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Non-symmetric fast decreasing polynomials and applications

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ABSTRACT

A polynomial P_n is called fast decreasing if $P_n(0) = 1$, and, on $[-1, 1]$, P_n decreases fast (in terms of n and the distance from 0) as we move away from the origin. This paper considers the version when P_n has to decrease only on some non-symmetric interval $[-a, 1]$ with possibly small a. In this case one gets a faster decrease, and this type of extension is needed in some problems, when symmetric fast decreasing polynomials are not sufficient. We shall apply such non-symmetric fast decreasing polynomials to find local bounds for Christoffel functions and for local zero spacing of orthogonal polynomials with respect to a doubling measure close to a local endpoint.

1. Introduction

Fast decreasing, or pin polynomials have been used in various places in mathematical analysis. They imitate in a best way the "Dirac delta" among polynomials of a given degree and they can serve to construct well-localized "partitions of unity" consisting of polynomials.

In [3], a fairly complete description of the possible degree of fast decreasing symmetric polynomials was given in the following form. Let Φ be an even function on $[-1, 1]$ such that Φ is increasing on $[0, 1]$, it is continuous from the right, and $\Phi(0+) \le 0$. Consider polynomials P such that $P(0) = 1$ and

$$|P(x)| \le e^{-\Phi(x)},$$

and let n_Φ denote the smallest degree for which such polynomials exist. Then, according to [3, Theorem 1],

$$\frac{1}{6} N_\Phi \le n_\Phi \le 12 N_\Phi,$$

where

$$N_\Phi = 2 \sup_{\Phi^{-1}(0) \le x < \Phi^{-1}(1)} \sqrt{\frac{\Phi(x)}{x^2}} + \int_{\Phi^{-1}(1)}^{1/2} \frac{\Phi(x)}{x^2} \mathrm{d}x + \sup_{1/2 \le x < 1} \frac{\Phi(x)}{-\log(1-x)} + 1.$$

The point is that this estimate is universal, in particular Φ can depend on some parameters. For example (see [3, Section 2]), if ψ is an increasing function on $[0, \infty)$ and $\psi(x) \le C\psi(x/2)$ there, then there are polynomials P_n of degree at most n such that

$$P_n(0) = 1, \qquad |P_n(x)| \le Ce^{-c\psi(nx)}, \quad x \in [-1, 1], \tag{1.1}$$

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for some constants C, c, independent of n, if and only if

$$\int_1^\infty \frac{\psi(u)}{u^2}\mathrm{d}u < \infty. \tag{1.2}$$

Another parametric choice is (see [3, Section 2]): if φ is an increasing function on $[0, 1]$ and $\varphi(x) \le C\varphi(x/2)$ there, then there are polynomials P_n of degree at most n such that

$$P_n(0) = 1, \qquad |P_n(x)| \le Ce^{-cn\varphi(x)}, \quad x \in [-1, 1], \tag{1.3}$$

for some constants C, c, independent of n, if and only if

$$\int_0^1 \frac{\varphi(u)}{u^2}\mathrm{d}u < \infty. \tag{1.4}$$

In this setting the decrease have to be the same on $[-1, 0]$ and on $[0, 1]$, but that is not important; if Φ is not an even function then similar results can be proven by considering the even function $\Phi^*(x) = \Phi(x) + \Phi(-x)$. However, what is important is a control of the polynomials on all of $[-1, 1]$, i.e. on a relatively large interval around 0 (where the polynomial takes the value 1).

If one needs to control the polynomials only on some interval $[-a, 1]$ with some small a, then things change: the decrease of P_n away from 0 can be faster due to the fact that P_n can behave arbitrarily to the left of $-a$.

In this paper we consider the analogue of (1.1) in this non-symmetric setting and give applications concerning Christoffel functions and zero spacing of orthogonal polynomials.

2. Non-symmetric fast decreasing polynomials

Let ψ be a nonnegative and increasing function on $[0, \infty)$, such that $\psi(0+) = 0$ and $\psi(x) \le M_0\psi(x/2)$ with some constant M_0.

Theorem 2.1. *Suppose that*

$$\int_1^\infty \frac{\psi(u)}{u^2}\mathrm{d}u < \infty. \tag{2.1}$$

Then there are constants $C, c > 0$ such that for all $a \in [0, 1/2]$ and for all n there are polynomials $P_n = P_{n,a}$ of degree at most n with the properties that $P_n(0) = 1$, $|P_n(x)| \le 2$, $x \in [-a, 1]$, and

$$|P_n(x)| \le C\exp\left(-c\psi\left(\frac{n|x|}{\sqrt{|x|}+\sqrt{a}}\right)\right), \quad x \in [-a, 1]. \tag{2.2}$$

We mention that the theorem is sharp from several points of view. Let us record here

Proposition 2.2. *If for a sequence $a_n \in [0, 1/4]$ there are polynomials P_n, $n = 1, 2, \ldots$ with properties $P_n(0) = 1$ and (2.2) (with $a = a_n$), then necessarily (2.1) must be true.*

A similar argument gives that if $\delta_n \to 0$, then there is a ψ for which (2.1) holds but

$$|P_n(x)| \le C\exp\left(-c\psi\left(\frac{n|x|}{\delta_n(\sqrt{|x|}+\sqrt{a})}\right)\right), \quad x \in [-a, 1]$$

is impossible.

The non-symmetric version of (1.3)–(1.4) is the following, in which φ is a nonnegative and increasing function on $[0, 1]$, such that $\varphi(0+) = 0$, and $\varphi(x) \le M_0\varphi(x/2)$ with some constant M_0.

Theorem 2.3. *Suppose that*

$$\int_0^1 \frac{\varphi(u)}{u^2}\mathrm{d}u < \infty. \tag{2.3}$$

Then there are constants $C, c > 0$ such that for all $a \in [0, 1/2]$ and for all n there are polynomials $P_n = P_{n,a}$ of degree at most n with the properties that $P_n(0) = 1$ and

$$|P_n(x)| \le C\exp\left(-cn\varphi\left(\frac{|x|}{\sqrt{|x|}+\sqrt{a}}\right)\right), \quad x \in [-a, 1]. \tag{2.4}$$

Theorem 2.3 is also sharp in the same sense as Proposition 2.2: if (2.4) is true with some sequence $\{P_n\}$ and $a = a_n$, then (2.3) must hold.

Proof of Theorem 2.1. We shall get these non-symmetric fast decreasing polynomials from the symmetric ones by a series of transformations.

I. *There are C_0, c_0 and for every n polynomials Q_n of degree at most n such that $Q_n(0) = 1$ and for $x \in [-1, 1]$ we have $0 \le Q_n(x) \le 1$ and*

$$Q_n(x) \le C_0 e^{-c_0\psi(n|x|)}.$$

This is just the example considered in (1.1) and (1.2).

