n$ about the point $x \in \mathbb{T}$. When ν is a nice measure on a C^2-smooth Jordan curve (homeomorphic image of the unit circle), then fine asymptotics for Christoffel functions were proven in [14]. No general estimate is known for less smooth curves let alone for curves with corners. Also, the case when the measure is lying on a Jordan arc (homeomorphic image of $[0,1]$) is open.

The main purpose of this paper is to show that (1.1) is valid if $[-1,1]$ is replaced by a quasismooth curve or arc in the complex plane, or by a finite union of them. The case when the curve or arc has Dini-smooth corners will be a special case. Applications will be given for estimating orthogonal polynomials. Other applications will be provided for Nikolskii-type inequalities in between different $L^p(\nu)$-norms of polynomials.

Our method relies on V. Andriewskii's work [1–3] who extended some results of [8] to quasismooth curves or arcs.

2. Main result

In order to claim the addressed extension we shortly review the notions and notations we are going to use. First, recall that $L \subset \mathbb{C}$ is a Jordan curve (arc) if it is a homeomorphic image of the unit circle (interval). We shall only consider rectifiable curves (arcs). Denote by $L(z_1, z_2)$ the (shorter) subarc of L that joins $z_1 \in L$ and $z_2 \in L$, and by $\left|L(z_1, z_2)\right|$ the arc length of $L(z_1, z_2)$.

Definition 2.1. The Jordan curve or arc L is *quasismooth* (in the sense of Lavrentiev), if there is a (Lavrentiev) constant Λ_L such that

$$\left|L(z_1, z_2)\right| \leqq \Lambda_L |z_1 - z_2|$$

holds for arbitrary $z_1, z_2 \in L$.

In the treatment of L the Green's function gets an important role. Let Ω be the exterior of L (the connected component of $\mathbb{C}_\infty \setminus L$ containing ∞) and let Φ be the conformal mapping of Ω onto the exterior of the unit disk $\{\, z \in \mathbb{C}_\infty : |z| > 1 \,\} =: \mathbb{D}^*$ with the normalization $\Phi(\infty) = \infty$ and $\Phi'(\infty) := \lim_{z\to\infty} \frac{\Phi(z)}{z} > 0$. The normalization guarantees that there is only one such map [12, Ch. 4.4] (Fig. 2). The Green's function of Ω with pole at ∞ coincides then with $\log |\Phi|$.

Let $\tilde{\Omega}$ denote the so-called Charatheodory compactification [10, Ch. 4.4] of Ω that is the union of Ω and the set of all prime ends belonging to null chains of Ω. Then Φ can be extended to a homeomorphism between $\tilde{\Omega}$ and the closure of the exterior of the unit disk $\{\, z \in \mathbb{C}_\infty : |z| \geqq 1 \,\}$ [10, Theorem 2.15].

For $\delta > 0$ let

$$L_\delta := \Big\{ \zeta \in \Omega : \left|\Phi(\zeta)\right| = 1 + \delta \Big\} \tag{2.1}$$

be the $(1 + \delta)$-level line of Φ and

$$\rho_\delta(z) := d(L_\delta, z) = \inf_{\zeta \in L_\delta} |z - \zeta| \tag{2.2}$$

the distance from z to this level line.

The notion of the doubling property for a measure with support on a Jordan curve or arc is a natural extension of the interval-case. Let

$$B(z, \delta) = \{\, w \mid |w - z| \leqq \delta \,\}$$

denote the closed disk of radius δ about the point z.

Definition 2.2. Let ν be a measure on the complex plane whose support is a quasismooth curve or arc L. ν is called a doubling measure on L, if there is a constant c_ν such that

$$c_\nu\,\nu\big(B(z,\delta)\big) \geqq \nu\big(B(z,2\delta)\big)$$

holds for any $z \in L$ and $\delta > 0$.

As an example, we show that the arc measure on a quasismooth curve or arc L is always a doubling measure. Indeed, let ν be that arc measure. The doubling property follows if we show that $\delta \leqq \nu\big(B(z,\delta)\big) \leqq C\delta$ with some C independent of $z \in L$ and of small $\delta > 0$. Let δ be sufficiently small, say smaller than $|L|/2\Lambda_L$. The left-inequality is clear, since $L \cap B(z,\delta)$ contains an arc that connects the center z of $B(z,\delta)$ with a boundary point on $B(z,\delta)$. On the other hand, $L \cap B(z,\delta)$ consists of countably many subarcs of L, the total length of which is at most as large as the length of the arc $L(z_1,z_2)$, where z_1 and z_2 are the first and last points (in some orientation of L) of contact of L with $B(z,\delta)$. Hence, by the quasismoothness of L,

$$\nu\big(B(z,\delta)\big) \leqq \big|L(z_1,z_2)\big| \leqq \Lambda_L|z_1-z_2| \leqq \Lambda_L 2\delta.$$

A similar argument shows that in the previous definition $B(z,\delta)$ and $B(z,2\delta)$ can be replaced by the connected component of $L \cap B(z,\delta)$ and $L \cap B(z,2\delta)$ respectively that contains z.

Take a direction of L (say, from ζ_1 to ζ_2 in the arc-case, if ζ_1 and ζ_2 denote the two endpoints of L; counterclockwise in the curve-case), and let $\delta > 0$ be such that $\sup_{z\in L}\rho_\delta(z) < |L|/2$. For a $z \in L$ let $z_{-\delta}$ be the point followed by z and z_δ that follows z such that $\big|L(z_{-\delta},z)\big| = \rho_\delta(z)/2$ and $\big|L(z,z_\delta)\big| = \rho_\delta(z)/2$ respectively. If there is no such point (in the arc case) then set $z_{-\delta} := \zeta_1$ and $z_\delta := \zeta_2$ respectively. With these set

$$l_\delta(z) := L(z_{-\delta},z_\delta), \tag{2.3}$$

and introduce the function

$$v_\delta(z) := \nu\big(l_\delta(z)\big)\,. \tag{2.4}$$

On quasismooth curves and arcs we introduce a further parameter in the definition of the Christoffel function:

Definition 2.3. Let L be a quasismooth curve or arc, ν a doubling measure on L, $p \in [1,\infty)$ and $t \in \mathbb{R}$. Then the function

$$\lambda_n(\nu,p,t,z) := \inf_{\substack{p_n\in P_n\\ p_n(z)=1}} \int \rho_{\frac{1}{n}}(\zeta)^t\big|p_n(\zeta)\big|^p\,d\nu(\zeta) \tag{2.5}$$

is called the n-th Christoffel function associated to ν with parameter (p,t).

Note that setting $t=0$ we obtain the classical L^p Christoffel function of ν.

Now we are ready to extend [8, (7.14)].

THEOREM 2.4. *Let L be a quasismooth curve or arc and ν a doubling measure on L. If $p \in [1,\infty)$, $t \in \mathbb{R}$ then there is a constant $c = c(L, c_\nu, p, t)$ such that*

$$\frac{1}{c}\,\rho_{\frac{1}{n}}(z)^t v_{\frac{1}{n}}(z) \leqq \lambda_n(\nu,p,t,z) \leqq c\,\rho_{\frac{1}{n}}(z)^t v_{\frac{1}{n}}(z) \tag{2.6}$$

is true for any $z \in L$ and $n \in \mathbb{N}$.

COROLLARY 2.5. *The same result holds if L is a finite union of quasismooth curves or arcs lying exterior to one another, and ν is a doubling measure on L remarking that $\rho_{\frac{1}{n}}(z)$ with respect to L means the function that is equal to $\rho_{\frac{1}{n}}(z)$ with respect to the connected component of L containing z.*

The case $t=0$ may be the most interesting one. It shows that the magnitude of the n-th Christoffel function at a point $z \in L$ is about as large as the ν-measure of $l_{\frac{1}{n}}(z)$.

3. Corollaries

3.1. Estimate for orthonormal polynomials. Using the fact that

$$\lambda_n(\nu,2,0,z) = \frac{1}{\sum_{k=0}^{n} \left|\pi_k(z)\right|^2}, \tag{3.1}$$

where π_k is the k-th orthonormal polynomial associated to ν (see e.g. [16, Theorem 1.4]) we immediately obtain the following corollaries.