We may assume this Q_n to be even, for otherwise we can consider $(Q_n(x) + Q_n(-x))/2$.

II. *Let τ be a fixed number such that $C_0 e^{-c_0\psi(\tau)} < 1/2$ (when there is no such τ then ψ is bounded and there is nothing to prove). For every n and for every $(8\tau/n)^2 \le a \le 1/4$ there are even polynomials $R_n = R_{n,a}$ of degree at most n with the properties*:

- $R_n(0) = 1$,
- $R_n(2\sqrt{a}\sqrt{1-a}) = 0$,
- $0 \le R_n(x) \le 1$, $x \in [-1, 1]$, and
- $R_n(x) \le C_0 e^{-c_0\psi(n|x|/4)}$, $x \in [-1, 1]$.

Indeed, put

$$R_n(x) = Q_{[n/2]}(x)\left(\frac{Q_{[n/4]}(x) - Q_{[n/4]}(2\sqrt{a}\sqrt{1-a})}{1 - Q_{[n/4]}(2\sqrt{a}\sqrt{1-a})}\right)^2, \tag{2.5}$$

and note that, by the choice of the τ, we have

$$\begin{aligned} Q_{[n/4]}(2\sqrt{a}\sqrt{1-a}) &\le C_0 \exp(-c_0\psi([n/4]2\sqrt{a}\sqrt{1-a})) \\ &\le C_0 \exp(-c_0\psi(\tau)) < 1/2, \end{aligned}$$

and hence the second factor on the right of (2.5) is at most 1 for all $x \in [-1, 1]$.

III. *For every n and for every $(8\tau/n)^2 \le a \le 1/4$ there are polynomials $S_n = S_{n,a}$ of degree at most n such that $S_n(a) = 1$, $0 \le S_n(x) \le 2$ for $x \in [0, 1]$ and*

$$0 \le S_n(x) \le 2C_0 \exp\left(-c_0\psi\left(\frac{n|x-a|}{32(\sqrt{x}+\sqrt{a})}\right)\right), \quad x \in [0, 1]. \tag{2.6}$$

Set

$$S_n(x) = R_n(\sqrt{x}\sqrt{1-a} - \sqrt{1-x}\sqrt{a}) + R_n(\sqrt{x}\sqrt{1-a} + \sqrt{1-x}\sqrt{a}).$$

Since R_n is a linear combination of powers x^{2k}, $k = 0, 1, \ldots$, this S_n is a polynomial of degree at most $n/2$. By the choice of R_n we clearly have $S_n(a) = 1$.

Let now $0 \le x \le 2a$. Then, since

$$|\sqrt{x}\sqrt{1-a} - \sqrt{1-x}\sqrt{a}| = \left|\frac{x-a}{\sqrt{x}\sqrt{1-a} + \sqrt{1-x}\sqrt{a}}\right| \ge \left|\frac{x-a}{2\sqrt{2a}}\right|,$$

and

$$|\sqrt{x}\sqrt{1-a} + \sqrt{1-x}\sqrt{a}| \ge \sqrt{a}/2,$$

we have

$$\begin{aligned} S_n(x) &\le C_0 \exp(-c_0\psi(n|x-a|/8\sqrt{2a})) + C_0 \exp(-c_0\psi(n\sqrt{a}/8)) \\ &\le 2C_0 \exp(-c_0\psi(n|x-a|/8\sqrt{2a})). \end{aligned} \tag{2.7}$$

On the other hand, if $2a \le x \le 1$, then

$$|\sqrt{x}\sqrt{1-a} - \sqrt{1-x}\sqrt{a}| \ge \sqrt{x}\sqrt{1-a} - \sqrt{x/2} \ge \sqrt{x}\left(\sqrt{3/4} - \sqrt{1/2}\right) \ge \sqrt{x}/8,$$

while

$$|\sqrt{x}\sqrt{1-a} + \sqrt{1-x}\sqrt{a}| \ge \sqrt{x}/2,$$

and so

$$\begin{aligned} S_n(x) &\le C_0 \exp(-c_0\psi(n\sqrt{x}/32)) + C_0 \exp(-c_0\psi(n\sqrt{x}/8)) \\ &\le 2C_0 \exp(-c_0\psi(n\sqrt{x}/32)). \end{aligned} \tag{2.8}$$

Now (2.7) and (2.8) prove (2.6).

IV. *For every n and for every $2(8\tau/n)^2 \le a \le 1/2$ there are polynomials $V_n = V_{n,a}^{\psi}$ of degree at most n such that $V_n(0) = 1$, $0 \le V_n(x) \le 2$ for $x \in [-a, 1]$, and*

$$0 \le V_n(x) \le 2C_0 \exp\left(-c_0\psi\left(\frac{n|x|}{128(\sqrt{|x|}+\sqrt{a})}\right)\right), \quad x \in [-a, 1]. \tag{2.9}$$

Indeed, for $V_n(x) = S_{n,a/2}((x+a)/2)$ we clearly have all the properties if we apply (2.6) (with x replaced by $(x+a)/2$ and a replaced by $a/2$) and notice that for $x \in [-a, 1]$

$$\sqrt{(x+a)/2} + \sqrt{a/2} \le 2(\sqrt{|x|} + \sqrt{a}).$$

Completion of the proof of Theorem 2.1. Apply IV to the function $\psi^*(u) = \psi(256u)$ rather than to $\psi(u)$ (the constants C_0, c_0, τ will now be different, and below their meaning is with respect to ψ^*). Then $P_{n,a}(x) = V_{n,a}^{\psi^*}(x)$ satisfies the requirements provided $2(8\tau/n)^2 \le a \le 1/2$. On the other hand, if $0 \le a \le 2(8\tau/n)^2$ then we set $P_{n,a} = V_{n,a^*}^{\psi^*}(x)$, where $a^* = (16\tau/n)^2$. For this we have

$$0 \le P_{n,a}(x) \le 2C_0 \exp\left(-c_0\psi\left(\frac{2n|x|}{\sqrt{|x|}+16\tau/n}\right)\right), \quad x \in [-a, 1], \tag{2.10}$$

and all we need to mention is that

$$\psi\left(\frac{2n|x|}{\sqrt{|x|}+16\tau/n}\right) \ge \psi\left(\frac{n|x|}{\sqrt{|x|}+\sqrt{a}}\right) - \psi(16\tau) \tag{2.11}$$

(check this separately for $|x| \le (16\tau/n)^2$ and the rest). $\square$

We skip the proof of Theorem 2.3, for it is very similar to the proof of Theorem 2.1, one just needs to use (1.3)–(1.4) instead of (1.1)–(1.2), and instead of the condition $(8\tau/n)^2 \le a \le 1/4$ one should use $(\varphi^{-1}(M/n))^2 \le a \le 1/4$ with some appropriately large M (and then (2.11) should read with $a^* = (\varphi^{-1}(M/n))^2$ as

$$\varphi\left(\frac{2|x|}{\sqrt{|x|}+\sqrt{a^*}}\right) \ge \varphi\left(\frac{|x|}{\sqrt{|x|}+\sqrt{a^*}}\right) - \varphi(\sqrt{a^*})$$

and here $n\varphi(\sqrt{a^*})$ is bounded in n).

Proof of Proposition 2.2. We only sketch the proof. In what follows we write a for a_n, but keep in mind that it can depend on n.

Suppose (2.2) is true. Then, with $b = 2a/(1+a)$ and

$$R_n(x) = P_n\left(\frac{1-x}{2}(1+a) - a\right),$$

we have $R_n(1-b) = 1$,

- $|R_n(x)| \le C_1 \exp(-c_1\psi(d_1 n|(1-b)-x|/\sqrt{b}))$ for $x \in [1-2b, 1]$,
- $|R_n(x)| \le C_1 \exp(-c_1\psi(d_1 n\sqrt{|1-x|}))$ for $x \in [-1, 1-2b]$,

with some constants C_1, c_1, d_1.

Set $B = \arccos(1-b)$ and $S_n(t) = R_n(\cos t)$. Then S_n is an even trigonometric polynomial of degree at most n, $S_n(\pm B) = 1$ and

- $|S_n(t)| \le C_2 \exp(-c_2\psi(d_2 n|t-B|))$ for $t \in [0, 2B]$,
- $|S_n(t)| \le C_2 \exp(-c_2\psi(d_2 nt))$ for $t \in [2B, \pi]$.

Hence, for

$$T_{2n}(u) = S_n(u-B)S_n(u+B)$$

we have $T_{2n}(0) = 1$ and

- $|T_{2n}(u)| \le C_3 \exp(-c_3\psi(d_3 n|u|))$ for $u \in [-B, B]$,
- $|T_{2n}(u)| \le C_3 \exp(-c_3\psi(d_3 n|u|))$ for $u \in [-\pi, \pi] \setminus [-B, B]$.

T_{2n} is again an even trigonometric polynomial, therefore

$$U_{2n}(v) = \frac{T_n(v-\pi/2) + T_n(v+\pi/2)}{T_n(0) + T_n(\pi)}$$

is also an even trigonometric polynomial such that $U_{2n}(\pi/2) = 1$ and

- $|U_{2n}(v)| \le C_4 \exp(-c_4\psi(d_4 n|v-\pi/2|))$ for $v \in [0, \pi]$.

Now set

$$Q_{2n}(x) = U_{2n}(\arccos x).$$

This is an algebraic polynomial of degree at most $2n$ such that $Q_{2n}(0) = 1$ and

- $|Q_{2n}(x)| \le C_5 \exp(-c_5\psi(d_5 n|x|))$ for $x \in [-1, 1]$.

Hence, by (1.1)–(1.2), we must have

$$\int_1^\infty \frac{\psi(d_5 u/2)}{u^2} \mathrm{d}u < \infty,$$

which is the same as (2.1). $\square$

3. Christoffel functions for locally doubling weights

As an application of Theorem 2.1, in this section we estimate the Christoffel function at a point by the measure of a neighborhood of that point.

We recall the definition of Christoffel functions. Let μ be a finite measure with compact support on the real line. The n-th Christoffel function associated with μ is defined as

$$\lambda_n(a, \mu) = \inf_{\substack{q(a)=1\\ \deg q \le n}} \int q^2(x)\, \mathrm{d}\mu(x),$$

where the minimum is taken for all polynomials of degree at most n taking the value 1 at a. This function plays an important role in the theory of orthogonal polynomials. In fact, if $\{p_k(\mu, \cdot)\}$ are the orthonormal polynomials with respect to μ then

$$\frac{1}{\lambda_n(a, \mu)} = \sum_{k=0}^{n} p_k(\mu, a)^2,$$

i.e. the reciprocal of λ_n, is given by the diagonal of the associated reproducing kernel. See Nevai and Simon [7,9] for various properties and applications of Christoffel functions.

The measure μ is called *doubling on the interval* $[A, B]$ if $\mu([A, B]) > 0$ and there is a constant L (called the doubling constant) such that

$$\mu(2I) \le L\mu(I) \tag{3.1}$$

for all intervals $2I \subset [A, B]$ (here $2I$ is the interval I enlarged twice from its center). In a similar, but slightly different fashion, μ is called *globally doubling on a set* K if (3.1) is true for every interval I centered at a point of K. One should exercise some care here: μ may be doubling on $[A, B]$ without being globally doubling on $[A, B]$ (consider for example, $[A, B] = [0, 1]$, $\mathrm{d}\mu(x) = |1 - x|\mathrm{d}x$ on $[0, 1]$ and $\mathrm{d}\mu(x) = \mathrm{d}x$ on $(1, 2]$). However, it is easy to see that μ is doubling on $[A, B]$ precisely if its restriction to $[A, B]$ is globally doubling there $[A, B]$.

It was shown in [5, (7.14)] that if the support of μ is $[-1, 1]$ and μ is doubling there, then we have[1]

$$\lambda_n(a, \mu) \sim \mu\left([a - \hat{\Delta}_n(a), a + \hat{\Delta}_n(a)]\right), \tag{3.2}$$

where

$$\hat{\Delta}_n(a) = \frac{\sqrt{1-a^2}}{n} + \frac{1}{n^2}.$$

The local analogue was given in [12, Lemma 6]: if μ is doubling on an interval $[A, B]$, then (without any assumption on its behavior outside $[A, B]$) we have $\lambda_n(a) \sim 1/n$ uniformly on every subinterval $[A + \varepsilon, B - \varepsilon]$. Now we show, with the help of the non-symmetric fast decreasing polynomials constructed in Section 2, the local behavior of λ_n around a local endpoint of the support.

Call a point A a "left endpoint" of the support of μ, if for some $\alpha > 0$ we have $\mu([A - \alpha, A)) = 0$ but $\mu([A, A + \beta)) > 0$ for all $\beta > 0$.

Theorem 3.1. *Let A be a "left endpoint" of the support of μ. Assume that μ is a doubling measure on some interval $[A, A + \beta]$, and let $\gamma < \beta$. Then uniformly in $a \in [A, A + \gamma]$ we have*

$$\lambda_n(a, \mu) \sim \mu\left([a - \Delta_n(a), a + \Delta_n(a)]\right), \tag{3.3}$$

where

$$\Delta_n(a) = \frac{\sqrt{a - A}}{n} + \frac{1}{n^2}.$$

[1] $A \sim B$ means that the ratio of the two sides is bounded from below and from above by two positive constants.