COROLLARY 3.1. *Let ν be a doubling measure on a quasismooth curve or arc L. Then*

$$\left|\pi_n(z)\right| \leqq \frac{\sqrt{c}}{\sqrt{v_{\frac{1}{n}}(z)}} \tag{3.2}$$

holds for every $z \in L$.

COROLLARY 3.2. *Let ν be a doubling measure on a quasismooth curve or arc L. Then*

$$\max_{0\leqq k\leqq n} \left|\pi_k(z)\right| \geqq \frac{1}{\sqrt{c}\sqrt{n}\sqrt{v_{\frac{1}{n}}(z)}}. \tag{3.3}$$

Moreover, if we know that $n \cdot v_{\frac{1}{n}}(z) \to 0$ then here we can delete "max":

COROLLARY 3.3. *Let ν be a doubling measure on a quasismooth curve or arc L. Then for any $z \in L$ for which $n \cdot v_{\frac{1}{n}}(z) \to 0$ holds there is an infinite subset $\mathbb{M} = \mathbb{M}(z)$ of $\mathbb{N}$ such that for all $n \in \mathbb{M}$*

$$|\pi_n(z)| \geqq \frac{1}{\sqrt{n}} \frac{1}{\sqrt{v_{\frac{1}{n}}(z)}}. \tag{3.4}$$

There is a discrepancy of order $\sqrt{n}$ in the estimates in the two Corollaries 3.1 and 3.2, but that is natural. Consider for example, the classical Jacobi polynomials at some point $z \in (-1, 1)$. Then $|\pi_k(z)| \leqq C$, while $nv_{1/n}(z) \sim 1$, so (3.3) gives then the correct order (and this example also explains why one needs the "max" in (3.3), since, in general, $\pi_k(z)$ can tend to zero along a subsequence of the k's). On the other hand, there are weights $w \geqq c > 0$ on $[-1, 1]$ such that if $d\nu(x) = w(x)\,dx$, then $\pi_n(0)/n^{1/2-\varepsilon} \to \infty$ along some subsequence of the n's for all $\varepsilon > 0$, see [11]. Although it is not clear if this ν is doubling, this example shows that, in general, nothing much better than (3.2) can be expected (note that in this case $v_{1/n}(z) \geqq c/n$).

Without the assumption $n \cdot v_{1/n}(z) \to 0$ we can only prove a weaker corollary.

COROLLARY 3.4. *Let ν be a doubling measure on a quasismooth curve or arc L. Then for any $z \in L$ and $\varepsilon > 0$ there is an infinite subset $\mathbb{M} = \mathbb{M}(z, \varepsilon)$ of $\mathbb{N}$ such that for all $n \in \mathbb{M}$*

$$|\pi_n(z)| \geqq \frac{1}{n^{1/2+\varepsilon}} \frac{1}{\sqrt{v_{\frac{1}{n}}(z)}}.$$

3.2. Christoffel functions on Dini-smooth curves and arcs. In the subsequent two corollaries we have some further assumption about the smoothness of L. We suppose that the considered curve or arc is Dini-smooth or has Dini-smooth corner at some point. Then we have an explicit form for the magnitude of $\rho_{\frac{1}{n}}(z)$.

DEFINITION 3.5. A Jordan curve or arc L is Dini-smooth, if it has a parametrization $\gamma(t)$ with non-zero and Dini-continuous derivative, that is

$$\int_0^\pi \frac{\omega(t)}{t}\,dt < \infty$$

holds for the modulus of continuity

$$\omega(\delta) := \sup_{\substack{t_1,t_2\in[0,2\pi] \\ |t_1-t_2|<\delta}} \left|\gamma'(t_1)-\gamma'(t_2)\right|$$

of the derivative γ'.

We have already mentioned that no estimate is known for Christoffel functions if the underlying curve has a corner; for example, the size of the Christoffel functions is unknown for the arc measure on the boundary of a square around the corners. We are going to give the precise order at Dini-smooth corners. We say that L has a corner at $\gamma(t_0) =: \zeta$ if the halftangents exist at $\gamma(t_0)$. If we speak of a corner with angle $\beta\pi$ then we always consider the angular domain (determined by the halftangents) which falls in the exterior of the curve in the curve-case and we always consider the angular domain with greater angle in the arc-case. So in the curve-case the magnitude of the angle falls between 0 and 2π while in the arc-case it ranges from π to 2π (see Fig. 1).

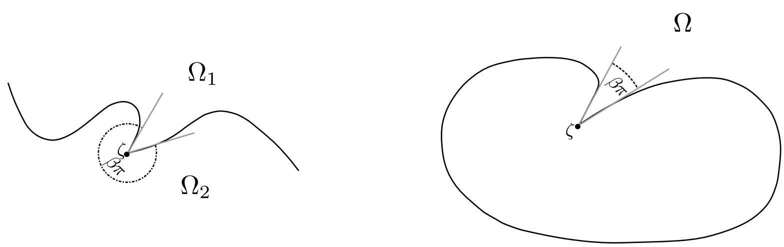

Fig. 1: Corner at ζ with angle $\beta\pi$. In the curve-case we always consider the angular domain lying the exterior of the curve, while in the arc-case we always consider the angular domain to which the grater angle belongs

The *corner at* $\gamma(t_0)$ *is Dini-smooth* if there are two subarcs of L ending and lying on the opposite sides of $\gamma(t_0)$ which are Dini-smooth, and, similarly, the *endpoint* ζ_i *is Dini-smooth* if it is also an endpoint of a Dini-smooth subarc of L. Note that if L is Dini-smooth then at any point (except for the endpoints) there is a Dini-smooth straight angle, so the following statements include the corner-free case, as well.

We introduce the following function at a Dini-smooth corner ζ with angle β which replaces $\rho_{\frac{1}{n}}(z)$ in Theorem 2.4. With l_δ from (2.3) we set

$$\Delta_n(z) := \begin{cases} \dfrac{1}{n^\beta} & \text{if } \ z \in l_{\frac{1}{n}}(\zeta) \\[2ex] \dfrac{|z-\zeta|^{1-\frac{1}{\beta}}}{n} & \text{if } \ z \in L \setminus l_{\frac{1}{n}}(\zeta). \end{cases}$$

REMARK 3.A. We can also define $\Delta_n(z)$ at a Dini-smooth endpoint, say, ζ_1. In this case $\beta := 2$ as if there were a corner at ζ_1 with angle 2π.

LEMMA 3.6. *Let L be a quasismooth curve or arc which has a Dini-smooth corner/endpoint at ζ with angle $\beta\pi$ ($0 < \beta < 2$ in the curve-case, $1 \leqq \beta < 2$ in the arc-case and $\beta := 2$ if ζ is an endpoint). Then there are an $\varepsilon = \varepsilon(L, \zeta) > 0$ and a constant $c_7 = c_7(L, \zeta, \varepsilon, \beta)$ such that for all points $z \in L$ with $|z - \zeta| \leqq \varepsilon$ the inequalities*

$$\frac{1}{c_7}\Delta_n(z) \leqq \rho_{\frac{1}{n}}(z) \leqq c_7\Delta_n(z)$$

hold.

Combining this lemma with Theorem 2.4 we immediately obtain

COROLLARY 3.7. *Let L be a quasismooth curve or arc which has a Dini-smooth corner/endpoint at ζ with angle $\beta\pi$ ($0 < \beta < 2$ in the curve-case, $1 \leqq \beta < 2$ in the arc-case and $\beta := 2$ if ζ is an endpoint). Then there are an $\varepsilon = \varepsilon(L, \zeta) > 0$ and a constant $c = c(L, c_\nu, p, t, \zeta, \varepsilon, \beta)$ such that for all points $z \in L$ with $|z - \zeta| \leqq \varepsilon$ the inequalities*

$$\frac{1}{c}\,\Delta_n(z)^t v_{\frac{1}{n}}(z) \leqq \lambda_n(\nu, p, t, z) \leqq c\,\Delta_n(z)^t v_{\frac{1}{n}}(z)$$

hold.