While Theorem 3.1 could be deduced from the global version (3.2), the proof we give for the upper estimate works also on general sets rather than just on intervals. When the doubling character is known only on a set K then naturally only an upper estimate can be given:

Theorem 3.2. *Let A be a "left endpoint" of the support of μ. Assume that μ is a globally doubling measure on some set $K \subset [A, A+\beta]$, and let $\gamma < \beta$. Then uniformly in $a \in K \cap [A, A+\gamma]$ we have*

$$\lambda_n(a, \mu) \le C\mu\left([a - \Delta_n(a), a + \Delta_n(a)]\right) \tag{3.4}$$

with some C independent of $a \in K$ and n.

Example. The Cantor measure is defined as follows. Do the standard triadic Cantor construction. At level l we have a set $\mathcal{C}_l$ consisting of 2^l intervals each of length 3^{-l}. Now let

$$\rho_l = (3/2)^l \cdot m_{|\mathcal{C}_l},$$

where m is the Lebesgue measure on $\mathbf{R}$, i.e. ρ_l puts equal uniform masses to each subinterval of $\mathcal{C}_l$. As $l \to \infty$ this ρ_l has a weak* limit ρ, called the Cantor measure. It is easy to see that ρ is supported on the Cantor set $\mathcal{C} = \cap_l \mathcal{C}_l$ and is globally doubling on $\mathcal{C}$ (but not, say, on [0, 1]), even though it is a singular continuous measure.

Let (p, q) denote any subinterval of $\mathcal{C}_l$. On applying Theorem 3.2 (and its obvious modification for right endpoints) we get the upper bound

$$\lambda_n(a, \rho) \le C_{p,q}\rho([a - \overline{\Delta}_n(a), a + \overline{\Delta}_n(a)]), \quad a \in (p, q)$$

with

$$\overline{\Delta}_n(a) = \frac{\sqrt{(a-p)(q-a)}}{n} + \frac{1}{n^2}.$$

Since $\rho(I) \le C_0|I|^{\log 2/\log 3}$ for any interval I with some absolute constant C_0, it follows that

$$\lambda_n(a, \rho) \le C'_{p,q}\left(\frac{\sqrt{(a-p)(q-a)}}{n} + \frac{1}{n^2}\right)^{\log 2/\log 3}, \quad a \in (p, q).$$

For example, at every endpoint of a contiguous subinterval to $\mathcal{C}$ we have $\lambda_n \le Cn^{-2\log 2/\log 3}$, and we believe that this is the correct order for λ_n at those points.

Before proving Theorems 3.1 and 3.2, let us mention an equivalent form of the doubling property, see [5, Lemma 2.1]:

Lemma 3.3. *The following conditions for a measure μ are equivalent:*

(1) *μ is doubling in $[A, B]$.*
(2) *There is an s and a K such that $\mu(I) \le K\left(|I|/|J|\right)^s \mu(J)$ for all intervals $J \subset I \subset [A, B]$.*[2]
(3) *There is an $s > 0$ and a K such that*

$$\mu(I) \le K\left(\frac{|I| + |J| + \mathrm{dist}\{I, J\}}{|J|}\right)^s \mu(J)$$

for all intervals I and $J \subset [a, b]$.

(4) *There is a σ and a κ such that $\mu(J) \le \kappa\left(|J|/|I|\right)^\sigma \mu(I)$ for all intervals $J \subset I \subset [A, B]$.*

Proof of Theorem 3.1. (3.3) holds on every interval $[A+\gamma', A+\gamma]$ with $0 < \gamma' < \gamma < \beta$ by [12, Lemma 6]. Therefore, we may assume $A = 0$, $\alpha = \beta = 1$ and $a \in [0, 1/4]$. So μ has no mass in $[-1, 0]$ but it is (non-zero and) doubling on [0,1], and we shall estimate the Christoffel function at an $a \in [0, 1/4]$. Note also that in this case

$$\Delta_n(a) = \frac{\sqrt{a}}{n} + \frac{1}{n^2}.$$

First we give a bound for $\lambda_n(a, \mu)$ from above. We apply Theorem 2.1 with $\psi(x) = \sqrt{x}$. According to that theorem, for any $0 \le a \le 1/2$ there are polynomials $P_{m,a}$ of degree at most m such that $P_{m,a}(0) = 1$, $|P_{m,a}(x-a)| \le 2$ on $[0, 1]$,

$$0 \le P_{m,a}(x-a) \le C\exp\left(-c\sqrt{\frac{m|x-a|}{\sqrt{a}}}\right), \quad 0 \le x \le 2a \tag{3.5}$$

[2] $|H|$ denotes the Lebesgue measure of the set H.

and

$$0 \le P_{m,a}(x-a) \le C \exp\left(-c\sqrt{m\sqrt{|x-a|}}\right), \quad 2a \le x \le 1 \tag{3.6}$$

with some absolute constants $c, C > 0$. Let $B \ge 2$ be such that $\mathrm{supp}(\mu) \subset [-B, B]$.

Next, we invoke the inequality (see [1, Proposition 4.2.3])

$$|q_n(x)| \le \|q_n\|_{[-1,1]} \left(|x| + \sqrt{x^2-1}\right)^n, \qquad \deg(q_n) \le n, \quad x \in \mathbf{R},$$

which implies for any interval $[\theta-\delta, \theta+\delta]$ the inequality

$$|q_n(x)| \le \|q_n\|_{[\theta-\delta,\theta+\delta]}(2 \cdot \mathrm{dist}(x,\theta)/\delta)^n, \qquad \deg(q_n) \le n, \quad x \in \mathbf{R} \setminus [\theta-\delta, \theta+\delta].$$

Since $0 \le P_{m,a}(x) \le 2$ on $[0, 1]$, we obtain from here that

$$P_{m,a}(x) \le 2(8B)^m \quad \text{for all } x \in [-B, B].$$

Consider now

$$U_{(2M+1)m}(x) := P_{m,a}(x-a)\left(1 - \frac{(x-a)^2}{(B+1)^2}\right)^{Mm},$$

where M will be chosen below, and for a given n set $p_n(x) := U_{(2M+1)m}(x)$ with $m = m(n) = \left[\frac{n}{2M+1}\right]$. Its degree is at most n, $p_n(a) = 1$, and since

$$\left(1 - \frac{(x-a)^2}{(B+1)^2}\right) \le 1 \quad \text{on } [-B, B],$$

we obtain

$$p_n(x) \le C \exp\left(-c\sqrt{\frac{m|x-a|}{\sqrt{a}}}\right), \quad x \in [0, 2a], \tag{3.7}$$

$$p_n(x) \le C \exp\left(-c\sqrt{m\sqrt{|x-a|}}\right), \quad x \in [2a, 1], \tag{3.8}$$

and on $[-B, B] \setminus [-1, 1/2]$ (recall that $0 \le a \le 1/4$)

$$|p_n(x)| \le 2(8B)^m \left(1 - \frac{1}{16(B+1)^2}\right)^{Mm}.$$

Now if M is chosen so large that

$$(8B)\left(1 - \frac{1}{16(B+1)^2}\right)^M < \frac{1}{e},$$

then we obtain

$$|p_n(x)| \le 2e^{-m} \le 2e^{-n/4M}, \quad x \in [-B, B] \setminus [-1, 1/2]. \tag{3.9}$$

First let $4/n^2 \le a \le 1/4$. Using the preceding estimates we can write

$$\begin{aligned} \lambda_n(a, \mu) &= \inf_{\substack{q(a)=1 \\ \deg q \le n}} \int q^2 \,\mathrm{d}\mu \le \int p_n^2 \,\mathrm{d}\mu \\ &= \int_{a-\Delta_n(a)}^{a+\Delta_n(a)} + \int_0^{a-\Delta_n(a)} + \int_{a+\Delta_n(a)}^{2a} + \int_{2a}^{1/2} + \int_{\mathrm{R}\setminus[-1,1/2]}, \end{aligned} \tag{3.10}$$

where we used that, by assumption, $\mu([-1, 0]) = 0$ and $a \in [0, 1/4]$. In the first integral $0 \le p_n(x) \le 2$ on $[0, 1]$, so

$$\int_{a-\Delta_n(a)}^{a+\Delta_n(a)} p_n^2 \mathrm{d}\mu \le \int_{a-\Delta_n(a)}^{a+\Delta_n(a)} 4 \,\mathrm{d}\mu = 4\mu([a-\Delta_n(a), a+\Delta_n(a)]). \tag{3.11}$$

The second and the third integrals are treated together, since we have similar estimates (see (3.7)) for p_n on the corresponding intervals:

$$\int_0^{a-\Delta_n(a)} + \int_{a+\Delta_n(a)}^{2a} \leq 2\int_{a+\Delta_n(a)}^{2a} C\exp\left(-c\sqrt{\frac{m|x-a|}{\sqrt{a}}}\right)\mathrm{d}\mu(x)$$

$$\leq 2C\sum_{i=1}^{H}\int_{a+i\Delta_n(a)}^{a+(i+1)\Delta_n(a)}\exp\left(-c\sqrt{\frac{m|x-a|}{\sqrt{a}}}\right)\mathrm{d}\mu(x), \tag{3.12}$$

where H is the positive integer for which $a+H\Delta_n(a) < 2a \leq a+(H+1)\Delta_n(a)$. The integrand on $[a+i\Delta_n(a), a+(i+1)\Delta_n(a)]$ is at most

$$\exp\left(-c\sqrt{\frac{m|x-a|}{\sqrt{a}}}\right) \leq \exp\left(-c\sqrt{\frac{mi\Delta_n(a)}{\sqrt{a}}}\right)$$

$$\leq \exp\left(-\frac{c}{2\sqrt{M}}\sqrt{\frac{ni\Delta_n(a)}{\sqrt{a}}}\right) \leq \exp\left(-\frac{c}{2\sqrt{M}}\sqrt{i}\right)$$

since $\frac{n}{4M} \leq m$ and $n\Delta_n(a) \geq \sqrt{a}$. Using this and the doubling property (Lemma 3.3, (3)) we obtain for (3.12)

$$\leq 2C\sum_{i=1}^{H}\exp\left(-\frac{c}{2\sqrt{M}}\sqrt{i}\right)\mu([a+i\Delta_n(a), a+(i+1)\Delta_n(a)])$$

$$\leq 2C\left(\sum_{i=1}^{\infty}K(i+1)^s e^{-\frac{c}{2\sqrt{M}}\sqrt{i}}\right)\mu([a-\Delta_n(a), a+\Delta_n(a)]), \tag{3.13}$$

where K and s depend only on the doubling constant of μ.

The estimate of the fourth integral is like the former one, but we use (3.8) instead of (3.7):

$$\int_{2a}^{1/2} \leq C\sum_{i=H}^{\hat{H}}\int_{a+i\Delta_n(a)}^{a+(i+1)\Delta_n(a)}\exp\left(-c\sqrt{m\sqrt{|x-a|}}\right)\mathrm{d}\mu(x),$$

where $\hat{H}$ is the constant for which $a+\hat{H}\Delta_n(a) < 1/2 \leq a+(\hat{H}+1)\Delta_n(a)$. Using that

$$m\sqrt{|x-a|} \geq m\sqrt{i\Delta_n(a)} \geq \frac{n}{4M}\sqrt{\frac{i}{n^2}} \geq \frac{\sqrt{i}}{4M}$$

on $[a+i\Delta_n(a), a+(i+1)\Delta_n(a)]$ we get from the doubling property (Lemma 3.3, (3))

$$\int_{2a}^{1/2} \leq C\left(\sum_{i=H}^{\infty}K(i+1)^s e^{-\frac{c}{2\sqrt{M}}\sqrt[4]{i}}\right)\mu([a-\Delta_n(a), a+\Delta_n(a)]). \tag{3.14}$$

Finally, we deal with the fifth integral. According to the doubling property (Lemma 3.3,(2)) we can see that, for large n,

$$\mu([a-\Delta_n(a), a+\Delta_n(a)]) \geq \frac{1}{K}\left(\frac{|[a-\Delta_n(a), a+\Delta_n(a)]|}{|[0,1]|}\right)^s\mu([0,1])$$

$$\geq c|\Delta_n(a)|^s \geq c\left(\frac{1}{n^2}\right)^s \geq c_1 e^{-n/4M}.$$

Therefore (3.9) gives

$$\int_{\mathbb{R}\setminus[-1,1/2]} p_n^2\,\mathrm{d}\mu \leq \mu(\mathbb{R}\setminus[-1,1/2])4e^{-n/4M} \leq C\mu([a-\Delta_n(a), a+\Delta_n(a)]). \tag{3.15}$$

From (3.11) and (3.13)–(3.15) we obtain

$$\lambda_n(a) \leq C\mu([a-\Delta_n(a), a+\Delta_n(a)]),$$

which is the upper estimate in (3.3).