In particular, if the curve or arc is piecewise Dini-smooth, then the previous corollaries are globally valid. We first give an appropriate form for $\Delta_n(z)$. Let $\zeta_1, \dots, \zeta_n$ be the corners of L with angles different from π where in the arc-case ζ_1, ζ_n continue denoting the endpoints of L. Let $\beta_1\pi, \dots, \beta_n\pi$ be the corresponding angles ($0 < \beta_i < 2$ in the curve-case; $\beta_i := 2$, if $i = 1$ or n and $1 < \beta_i < 2$, if $1 < i < n$ in the arc-case). With

$$\Delta_n(z) := \begin{cases} \dfrac{1}{n^{\beta_i}} & \text{if} \quad z \in l_{\frac{1}{n}}(\zeta_i) \ (i = 1, 2, \dots, n) \\ \dfrac{\prod_{i=1}^n |z - \zeta_i|^{1 - \frac{1}{\beta_i}}}{n} & \text{if} \quad z \in L \setminus \left(\cup_{i=1}^n l_{\frac{1}{n}}(\zeta_i) \right) \end{cases}$$

we get the global variant of Lemma 3.6:

LEMMA 3.8. *If L is a piecewise Dini-smooth curve or arc, then there exists a constant $c_8 = c_8(L)$ such that*

$$\frac{1}{c_8}\Delta_n(z) \leqq \rho_{\frac{1}{n}}(z) \leqq c_8\Delta_n(z)$$

for all $z \in L$.

COROLLARY 3.9. *If L is a piecewise Dini-smooth curve or arc, then there exists a constant $c = c(L, c_\nu, p, t)$ such that*

$$\frac{1}{c}\Delta_n(z)^t v_{\frac{1}{n}}(z) \leqq \lambda_n(\nu, p, t, z) \leqq c\,\Delta_n(z)^t v_{\frac{1}{n}}(z)$$

holds for all $z \in L$.

Theorem 1.1 corresponds to the choice $L = [-1, 1]$, $p = 2$ and $t = 0$.

3.3. Nikolskii-type inequalities. If $1 \leqq p < q$, then, by Hölder's inequality, we can estimate the L^p norm by L^q norm from above. In the opposite direction the so-called Nikolskii-type inequalities are used for polynomials. With the help of Theorem 2.4 we can create such inequalities for doubling measures supported on a quasismooth curve or arc. So these results partly overlap with [2, Theorem 6] in which Andrievskii proved an unweighted Nikolskii-type inequality over rectifiable curves in another way.

Introduce the following notations:

$$M_n := \sup_{\zeta \in L} \frac{1}{v_{\frac{1}{n}}(\zeta)},$$

and for a function f on L let

$$\|f\|_\infty := \sup_{\zeta \in L} |f(\zeta)|, \quad \|f\|_{\nu,p} := \left(\int_L |f(\zeta)|^p\, d\nu(\zeta)\right)^{\frac{1}{p}}.$$

With these we have the following Nikolskii-type inequalities.

COROLLARY 3.10. *Let L be a quasismooth curve or arc and ν a doubling measure on L. If $1 \leqq p < q$, then there is a constant $c = c(p, q)$ independent of n such that*

$$\|p_n\|_\infty \leqq c M_n^{\frac{1}{p}} \|p_n\|_{\nu,p}, \tag{3.5}$$

as well as

$$\|p_n\|_{\nu,q} \leqq M_n^{\frac{1}{p}-\frac{1}{q}} \|p_n\|_{\nu,p} \tag{3.6}$$

for every polynomial p_n of degree at most n.

REMARK 3.B. By Lemma 4.10 and (4.6′) below, the magnitude of M_n is at most $n^{2\alpha_4}$ (with the constant α_4 in (4.6′)) which is n^2 in the unweighted case. Moreover, if we know that L is a piecewise Dini-smooth curve with angles $\beta_1\pi, \ldots, \beta_N\pi$ and $\beta := \max(\beta_1, \ldots, \beta_N, 1)$, then this magnitude is $n^{\beta\alpha_4}$. So our corollary includes the classical results for the unit circle as well as for $[-1, 1]$ (up to constants).

4. Proofs

Before the proofs, we list some properties of ρ_δ (see (2.2)) which are simple consequences of [4, Theorem 4.1 (p. 97) and Lemma 5.3 (p. 147)], then mention two straightforward implications of the doubling property and finally cite some results of Andrievskii [1,2] necessary for the proofs.

Recall that Φ is the conformal map taking the unbounded component Ω of $\mathbb{C} \setminus L$ onto the outside $\mathbb{D}^*$ of the unit circle. If $z \in L$ then let $\tilde{z}_\delta$ denote $\Phi^{-1}\big((1+\delta)\Phi(z)\big)$ in the curve-case. In the arc-case we have to be a bit more elaborate: Let ζ_1 and ζ_2 be the two endpoints of L. Their images under Φ are $e^{i\theta_1}$ and $e^{i\theta_2}$ respectively for some appropriate $0 \leqq \theta_1 < \theta_2 < \theta_1 + 2\pi$. Every other point z of L is the impression of two prime ends Z_1 and Z_2 of $\tilde{\Omega}$. We let

$$\Delta_1 := \{z \in \mathbb{D}^* : \theta_1 < \arg z < \theta_2\}, \quad \Delta_2 := \mathbb{D}^* \setminus \bar{\Delta}_1,$$

$$\tilde{\Omega}_j := \Phi^{-1}(\bar{\Delta}_j), \quad L_\delta^j := L_\delta \cap \tilde{\Omega}_j, \quad \rho_\delta^j(z) := d(z, L_\delta^j).$$

Then set $\tilde{z}_\delta^j := \Phi^{-1}\big((1+\delta)\Phi(Z_j)\big)$ and

$$\tilde{z}_\delta := \begin{cases} \tilde{z}_\delta^1 & \text{if } \rho_\delta^1(z) \leqq \rho_\delta^2(z) \\ \tilde{z}_\delta^2 & \text{if } \rho_\delta^2(z) < \rho_\delta^1(z) \end{cases}$$

(see Fig. 2).

Lemma 4.1 ([1, (3.1), (3.2), (3.4), (3.5)], [2, (3.1),(3.3),(3.4),(3.5)]). *Let L be a quasismooth curve or arc, $z, z_1, z_2 \in L$. Then there are constants $c_1 = c_1(L)$, $c_2 = c_2(L)$, $c_3 = c_3(L)$, $c_4 = c_4(L)$, $\alpha_1 = \alpha_1(L)$, $\alpha_2 = \alpha_2(L)$ such that for any $\delta > 0$ the following relations hold*:

$$\frac{1}{c_1}|z - \tilde{z}_\delta| \leqq \rho_\delta(z) \leqq c_1|z - \tilde{z}_\delta|; \tag{4.1}$$

if $|z_1 - z_2| \leqq \rho_\delta(z_1)$, *then*

$$\frac{1}{c_2}\rho_\delta(z_1) \leqq \rho_\delta(z_2) \leqq c_2\rho_\delta(z_1); \tag{4.2}$$

if $|z_1 - z_2| > \rho_\delta(z_1)$, *then*

$$\frac{\rho_\delta(z_2)}{|z_1 - z_2|} \leqq c_3 \left(\frac{\rho_\delta(z_1)}{|z_1 - z_2|}\right)^{\alpha_1}; \tag{4.3}$$

and for $0 < \delta_1 < \delta_2 \leqq 1$

$$\frac{1}{c_4}\left(\frac{\delta_2}{\delta_1}\right)^{\frac{1}{\alpha_2}} \leqq \frac{\rho_{\delta_2}(z)}{\rho_{\delta_1}(z)} \leqq c_4 \left(\frac{\delta_2}{\delta_1}\right)^{\alpha_2}. \tag{4.4}$$