When $0 \leq a \leq 4/n^2$, then the argument is similar if, instead of (3.10), we use the splitting

$$\int p_n^2\mathrm{d}\mu = \int_0^{a+\Delta_n(a)} + \int_{a+\Delta_n(a)}^{1/2} + \int_{\mathbf{R}\setminus[0,1/2]}.$$

The corresponding lower estimate for $\lambda_n(a, \mu)$ in (3.3) is immediate from (3.2). Indeed, according to our assumptions, μ is a doubling measure on $[0, 1]$, so taking the restriction $\nu = \mu|_{[0,1]}$ we get with

$$\tilde{\Delta}_n(x) = \frac{2\sqrt{x - x^2}}{n} + \frac{1}{n^2}$$

for $a \le \frac{1}{2}$ (transform (3.2) to the interval $[0, 1]$ by a linear transformation)

$$\begin{aligned} \lambda_n(a, \mu) &= \inf_{\substack{q(a)=1\\ \deg q \le n}} \int q^2(x)\,\mathrm{d}\mu(x) \ge \inf_{\substack{q(a)=1\\ \deg q \le n}} \int_0^1 q^2(x)\,\mathrm{d}\mu(x) = \lambda_n(a, \nu) \\ &\ge \frac{1}{C_0} \int_{a-\tilde{\Delta}_n(a)}^{a+\tilde{\Delta}_n(a)} \mathrm{d}\nu(x) \ge \frac{1}{C_0} \int_{a-\Delta_n(a)}^{a+\Delta_n(a)} \mathrm{d}\nu(x) \\ &= \frac{1}{C_0} \mu([a - \Delta_n(a), a + \Delta_n(a)]). \end{aligned} \tag{3.16}$$

This proves the lower estimate in (3.3), and the proof is complete. □

We skip the proof of Theorem 3.2, for it agrees with the proof of the upper estimate given in the preceding proof. Indeed, in that proof we only needed that if μ is doubling on a set K then for all intervals I centered at a point of K and for all $\lambda \ge 1$ we have

$$\mu(\lambda I) \le C\lambda^s \mu(I),$$

with some constant C independent of I and λ, which is clearly true with $s = \log L/\log 2$.

4. Local zero spacing of orthogonal polynomials

Let μ be a measure on the real line with compact support, $\{p_n\}$ the orthonormal polynomials with respect to μ and let $x_{n,1} < \cdots < x_{n,n}$ be the zeros of p_n. In this section, using Theorem 3.1, we give matching upper and lower bounds for $x_{n,k+1} - x_{n,k}$ around local endpoints of the support where the weight is doubling.

If μ is supported on $[-1, 1]$ and it is doubling there, then by [6, Theorem 1]

$$x_{n,k+1} - x_{n,k} \sim \frac{\sqrt{1 - x_{n,k}^2}}{n} + \frac{1}{n^2}, \quad k = 1, \ldots, n-1. \tag{4.1}$$

Actually, this is also true for $k = 0$ and $k = n$ if we set $x_{n,0} = -1$ and $x_{n,n+1} = 1$, i.e. the first and last zeros are of distance $\sim 1/n^2$ from the endpoints of the intervals. In this result a global property implies quasi-uniform spacing for the zeros over the whole support of the measure.

Last and Simon [4] considered zero spacing using information only around the zeros in question. Roughly speaking, they showed that if μ is absolutely continuous in a neighborhood of E_0 and its density behaves like $|x - E_0|^q$ there, then $x_n^{(1)}(E_0) - x_n^{(-1)}(E_0) \sim 1/n$ for the zeros $x_n^{(\pm 1)}(E_0)$ enclosing E_0. As a generalization, Varga showed in [12] that if μ is a doubling measure on an interval $[a, b]$ then

$$x_{n,k+1} - x_{n,k} \sim \frac{1}{n}$$

uniformly for $x_{n,k} \in [a + \varepsilon, b - \varepsilon]$ for all $\varepsilon > 0$.

In this section we prove the analogue of this last result for a local endpoint.

Theorem 4.1. *Let A be a "left endpoint" for the support of μ, and assume that μ is a doubling measure on some interval $[A, A+\beta]$. Then for any $\gamma < \beta$*

$$x_{n,k+1} - x_{n,k} \sim \Delta_n(x_{n,k}) = \frac{\sqrt{x_{n,k} - A}}{n} + \frac{1}{n^2} \tag{4.2}$$

uniformly in $x_{n,k}, x_{n,k+1} \in [A, A + \gamma]$.

This theorem and Theorem 3.1 have a simple consequence concerning the quotient of adjacent Cotes numbers. Recall that the Cotes numbers are the values of the Christoffel function at the zeros of orthonormal polynomials: $\lambda_{n,k} := \lambda_n(x_{n,k})$.

Corollary 4.2. *Assume that μ has the doubling property on $[A, A+\beta]$ and vanishes on some interval $[A-\alpha, A]$. Then, for every $\gamma < \beta$, there is a constant D_γ such that*

$$\frac{1}{D_\gamma} \le \frac{\lambda_{n,k}}{\lambda_{n,k+1}} \le D_\gamma, \tag{4.3}$$

whenever $x_{n,k}, x_{n,k+1} \in [A, A+\gamma]$.

This is the local version of [6, Theorem 2].

Exactly as in [6, Theorem 3], Theorem 4.1 and Corollary 4.2 have a converse:

Theorem 4.3. *Assume that μ vanishes on $[A-\alpha, A]$, and that (4.2) and (4.3) hold on every interval $[A, A+\gamma]$, $\gamma < \beta$. Then μ has the doubling property on every such interval.*

Proof of Theorem 4.1. The proof follows that of theorem 1 in [6]. We begin the proof by the following variant of Lemma 4 in [6]: for $A \le y \le x$, if (see (4.2) for the definition of Δ_n)

$$x - y \le S(\Delta_n(x) + \Delta_n(y)), \quad S \ge 1,$$

then

$$\Delta_n(x) \le 16S\Delta_n(y). \tag{4.4}$$

This can be obtained by simple calculation as in [6, Lemma 4].

By [12, Theorem 1], (4.2) is true on any interval $[A+\gamma', A+\gamma]$, $0 < \gamma' < \gamma < \beta$, therefore it can be assumed again that $\alpha = \beta = 1$ and $\gamma = 1/4$ (apply a linear transformation if necessary).

We begin with the upper estimate of $x_{n,k+1} - x_{n,k}$. We need the following well known Markov inequality (see [2]):

$$\sum_{j=1}^{k-1} \lambda_{m,j} \le \mu((-\infty, x_{m,k})) \le \mu((-\infty, x_{m,k}]) \le \sum_{j=1}^{k} \lambda_{m,j} \tag{4.5}$$

connecting the measure, the zeros of the orthogonal polynomials and the Cotes numbers. If we apply this with $k+1$ and k and subtract the resulting inequalities, then it follows that

$$\mu([x_{m,k}, x_{m,k+1}]) \le \lambda_{m,k} + \lambda_{m,k+1}. \tag{4.6}$$

Let $x_{n,k}, x_{n,k+1} \in [0, 1/4]$ and $\Delta_{n,k} := \Delta_n(x_{n,k})$. We may assume $x_{n,k+1} - x_{n,k} \ge 2\Delta_{n,k}$, for otherwise there is nothing to prove. Then

$$x_{n,k} + \Delta_{n,k} \le x_{n,k+1} - \Delta_{n,k}.$$

Let

$$E_1 = \left[x_{n,k} - \Delta_{n,k}, x_{n,k} + \Delta_{n,k}\right], \qquad E_2 = \left[x_{n,k+1} - \Delta_{n,k}, x_{n,k+1} + \Delta_{n,k}\right]$$

and

$$I = \left[x_{n,k} - \Delta_{n,k}, x_{n,k+1} + \Delta_{n,k}\right].$$

If we can estimate $|I|$ by a constant times $\Delta_{n,k}$ from above, then we are done.