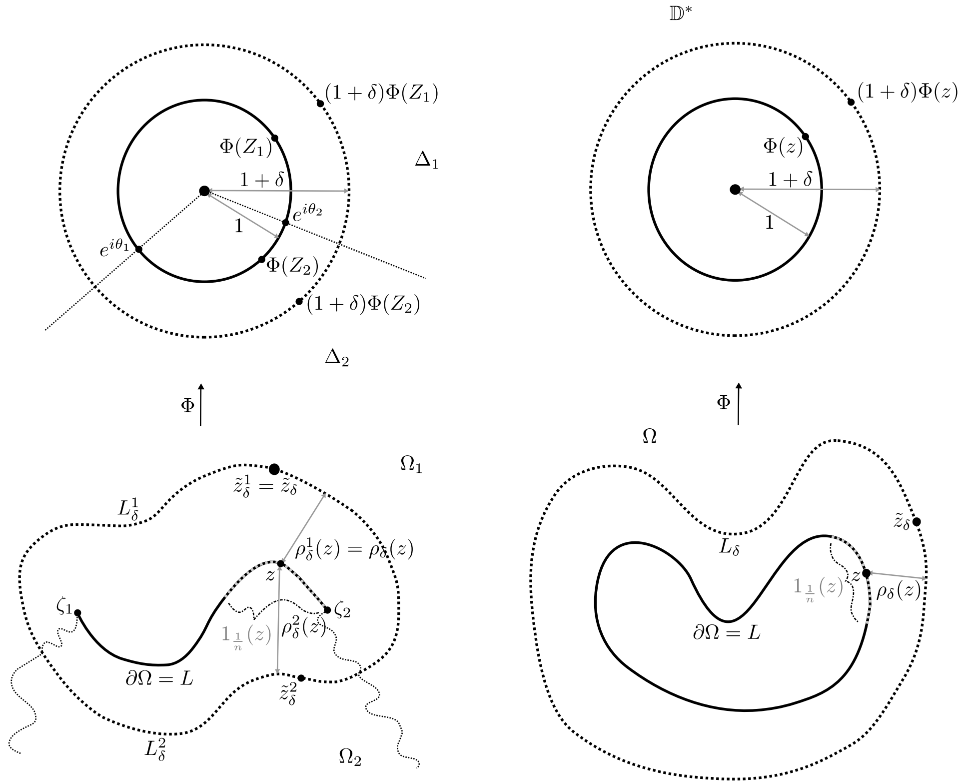

Fig. 2: In the arc-case every point of L except the two endpoints is the impression of two prime ends, e.g. z is the impression of both Z_1 and Z_2. Their images lie on the distinct subarcs of the unit circle with endpoints $e^{i\theta_1}$ and $e^{i\theta_2}$ which are the images of the endpoints of L. $e^{i\theta_1}$, the origin and $e^{i\theta_2}$ determine two angular domains. Their parts lying outside of the unit disk are denoted by Δ_1 and Δ_2, and we set $\Omega_1 := \Phi^{-1}(\Delta_1)$ and $\Omega_2 := \Phi^{-1}(\Delta_2)$. In the curve-case the notation is similar but it becomes simpler, because every point of L is an impression of precisely one prime end

We emphasize that the constants are independent of δ, z and ζ.

Recall now the definition of v_δ from (2.4), for which we claim

LEMMA 4.2 ([1, Lemma 4], [2, (4.2)], cf. [8, Lemma 2.1 (vi), (vii), (viii)]). *If ν is a doubling measure on a quasismooth curve or arc L then there are constants $c_5 = c_5(L, c_\nu)$ and $\alpha_3 = \alpha_3(L, c_\nu)$ such that for $0 < \delta < 1$*

$$v_\delta(z_1) \leqq c_5 \left(1 + \frac{|z_1 - z_2|}{\rho_\delta(z_2)}\right)^{\alpha_3} v_\delta(z_2) \tag{4.5}$$

for any point pair z_1, $z_2 \in L$. Furthermore, with some $c_6 = c_6(L, c_\nu)$ and $\alpha_4 = \alpha_4(L, c_\nu)$

$$\frac{1}{c_6}\left(\frac{\delta_2}{\delta_1}\right)^{\frac{1}{\alpha_4}} \leqq \frac{v_{\delta_2}(z)}{v_{\delta_1}(z)} \leqq c_6\left(\frac{\delta_2}{\delta_1}\right)^{\alpha_4} \tag{4.6}$$

for every $z \in L$ and $0 < \delta_1 < \delta_2 < 1$. Finally, if J is a subarc of L and E is a subarc of J, then

$$\frac{1}{c_6}\left(\frac{|J|}{|E|}\right)^{\frac{1}{\alpha_4}} \leqq \frac{\nu(J)}{\nu(E)} \leqq c_6\left(\frac{|J|}{|E|}\right)^{\alpha_4}. \tag{4.6$'$}$$

Noting that the roles of z_1 and z_2 are symmetric, (4.5) combined with (4.2) gives

- if $|z_1 - z_2| \leqq \rho_\delta(z_1)$ then

$$\frac{1}{c_5 2^{\alpha_3}} v_\delta(z_2) \leqq v_\delta(z_1) \leqq c_5(1 + c_2)^{\alpha_3} v_\delta(z_2); \tag{4.5a}$$

- if $|z_1 - z_2| > \rho_\delta(z_1)$, even if $|z_1 - z_2| \leqq \rho_\delta(z_2)$, then

$$\frac{|z_1 - z_2|}{\rho_\delta(z_2)} \geqq \frac{1}{c_2},$$

so

$$v_\delta(z_1) \leqq c_5(2c_2)^{\alpha_3}\left(\frac{|z_1 - z_2|}{\rho_\delta(z_2)}\right)^{\alpha_3} v_\delta(z_2). \tag{4.5b}$$

The following theorem provides that $d\nu(\zeta)$ can be replaced by $\frac{v_{1/n}(\zeta)}{\rho_{1/n}(\zeta)}|d\zeta|$ during the proofs.

THEOREM 4.3 ([1, Lemma 2], [2, (4.21)]). *Let L be a quasismooth curve or arc and ν a doubling measure on L with doubling constant c_ν. Then for any $p \in [1, \infty)$, $t \in \mathbb{R}$ there is a constant $c_a = c_a(L, c_\nu, p, t)$ such that for every polynomial p_n of degree at most n*

$$\frac{1}{c_a}\int_L |p_n(\zeta)|^p \rho_{\frac{1}{n}}(\zeta)^t \frac{v_{\frac{1}{n}}(\zeta)}{\rho_{\frac{1}{n}}(\zeta)}|d\zeta| \tag{4.7}$$

$$\leqq \int_L |p_n(\zeta)|^p \rho_{\frac{1}{n}}(\zeta)^t \, d\nu(\zeta) \leqq c_a \int_L |p_n(\zeta)|^p \rho_{\frac{1}{n}}(\zeta)^t \frac{v_{\frac{1}{n}}(\zeta)}{\rho_{\frac{1}{n}}(\zeta)}|d\zeta|.$$

THEOREM 4.4 (Bernstein inequality, [1, Theorem 1], [2, Theorem 1]). *Let L be a quasismooth curve or arc and ν a doubling measure on L with doubling constant c_ν. Then for any $p \in [1, \infty)$, $t \in \mathbb{R}$ there is a constant $c_B = c_B(L, c_\nu, p, t)$ such that for any polynomial p_n of degree at most n*

$$\int |p_n'|^p \rho_{\frac{1}{n}}^{p+t} \, \mathrm{d}\nu \leqq c_B \int |p_n|^p \rho_{\frac{1}{n}}^{t} \, d\nu. \tag{4.8}$$

The basic idea of the upper estimate for the Chrisoffel function is to find an appropriate polynomial with small norm. In the present paper we are going to use the so-called Dzjadyk kernel $K_{1,1,2,n}(\xi, \zeta)$ (for precise definition see e.g. [5]). This is a polynomial of ζ with degree $10n - 1$ and coefficients depending on ξ. The choice of ξ depends on the point of L where we investigate the Christoffel function. We summarize Andrievskii's calculation with respect to such appropriate ξ in a lemma.

LEMMA 4.5 ([1, (3.9)],[2, (4.6)]). *If L is a quasismooth curve or arc then there exits a constant $c_J = c_J(L)$ and for any $z \in L$ there is a $\xi = \xi(z)$ such that*

$$\frac{1}{c_J} \frac{1}{|z - \zeta| + \rho_{\frac{1}{n}}(z)} \leqq \left| K_{1,1,2,n}(\xi, \zeta) \right| \leqq c_J \frac{1}{|z - \zeta| + \rho_{\frac{1}{n}}(z)} \tag{4.9}$$

for every $\zeta \in L$.