We obtain from the doubling property of μ and from (4.6)

$$\mu(I) \le L\mu([x_{m,k+1}, x_{m,k}]) \le L\left(\lambda_{m,k+1} + \lambda_{m,k}\right).$$

Now we apply Theorem 3.1 to continue this as

$$\le LC(\mu(E_1) + \mu(E_2)) \le 2LC\kappa \left(\frac{|E_1|}{|I|} + \frac{|E_2|}{|I|}\right)^\sigma \mu(I)$$

where, in the last estimate, Lemma 3.3, (4) was used. Therefore,

$$|I| \le \sqrt[\sigma]{2LC\kappa}(|E_1| + |E_2|),$$

and then (4.4) with $S = \sqrt[\sigma]{2LC\kappa}$ gives the upper bound

$$x_{n,k+1} - x_{n,k} \le C\Delta_{n,k}.$$

As for the lower estimate, we may assume that $x_{n,k+1} - x_{n,k} = \delta\Delta_{n,k}$ with some $\delta \le \frac{1}{2}$. Define the polynomial q_{n-2} such that

$$p_n(x) = q_{n-2}(x)(x - x_{n,k})(x - x_{n,k+1}).$$

Using that p_n is orthogonal to all polynomials of degree at most $n-1$ we obtain

$$0 = \int p_n q_{n-2}\,\mathrm{d}\mu = \int q_{n-2}^2(x)(x-x_{n,k})(x-x_{n,k+1})\,\mathrm{d}\mu(x)$$
$$= \int_{x_{n,k}}^{x_{n,k+1}} + \int_{\mathbf{R}\setminus[x_{n,k},x_{n,k+1}]}. \tag{4.7}$$

Note that the integrand is negative only on $[x_{n,k}, x_{n,k+1}]$. Since $x_{n,k+1} - x_{n,k} = \delta\Delta_{n,k}$ with $\delta \le 1/2$, we get

$$\int_{x_{n,k}}^{x_{n,k+1}} q_{n-2}^2(x)(x-x_{n,k})(x-x_{n,k+1})\,\mathrm{d}\mu(x) = -\int_{x_k}^{x_{k+1}} q_{n-2}^2(x)|x-x_{n,k}||x-x_{n,k+1}|\,\mathrm{d}\mu(x)$$
$$\ge -\delta^2\Delta_{n,k}^2 \int_{x_{n,k}}^{x_{n,k+1}} q_{n-2}^2\,\mathrm{d}\mu. \tag{4.8}$$

For the second integral we use the assumption $\delta \le \frac{1}{2}$ and Remez' inequality [5, (7.16)]: if μ doubling on [0, 1], then for every Λ there is a C_Λ such that for $[\eta, \vartheta] \subset [0, 1]$ and for an arbitrary polynomial r_n of degree at most n

$$\int_0^1 r_n^2\,\mathrm{d}\mu \le C_\Lambda \int_{[0,1]\setminus[\eta,\vartheta]} r_n^2\,\mathrm{d}\mu \tag{4.9}$$

holds, provided $|\arccos([2\eta-1, 2\vartheta-1])| \le \Lambda/n$. We are going to apply this with

$$[\eta, \theta] = [x_{n,k} - 2\Delta_{n,k}, x_{n,k} + 2\Delta_{n,k}] \cap [0, 1].$$

Because of the definition of $\Delta_{n,k}$, we have $|\arccos([2\eta-1, 2\vartheta-1])| \le \Lambda/n$, so (4.9) is applicable, and we obtain

$$\int_{\mathbf{R}\setminus[x_{n,k},x_{n,k+1}]} q_{n-2}^2(x)(x-x_{n,k})(x-x_{n,k+1})\mathrm{d}\mu(x)$$
$$\ge \int_{[0,1]\setminus[x_{n,k}-2\Delta_{n,k},x_{n,k}+2\Delta_{n,k}]} q_{n-2}^2(x)(x-x_{n,k})(x-x_{n,k+1})\,\mathrm{d}\mu(x)$$
$$\ge \Delta_{n,k}^2 \int_{[0,1]\setminus[x_{n,k}-2\Delta_{n,k},x_{n,k}+2\Delta_{n,k}]} q_{n-2}^2\,\mathrm{d}\mu \ge \frac{\Delta_{n,k}^2}{C_\Lambda}\int q_{n-2}^2\,\mathrm{d}\mu$$
$$\ge \frac{\Delta_{n,k}^2}{C_\Lambda}\int_{[x_{n,k},x_{n,k+1}]} q_{n-2}^2\,\mathrm{d}\,\mu. \tag{4.10}$$

From (4.7), (4.8) and (4.10) we get

$$0 \ge \left(\frac{1}{C_\Lambda} - \delta^2\right)\Delta_{n,k}^2 \int_{x_{n,k}}^{x_{n,k+1}} q_{n-2}^2\,\mathrm{d}\mu.$$

But this is possible only if $\delta \ge \frac{1}{\sqrt{C_\Lambda}}$, hence

$$x_{n,k+1} - x_{n,k} \ge \frac{1}{\sqrt{C_\Lambda}}\Delta_{n,k}$$

follows. □

The proof of Corollary 4.2 is much the same as that of theorem 2 in [6] once Theorems 3.1 and 4.1 are available. We also skip the proof of Theorem 4.3, since the proof of [6, Theorem 3] can be adjusted to the local setting considered here; the necessary changes are very similar to what was done in the proof of Theorem 3.1.