Taking an appropriate power of $\left| K_{1,1,2,n}(\xi, \zeta) \right|$ it can be achieved that this power is small enough outside a neighbourhood $B(z, d)$ compared to its value at z. The calculation for $K_{1,1,2,n}(\xi, \zeta)$ is much simplified by:

LEMMA 4.6 ([1, (3.18)], [2, (3.6)]). *Let L be a quasismooth curve or arc, $b > 1$ and $d > 0$. Then for every $z \in L$*

$$\int_{L \setminus B(z,d)} \frac{1}{|\zeta - z|^b} |d\zeta| \leqq |L|^{1-b} + \frac{2\Lambda_L b}{(b-1)d^{b-1}}, \tag{4.10}$$

where Λ_L is the Lavrentiev constant of L (see Definition 2.1).

Recall that here $B(z, \delta)$ is the disk of radius δ about the point z.

PROOF OF THEOREM 2.4. We only give the proof for the arc-case, because the curve-case is similar and actually simpler since there are no endpoints. By virtue of (4.7), in L^p-norms $d\nu(\zeta)$ can be replaced by

$$\frac{v_{\frac{1}{n}}(\zeta)}{\rho_{\frac{1}{n}}(\zeta)} |d\zeta|,$$

and we do so in the proof. We deal with the upper estimate and the lower estimate separately.

We begin with the upper estimate. The proof is based on fast decreasing polynomials as in the proof of [8, Theorem 4.3], [17, Lemma 6] or [15, Theorem 3.1]. Here this role is played by the Dzjadyk kernel (see Lemma 4.5). Let m be a fixed positive integer defined later, $\hat{n}$ the integer part of $\frac{n}{10m}$ and for a $z \in L$ let K_n denote the polynomial

$$\frac{K_{1,1,2,\hat{n}}(\xi,\zeta)}{K_{1,1,2,\hat{n}}(\xi,z)}$$

with the $\xi = \xi(z)$ from Lemma 4.5. This $(K_n)^m$ is a polynomial of ζ of degree at most n and $K_n(z)^m = 1$. We introduce the following notation which make our calculations more transparent: if $g, h : \mathbf{M} \to \mathbb{R}$ are two real valued functions on a set $\mathbf{M}$ which is obvious from the actual situation, then $g(u) \preceq h(u)$ denotes the fact that there is a constant κ independent of u such that $g(u) \leqq \kappa\, h(u)$ holds for every $u \in \mathbf{M}$, while $g(u) \asymp h(u)$ means that $g(u) \preceq h(u)$ and $h(u) \preceq g(u)$. By (4.9), (4.4) and the fact that $\left|l_{\frac{1}{n}}(z)\right| = \rho_{\frac{1}{n}}(z)$ (see (2.3)) we get

$$\begin{aligned} \left|K_n(\zeta)\right| &\leqq c_J^2 \frac{\rho_{\frac{1}{\hat{n}}}(z)}{|z-\zeta| + \rho_{\frac{1}{\hat{n}}}(z)} \preceq \frac{\rho_{\frac{1}{n}}(z)}{|z-\zeta| + \rho_{\frac{1}{n}}(z)} \\ &\preceq \begin{cases} 1 & \text{if } \ \zeta \in l_{\frac{1}{n}}(z) \\ \rho_{\frac{1}{n}}(z)\dfrac{1}{|z-\zeta|} & \text{if } \ \zeta \in L \setminus l_{\frac{1}{n}}(z). \end{cases} \end{aligned} \tag{4.11}$$

The definition of Christoffel function (2.5) and (4.7) show that

$$\lambda_n(\nu,p,t,z) \preceq \int \left|K_n(\zeta)^m\right|^p \rho_{\frac{1}{n}}(\zeta)^{t-1} v_{\frac{1}{n}}(\zeta)|d\zeta|. \tag{4.12}$$

In what follows we estimate the right-hand side of (4.12) to get the appropriate upper bound for λ_n. We break the integral into two parts:

$$\int \left|K_n(\zeta)^m\right|^p \rho_{\frac{1}{n}}(\zeta)^{t-1} v_{\frac{1}{n}}(\zeta)|d\zeta| = \int_{l_{\frac{1}{n}}(z)} + \int_{L\setminus l_{\frac{1}{n}}(z)}.$$

With (4.11) in hand and using (4.2) as well as (4.5a), the first part can be simply treated:

$$\begin{aligned} \int_{l_{\frac{1}{n}}(z)} \left|K_n(\zeta)^m\right|^p \rho_{\frac{1}{n}}(\zeta)^{t-1} v_{\frac{1}{n}}(\zeta)|d\zeta| &\preceq \int_{l_{\frac{1}{n}}(z)} \rho_{\frac{1}{n}}(\zeta)^{t-1} v_{\frac{1}{n}}(\zeta)|d\zeta| \\ &\preceq \rho_{\frac{1}{n}}(z)^t v_{\frac{1}{n}}(z). \end{aligned}$$

Regarding the other part we make use of (4.11), (4.3) and (4.5b):

$$\int_{L\setminus l_{\frac{1}{n}}(z)} \left|K_n(\zeta)^m\right|^p \rho_{\frac{1}{n}}(\zeta)^{t-1} v_{\frac{1}{n}}(\zeta)|d\zeta|$$

$$\preceq \int_{L\setminus l_{\frac{1}{n}}(z)} \left(\rho_{\frac{1}{n}}(z)\frac{1}{|z-\zeta|}\right)^{mp} \rho_{\frac{1}{n}}(z)^{\alpha_1(t-1)}$$

$$\cdot\, |z-\zeta|^{(1-\alpha_1)(t-1)} \left(\frac{|z-\zeta|}{\rho_{\frac{1}{n}}(z)}\right)^{\alpha_3} v_{\frac{1}{n}}(z)|d\zeta|$$

$$\asymp \rho_{\frac{1}{n}}(z)^t v_{\frac{1}{n}}(z)\rho_{\frac{1}{n}}(z)^{mp+\alpha_1(t-1)-\alpha_3-t}$$

$$\cdot \int_{L\setminus l_{\frac{1}{n}}(z)} \left(\frac{1}{|z-\zeta|}\right)^{mp-(1-\alpha_1)(t-1)-\alpha_3} |d\zeta|.$$

Introduce the notation b for $mp-\alpha_1-\alpha_3+t(\alpha_1-1)+1$, remember that $\left|l_{\frac{1}{n}}(z)\right| = \rho_{\frac{1}{n}}(z)$, and apply (4.10) for the integral in the last expression. This way we continue the preceding expression:

$$\preceq \rho_{\frac{1}{n}}(z)^t v_{\frac{1}{n}}(z)\rho_{\frac{1}{n}}(z)^{b-1} \int_{L\setminus l_{\frac{1}{n}}(z)} \left(\frac{1}{|z-\zeta|}\right)^b |d\zeta|$$

$$\preceq \rho_{\frac{1}{n}}(z)^t v_{\frac{1}{n}}(z)\rho_{\frac{1}{n}}(z)^{b-1} \left(|L|^{1-b} + \frac{2\Lambda_L b}{(b-1)\rho_{\frac{1}{n}}(z)^{b-1}}\right).$$

We are almost ready, only we should achieve that $b \geqq 1$, but this is possible, if m is taken at least as large as $\frac{\alpha_1+\alpha_3-t(\alpha_1-1)}{p}$, e.g. let m be the upper integer part of $\frac{\alpha_1+\alpha_3-t(\alpha_1-1)}{p}$. With this choice the expression

$$\rho_{\frac{1}{n}}(z)^{b-1} \left(|L|^{1-b} + \frac{2\Lambda_L b}{(b-1)\rho_{\frac{1}{n}}(z)^{b-1}}\right)$$

is bounded above in n, so there is $c = c(L, c_\nu, p, t)$ independent of n such that

$$\int \left|K_n(\zeta)^m\right|^p \rho_{\frac{1}{n}}(\zeta)^{t-1} v_{\frac{1}{n}}(\zeta)|d\zeta| \leqq c\rho_{\frac{1}{n}}(z)^t v_{\frac{1}{n}}(z),$$

which proves the upper estimate.

Next, we turn to the lower estimate. Let $z \in L$ be a point and let $p_n(\zeta)$ be a polynomial of degree at most n for which $p_n(z) = 1$ holds. We prove that there is a constant $\tilde{c} = \tilde{c}(L, c_\nu, p, t)$ independent of n and z such that

$$\int_L |p_n(\zeta)|^p \rho_{\frac{1}{n}}(\zeta)^t \frac{v_{\frac{1}{n}}(\zeta)}{\rho_{\frac{1}{n}}(\zeta)} |d\zeta| \geqq \tilde{c} \rho_{\frac{1}{n}}(z)^t v_{\frac{1}{n}}(z). \tag{4.13}$$

In view of (4.7) this will give the lower estimate for the Christoffel function in Theorem 2.4. Let

$$\hat{c} := 2^p c_2^{p+t+1} c_5 (1 + c_2)^{\alpha_3} c_B$$

(c_2 from Lemma 4.1, c_5 and α_3 from (4.5), c_B from Theorem 4.4); if

$$\int_L |p_n(\zeta)|^p \rho_{\frac{1}{n}}(\zeta)^{t-1} v_{\frac{1}{n}}(\zeta) |d\zeta| < \frac{1}{\hat{c}} \rho_{\frac{1}{n}}(z)^t v_{\frac{1}{n}}(z) \tag{4.14}$$

does not hold then we are done with $\tilde{c} = \frac{1}{\hat{c}}$, so we may assume (4.14). Clearly,

(4.15)
$$\int |p_n(\zeta)|^p \rho_{\frac{1}{n}}(\zeta)^{t-1} v_{\frac{1}{n}}(\zeta) |d\zeta| \geqq \int_{l_{\frac{1}{n}}(z)} |p_n(\zeta)|^p \rho_{\frac{1}{n}}(\zeta)^{t-1} v_{\frac{1}{n}}(\zeta) |d\zeta| \geqq \dots .$$

If $\zeta \in l_{\frac{1}{n}}(z)$, then by applying Hölder's inequality we obtain

$$|p_n(\zeta) - p_n(z)| = \left| \int_{L(z,\zeta)} p_n'(s)\, ds \right|$$

$$\leqq \int_{l_{\frac{1}{n}}(z)} |p_n'(s)| |ds| \leqq \left(\int_{l_{\frac{1}{n}}(z)} |p_n'(s)|^p |ds| \right)^{\frac{1}{p}} \rho_{\frac{1}{n}}(z)^{\frac{p-1}{p}}$$

$$= \left(\frac{1}{\rho_{\frac{1}{n}}(z)^{p+t-1} v_{\frac{1}{n}}(z)} \right)^{\frac{1}{p}} \left(\int_{l_{\frac{1}{n}}(z)} |p_n'(s)|^p \rho_{\frac{1}{n}}(z)^{p+t-1} v_{\frac{1}{n}}(z) |ds| \right)^{\frac{1}{p}} \rho_{\frac{1}{n}}(z)^{\frac{p-1}{p}}.$$

Applying (4.2) and (4.5a) we can replace $\rho_{\frac{1}{n}}(z)$ by $\rho_{\frac{1}{n}}(s)$ and $v_{\frac{1}{n}}(z)$ by $v_{\frac{1}{n}}(s)$ in the integral over $l_{\frac{1}{n}}(z)$, so the inequality is continued as

$$\leqq c_2^{1+\frac{t-1}{p}} c_5^{\frac{1}{p}} (1 + c_2)^{\frac{\alpha_3}{p}} \left(\frac{1}{\rho_{\frac{1}{n}}(z)^{p+t-1} v_{\frac{1}{n}}(z)} \right)^{\frac{1}{p}}$$

$$\cdot \left(\int_{l_{\frac{1}{n}}(z)} |p_n'(s)|^p \rho_{\frac{1}{n}}(s)^{p+t-1} v_{\frac{1}{n}}(s) |ds| \right)^{\frac{1}{p}} \rho_{\frac{1}{n}}(z)^{\frac{p-1}{p}},$$

and here we can replace the integral over $l_{\frac{1}{n}}(z)$ by the same integral over L. By the Bernstein inequality (Theorem 4.4), then by (4.14), we can continue as

(4.16)

$$\leqq \frac{\hat{c}^{\frac{1}{p}}}{2}\left(\frac{1}{\rho_{\frac{1}{n}}(z)^{p+t-1}v_{\frac{1}{n}}(z)}\right)^{\frac{1}{p}}\left(\int_L |p_n(s)|^p\rho_{\frac{1}{n}}(s)^{t-1}v_{\frac{1}{n}}(s)|ds|\right)^{\frac{1}{p}}\rho_{\frac{1}{n}}(z)^{\frac{p-1}{p}}$$

$$\leqq \frac{1}{2}\left(\frac{1}{\rho_{\frac{1}{n}}(z)^{p+t-1}v_{\frac{1}{n}}(z)}\right)^{\frac{1}{p}}\left(\rho_{\frac{1}{n}}(z)^t v_{\frac{1}{n}}(z)\right)^{\frac{1}{p}}\rho_{\frac{1}{n}}(z)^{\frac{p-1}{p}} \leqq \frac{1}{2}.$$

In view of $p_n(z)=1$ this shows that $|p_n(\zeta)| \geqq 1/2$ for all $z \in l_{1/n}(z)$. Using this, (4.2) and (4.5a) we continue (4.15) as

$$\dots \geqq \int_{l_{\frac{1}{n}}(z)}\left(\frac{1}{2}\right)^p \rho_{\frac{1}{n}}(\zeta)^{t-1}v_{\frac{1}{n}}(\zeta)|d\zeta| \tag{4.17}$$

$$\geqq \frac{1}{c_2^{t-1}}\,\frac{1}{c_5(1+c_2)^{\alpha_3}}\int_{l_{\frac{1}{n}}(z)}\left(\frac{1}{2}\right)^p \rho_{\frac{1}{n}}(z)^{t-1}v_{\frac{1}{n}}(z)|d\zeta|$$

$$= \frac{1}{c_2^{t-1}}\,\frac{1}{c_5(1+c_2)^{\alpha_3}}\left(\frac{1}{2}\right)^p \rho_{\frac{1}{n}}(z)^t v_{\frac{1}{n}}(z),$$

which proves (4.13) (say) with

$$\tilde{c} = \frac{1}{c_2^{t-1}}\,\frac{1}{c_5(1+c_2)^{\alpha_3}}\left(\frac{1}{2}\right)^p,$$

and with them the lower estimate in Theorem 2.4. □

Proof of Corollary 2.5. If $L_1,\dots,L_N$ denote the connected components of L (that is $L_1,\dots,L_N$ are curves or arcs) then we can apply Theorem 2.4 to each one separately with some constants $c_{L_1},\dots,c_{L_N}$. Then setting $c := \max(c_{L_1},\dots,c_{L_N})$ we get for $z \in L$

$$\inf_{\substack{p_n\in P_n\\ p_n(z)=1}}\int_L \rho_{\frac{1}{n}}(\zeta)^t|p_n(\zeta)|^p\,d\nu(\zeta) \geqq \inf_{\substack{p_n\in P_n\\ p_n(z)=1}}\int_{L_z} \geqq \frac{1}{c_{L_z}}\,\rho_{\frac{1}{n}}(z)^t v_{\frac{1}{n}}(z)$$

$$\geqq \frac{1}{c}\,\rho_{\frac{1}{n}}(z)^t v_{\frac{1}{n}}(z),$$

where L_z is the component of L that contains the point z. This gives the necessary lower estimate for the Christoffel function.

The upper estimate follows the proof of the right-hand side of (2.6) for the point $z \in L_z$. We construct the polynomial K_n for L_z as in (4.11). If L consists of one component (that is $L = L_z$) we are done. If there are other components, then we have to ensure that the integral of the absolute value of the polynomial is small enough on these components too. Therefore we multiply K_n by an appropriate auxiliary polynomial. To find this polynomial we invoke the following two theorems:

Theorem 4.7 ([12, Bernstein's lemma, Theorem 5.5.7, p. 156]). *Let K be a compact subset of $\mathbb{C}$ with positive logarithmic capacity, and let D be a compact subset in $\mathbb{C} \setminus K$. Then there is a constant $\eta = \eta(K, D)$ such that for any polynomial q that is not constant*

$$\left| q(z) \right| \leqq \eta^{\deg q} \sup_{\zeta \in K} \left| q(\zeta) \right|$$

is valid for every $z \in D$.

In this theorem the only assumption on K is that it should have positive logarithmic capacity, which is true for any Jordan curve or arc, or for unions of them.

Theorem 4.8 ([12, Bernstein–Walsh theorem, Theorem 6.3.1, p. 170]). *Let K be a compact subset of $\mathbb{C}$ such that $\mathbb{C} \setminus K$ is connected. If f is a function holomorphic on an open neighbourhood U of K then there are a polynomial sequence $\{q_n\}_{n \in \mathbb{N}}$ $(\deg q_n \leqq n)$, $\mathcal{M} > 0$ and $\theta \in (0,1)$ such that*

$$\sup_{z \in K} \left| f(z) - q_n(z) \right| \leqq \mathcal{M} \theta^n$$

holds for every n.

Next, we recall the notion of polynomial convex hull.

Definition 4.9. If K is a compact subset of $\mathbb{C}$ and Ω denotes the unbounded connected component of $\mathbb{C} \setminus K$, then the set $\mathbb{C} \setminus \Omega$ is called the polynomial convex hull of K. It is the union of K with all the bounded components of $\mathbb{C} \setminus K$.

From Theorem 4.7 by (4.11) we get that there exists a constant η_{L_z} such that

$$\sup_{L \setminus L_z} \left| K_n(z) \right| \leqq C_{L_z} \eta_{L_z}^n, \tag{4.18}$$

where

$$C_{L_z} = \sup_{\substack{\zeta \in L_z \\ n \in \mathbb{N}}} \left| K_n(z) \right| < \infty.$$

On the other hand, let $U_1, \dots, U_N$ denote some disjoint neighbourhoods of the polynomial convex hulls of $L_1, \dots, L_N$ and set

$$f(\zeta) := \begin{cases} 1 & \text{if } \zeta \in U_z \\ 0 & \text{if } \zeta \in \bigcup_{i=1}^{N} U_i \setminus U_z, \end{cases}$$

where U_z is the neighbourhood of the polynomial convex hull of L_z. Since f is holomorphic on an open neighbourhood of L, we can apply Theorem 4.8 and obtain that there are a polynomial sequence $\{q_n\}_{n\in\mathbb{N}}$ ($\deg q_n \leqq n$), $\mathcal{M} > 0$ and $\theta \in (0,1)$ such that

$$\left| \frac{q_n(\zeta)}{q_n(z)} \right| \leqq \begin{cases} \dfrac{1+\mathcal{M}\theta^n}{|1-\mathcal{M}\theta^n|} \asymp 1 & \text{if } \zeta \in L_z \\ \dfrac{\mathcal{M}\theta^n}{|1-\mathcal{M}\theta^n|} \asymp \theta^n & \text{if } \zeta \in L \setminus L_z. \end{cases} \tag{4.19}$$

Let

$$\mathcal{K}_n(\zeta) = \left(K_{[\frac{n}{\tau+1}]}(\zeta) \right)^m \frac{q_{[\frac{\tau n}{\tau+1}]}(\zeta)}{q_{[\frac{\tau n}{\tau+1}]}(z)}, \tag{4.20}$$

where m is chosen as in the proof of Theorem 2.4 while τ is set subsequently.

We shall also need

LEMMA 4.10 ([4, Corollary 2.7, p. 61]). *Let K be an arbitrary continuum* (*that is a connected compact set with infinite cardinality*) *with connected complement. If L denotes the boundary of K, then*

$$\delta^2 \preceq d(L, L_\delta) \preceq 1.$$

Here $d(L, L_\delta)$ denotes the distance between L and the level line L_δ associated with L.

Now using (4.18), (4.19) and the facts that $\rho_{\frac{1}{n}}(\zeta) \preceq 1$ and $v_{\frac{1}{n}}(\zeta) \leqq |L| \preceq 1$ we obtain

$$\int_{L\setminus L_z} \left| \mathcal{K}_n(\zeta) \right|^p \rho_{\frac{1}{n}}(\zeta)^{t-1} v_{\frac{1}{n}}(\zeta) \, |d\zeta| \tag{4.21}$$

$$\preceq \int_{L\setminus L_z} \eta_{L_z}^{\frac{nmp}{\tau+1}} \theta^{\frac{n\tau mp}{\tau+1}} \rho_{\frac{1}{n}}(\zeta)^{t-1} v_{\frac{1}{n}}(\zeta)\,|d\zeta| \preceq \left((\eta_{L_z}\theta^\tau)^{\frac{mp}{\tau+1}}\right)^n.$$

Choose $\tau > 0$ such that $\eta_{L_z}\theta^\tau < 1$ holds, e.g. $\tau := \frac{\log 1/\eta}{\log\theta} + 1$, so the previous integral is exponentially small.

By (4.11), (4.18), (4.19) and (4.20) we get

$$\mathcal{K}_n(\zeta) = \begin{cases} 1 & \text{if } \ \zeta = z \\ \preceq 1 & \text{if } \ \zeta \in l_{\frac{1}{n}}(z) \\ \preceq \left(\rho_{\frac{1}{n}}(z)\dfrac{1}{|z-\zeta|}\right)^m & \text{if } \ \zeta \in L_z \setminus l_{\frac{1}{n}}(z) \\ \preceq (\eta_{L_z}\theta^\tau)^{\frac{nm}{\tau+1}} & \text{if } \ \zeta \in L \setminus L_z. \end{cases}$$

Considering that $\deg \mathcal{K}_n \leqq n$ the n-th Christoffel function can be estimated from above as follows:

$$\lambda_n(\nu,p,t,z) \preceq \int_L |\mathcal{K}_n(\zeta)|^p \rho_{\frac{1}{n}}(\zeta)^t \frac{v_{\frac{1}{n}}(\zeta)}{\rho_{\frac{1}{n}}(\zeta)}\,|d\zeta| \tag{4.22}$$

$$= \int_{l_{\frac{1}{n}}(z)} + \int_{L_z\setminus l_{\frac{1}{n}}(z)} + \int_{L\setminus L_z} \preceq \rho_{\frac{1}{n}}(z)^t v_{\frac{1}{n}}(z),$$

where to get the last inequality we follow the proof of Theorem 2.4 considering (4.19) to establish an estimate for the integral over $l_{1/n}(z)$ and $L_z \setminus l_{1/n}(z)$ respectively. As to the integral over $L \setminus L_z$ we use (4.21) and the estimate

$$\rho_{\frac{1}{n}}(\zeta)^t v_{\frac{1}{n}}(\zeta) \succeq \frac{1}{n^{2t}}\frac{1}{n^{2\alpha_4}} \succeq (\eta_{L_z}\theta^\tau)^{\frac{n}{\tau+1}},$$

which is the consequence of the application of (4.6′) to $l_{\frac{1}{n}}(\zeta)$ and L, as well as of Lemma 4.10. $\square$

Corollaries 3.1 and 3.2 are trivial consequences of (3.1), while to Corollary 3.3 we only have to use the simple observation that if for a sequence $\{a_n\}_{n=1}^\infty$

$$\max_{1\leqq k\leqq n} a_k \to \infty$$

holds, then there is an infinite subset $\mathbb{M} \subset \mathbb{N}$ such that for every $n \in \mathbb{M}$

$$a_n = \max_{1\leqq k\leqq n} a_k.$$

Proof of Corollary 3.4. Fix $z \in L$ and $\varepsilon > 0$. Assume that

$$|\pi_n(z)| < \frac{1}{n^{1/2+\varepsilon}} \frac{1}{\sqrt{v_{\frac{1}{n}}(z)}} \tag{4.23}$$

holds except for finitely many n. From this assumption we derive a contradiction. Let the maximum of the indices of the exceptions be $m = m(z, \varepsilon)$. By Theorem 2.4 and (3.1)

$$\sum_{k=0}^{Sm} |\pi_k(z)|^2 \leqq \frac{c}{v_{\frac{1}{Sm}}(z)}$$

where S is a positive integer chosen later. Let T be another positive integer also defined later. According to (4.6)

$$\frac{1}{v_{\frac{1}{Sm}}(z)} \leqq \frac{c_6}{T^{\frac{1}{\alpha_4}}} \frac{1}{v_{\frac{1}{TSm}}(z)}.$$

Substituting this into the previous inequality and using again Theorem 2.4 for $v_{\frac{1}{TSm}}$ we get

$$\left(\frac{1}{c} - \frac{cc_6}{T^{\frac{1}{\alpha_4}}}\right) \frac{1}{v_{\frac{1}{TSm}}(z)} \leqq \sum_{k=Sm+1}^{TSm} |\pi_k(z)|^2.$$

By (4.23) and by the monotonicity of $v_{1/k}(z)$ in k we can continue this as

$$< \sum_{k=Sm+1}^{TSm} \frac{1}{k^{2\varepsilon+1}} \frac{1}{v_{\frac{1}{k}}(z)} \leqq \sum_{k=Sm+1}^{TSm} \frac{1}{k^{2\varepsilon+1}} \frac{1}{v_{\frac{1}{TSm}}(z)} \leqq \frac{1}{v_{\frac{1}{TSm}}(z)} \sum_{k=Sm+1}^{\infty} \frac{1}{k^{2\varepsilon+1}}.$$

Now if we choose T such that

$$\left(\frac{1}{c} - \frac{cc_6}{T^{\frac{1}{\alpha_4}}}\right) > \frac{1}{2c}$$

and S such that

$$\sum_{k=Sm+1}^{\infty} \frac{1}{k^{2\varepsilon+1}} < \frac{1}{2c},$$

then we obtain the desired contradiction. $\square$

Lemma 3.6 and Lemma 3.8 are simple consequences of [5, Lemma 2.10, Lemma 2.11 and Lemma 2.12]. Because of the similarity we only cite [5, Lemma 2.10] and prove Lemma 3.6 here.

Lemma 4.11 ([5, Lemma 2.10]). *Let L be a quasismooth curve which has a Dini-smooth corner at $\zeta \in L$ with angle $\beta\pi$, $0<\beta<2$. Then there is a constant $\varepsilon = \varepsilon(L,\zeta)$ such that for all points $z \in L$, $z' \in (L \cup \Omega)$ with $|z-z'| \leqq |z-\zeta| \leqq \varepsilon$ the inequalities*

$$\frac{1}{r_1}\left|\frac{\Phi(z)-\Phi(\zeta)}{\Phi(z)-\Phi(z')}\right| \leqq \left|\frac{z-\zeta}{z-z'}\right| \leqq r_1\left|\frac{\Phi(z)-\Phi(\zeta)}{\Phi(z)-\Phi(z')}\right|, \tag{4.24}$$

$$\frac{1}{r_2}\big|\Phi(z)-\Phi(\zeta)\big|^{\beta} \leqq |z-\zeta| \leqq r_2\big|\Phi(z)-\Phi(\zeta)\big|^{\beta} \tag{4.25}$$

hold with some constants $r_i = r_i(L,\zeta,\varepsilon,\beta) > 0$, $i \in \{1;2\}$.

[5, Lemma 2.11 and Lemma 2.12] are the appropriate variants of [5, Lemma 2.10] for arc and for the endpoints of an arc respectively.

Remark 4.A. It can be seen from the proof of [5, Lemma 2.10] that if L is piecewise Dini-smooth with finitely many corners (with angles different from π) then ε, r_1, r_2 can be chosen independently of ζ and of the angles, so the previous inequalities are valid for every $\zeta \in L$ with the same ε, r_1 and r_2.

Applying the lemma we can reason in the following way (cf. [3, Lemma 3]):

Proof of Lemma 3.6. We deal with only the curve-case, because the proof for the arc-case or the endpoint follows the same way but applying [5, Lemma 2.11] and [5, Lemma 2.12] respectively.

We shall use the constants α_j and c_j from Lemmas 4.1 and 4.2.

Let $z \in l_\varepsilon(\zeta)$. It can be assumed that $\rho_{\frac{1}{n}}(z) < \varepsilon$ for any $z \in L$ (this is true if n is large enough).

- If $|z-\zeta| \leqq \rho_{\frac{1}{n}}(z)$ then, by (4.2),

$$\frac{1}{c_2}\rho_{\frac{1}{n}}(\zeta) \leqq \rho_{\frac{1}{n}}(z) \leqq c_2\rho_{\frac{1}{n}}(\zeta), \tag{4.26}$$

- if $|z-\zeta| > \rho_{\frac{1}{n}}(z)$ then, with $K := (c_1c_4)^{\alpha_2}$, (4.4) and (4.1) with $z' = \tilde{z}_{\frac{1}{Kn}}$ show

$$|z-\zeta| \geqq \frac{1}{c_4}\left(\frac{Kn}{n}\right)^{1/\alpha_2}\rho_{\frac{1}{Kn}}(z) \geqq \left|z-\tilde{z}_{\frac{1}{Kn}}\right|,$$

so we can apply (4.24) with $z' = \tilde{z}_{\frac{1}{Kn}}$. Therefore, by applying (4.4), (4.1), (4.24) and (4.25) in this order we obtain

$$\rho_{\frac{1}{n}}(z) \leqq c_4K^{\alpha_2}\rho_{\frac{1}{Kn}}(z) \leqq c_1c_4K^{\alpha_2}\left|z-\tilde{z}_{\frac{1}{Kn}}\right| \tag{4.27}$$

$$\leqq r_1 c_1 c_4 K^{\alpha_2} \frac{|z-\zeta| \left|\Phi(z) - \left(1+\frac{1}{Kn}\right)\Phi(z)\right|}{\left|\Phi(z)-\Phi(\zeta)\right|}$$

$$\leqq r_1 r_2^{1/\beta} c_1 c_4 K^{\alpha_2} \frac{|z-\zeta|^{1-\frac{1}{\beta}}}{Kn},$$

and similarly $\rho_{\frac{1}{n}}(z) \succeq \frac{|z-\zeta|^{1-\frac{1}{\beta}}}{n}$.

To finish the proof, in (4.26) we have to show that $\rho_{\frac{1}{n}}(\zeta) \asymp \frac{1}{n^\beta}$, but this easily follows from (4.24) and (4.25). Indeed, let z be a point on L such that $|z-\zeta| = \rho_{\frac{1}{n}}(\zeta)$. If $H \geqq (c_1c_2c_4)^{\alpha_2}$ then by (4.1), (4.2) and (4.4)

$$\frac{1}{c_1c_2c_4H^{\alpha_2}} \rho_{\frac{1}{n}}(\zeta) \leqq \left|z - \tilde{z}_{\frac{1}{Hn}}\right| \leqq \rho_{\frac{1}{n}}(\zeta), \tag{4.28}$$

so (4.24) can be applied with $z' = \tilde{z}_{\frac{1}{Hn}}$. Considering (4.28) we obtain that

$$\frac{1}{r_1H}\frac{1}{n} \leqq \left|\Phi(z)-\Phi(\zeta)\right| \leqq \frac{r_1c_1c_2c_4H^{\alpha_2}}{H}\frac{1}{n}.$$

Substituting this into (4.25) we get

$$\frac{1}{r_2}\frac{1}{(r_1H)^\beta}\frac{1}{n^\beta} \leqq |z-\zeta| = \rho_{\frac{1}{n}}(\zeta) \leqq r_2\left(\frac{r_1c_1c_2c_4H^{\alpha_2}}{H}\right)^\beta \frac{1}{n^\beta}.$$

Now (4.26) and (4.27) imply the existence of a constant c_7 such that

$$\frac{1}{c_7}\Delta_n(z) \leqq \rho_{\frac{1}{n}}(z) \leqq c_7 \Delta_n(z). \quad \square$$

PROOF OF COROLLARY 3.10. By (2.6) we have

$$v_{\frac{1}{n}}(z) \preceq \int_L \left|\frac{p_n(\zeta)}{p_n(z)}\right|^p d\nu(\zeta), \tag{4.29}$$

that is

$$\left|p_n(z)\right|^p \preceq \frac{1}{v_{\frac{1}{n}}(z)} \|p_n\|_{\nu,p}^p.$$

Taking the supremum over L and then taking the p-th root we get (3.5). To obtain (3.6) we use (3.5):

$$\|p_n\|_{\nu,q} \leqq \left(\int_L \|p_n\|_\infty^{q-p} \left| p_n(\zeta) \right|^p d\nu(\zeta) \right)^{\frac{1}{q}}$$

$$\leqq M_n^{\frac{q-p}{pq}} \|p_n\|_{\nu,p}^{\frac{q-p}{q}} \|p_n\|_{\nu,p}^{\frac{p}{q}} = M_n^{\frac{1}{p}-\frac{1}{q}} \|p_n\|_{\nu,p}. \quad \square$$

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