5. Remark to Theorem 4.1

Theorem 4.1 is a local version of (4.1) (proved in [6, Theorem 1], where μ was assumed to be doubling on its support $[-1, 1]$), and the zero spacing $x_{n,k+1} - x_{n,k}$ in Theorem 4.1 follows precisely the same pattern as that in [6, Theorem 1] *once the zeros $x_{n,k}$, $x_{n,k+1}$ belong to the interval where the measure is doubling*. We have already mentioned that theorem 1 in [6], i.e. (4.1), also tells us that if μ is supported on $[-1, 1]$ and it is doubling there, then the distance from the smallest zero to the left endpoint of the support is about $1/n^2$. The proof of Theorem 4.1 gives also that if A is the smallest element of the support and μ is doubling on some interval $[A, A+\beta]$, then, for large n,

$$x_{n,1} - A \sim \Delta_n(x_{n,1}) \sim 1/n^2.$$

In other words, in this case the distance from the smallest zero to the left endpoint A is again about $1/n^2$, just as it was in the global case in (4.1). Now we show that this is not necessarily true for local endpoints. We exhibit an example when the support of the measure consists of two disjoint intervals $[-2, -1]$ and $[0, d]$, but for infinitely many n the smallest positive zero of the corresponding orthogonal polynomials is very close to 0, much closer than $1/n^2$.

Example 5.1. There is a $0 < d < 1$ such that if μ is the restriction of the Lebesgue-measure onto $[-2, -1] \cup [0, d]$, then for infinitely many n we have for the smallest positive zero x_{n,j_0} of $p_n(\mu, \cdot)$ the inequality

$$\frac{1}{2}e^{-n} \le x_{n,j_0} \le 2e^{-n}. \tag{5.1}$$

The proof can be easily modified to yield the following stronger statement: if $\delta_n = o(n^{-2})$ is any sequence, then there is a d such that for $\mu = m\big|_{[-2,-1]\cup[0,d]}$ and for some subsequence $\{n_k\}$ of the natural numbers we have

$$\lim_{k\to\infty} x_{n_k,j_0}/\delta_{n_k} = 1.$$

Proof. We need the following results in the construction.

Let ν_n be the measure that places mass $\frac{1}{n}$ to every zero of the n-th orthogonal polynomial $p_n(\mu, \cdot)$ (so-called normalized counting measure on the zeros).

Denote by ω_S the equilibrium measure of a compact set $S \subset \mathbf{R}$ of positive capacity (see [8] for the concept of equilibrium measure).

Lemma 5.2. *If μ is the restriction of the linear Lebesgue measure on some set S consisting of finitely many intervals, then $\nu_n \to \omega_S$ in the* weak* *topology of measures on the complex plane.*

This follows from [10, Theorem 3.1.4] and from any of the regularity criteria given in [10, Chapter 4.].

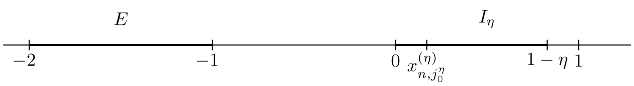
Lemma 5.3 *([11, Section 3]). Let $[a_1, b_1], \ldots, [a_l, b_l]$ be pairwise disjoint intervals and $\varepsilon \le b_l - a_l$. If ω_ε denotes the equilibrium measure for $[a_1, b_1] \cup \cdots \cup [a_{l-1}, b_{l-1}] \cup [a_l, b_l - \varepsilon]$, then*

1. *$\omega_\varepsilon([a_l, b_l - \varepsilon])$ is strictly monotone decreasing in ε,*
2. *$\omega_\varepsilon([a_i, b_i])$ strictly monotone increasing in ε for every $1 \le i \le l - 1$.*

Lemma 5.4. *Let m_ε denote the normalized Lebesgue measure on the previous interval system. Then the zeros of the orthogonal polynomials associated with m_ε are continuous functions of ε.*

This is obvious, since the Gram–Schmidt process shows that the coefficients of the n-th orthogonal polynomials are continuous functions of ε.

After these we turn to the construction. In Lemma 5.3 we set $l = 2$, $[a_1, b_1] = [-2, -1]$ and $[a_2, b_2] = [0, 1]$. Let $E = [-2, -1]$ and $I = [0, 1]$ be these two intervals and m_η the normalized Lebesgue measure on $E \cup I_\eta$, where $I_\eta = [0, 1 - \eta]$, $0 < \eta < 1/2$. Let $x_{n,k}^{(\eta)}$, $k = 1, 2, \ldots, n$ denote zeros (in increasing order) of the n-th orthogonal polynomial $p_n(m_\eta, \cdot)$ associated with m_η, and let $x_{n,j_0^\eta}^{(\eta)}$ be the smallest positive zero of $p_n(m_\eta, \cdot)$. For large n this exists, and by Theorem 4.1 we know that $x_{n,j_0^\eta+1}^{(\eta)} \ge c/n^2$ with some $c > 0$ independent of $\eta < 1/2$ and n.



If $\eta' > \eta$, then, by Lemmas 5.2 and 5.3, for large n, say for $n \ge N_{\eta,\eta'}$, there are at least two more zeros of $p_n(m_{\eta'}, \cdot)$ in E than $p_n(m_\eta, \cdot)$ has there (the proportion of the zeros lying in E is larger for $p_n(m_{\eta'}, \cdot)$ than for $p_n(m_\eta, \cdot)$). This means that $x_{n,j_0^\eta+1}^{(\eta')} \in E$, while $x_{n,j_0^\eta+1}^{(\eta)} \in I_\eta$ by definition. Hence, no matter how $n \ge N_{\eta,\eta'}$ is fixed, if ε is moving from η to η', the zero $x_{n,j_0^\eta+1}^{(\varepsilon)}$ moves from the interval $[c/n^2, \infty)$ to the interval $(-\infty, -1]$ in a continuous manner. So there is an $\eta < \varepsilon < \eta'$ such that $x_{n,j_0^\eta+1}^{(\varepsilon)} = e^{-n}$. Note that, in this case, necessarily $j_0^\varepsilon = j_0^\eta + 1$, since there cannot be a positive zero of $p_n(m_\varepsilon, \cdot)$ smaller than $e^{-n} = x_{n,j_0^\eta+1}^{(\varepsilon)}$, for then, by Theorem 4.1, $x_{n,j_0^\eta+1}^{(\varepsilon)}$ would have to be larger than c/n^2. Thus, $x_{n,j_0^\varepsilon}^{(\varepsilon)} = e^{-n}$.

Based on this, we can easily define sequences $0 = \varepsilon_0 < \varepsilon_1 < \cdots < 1/2$ and integers $n_0 < n_1 < \cdots$ such that

$$x_{n_k,j_0^{\varepsilon m}}^{(\varepsilon m)} = e^{-n_k}(1 + O(k^{-1})) \tag{5.2}$$

for all $m \geq k$, the O being uniform in m and k. Indeed, if ε_m, n_m are already given, then select an $\varepsilon'_m > \varepsilon_m$ so small that for $\varepsilon_m \leq \varepsilon \leq \varepsilon'_m$ we have

$$|x^{(\varepsilon)}_{n_k,j_0^{\varepsilon}} - x^{(\varepsilon_m)}_{n_k,j_0^{\varepsilon_m}}| < e^{-n_k}/m^2 \quad \text{for all } k \leq m, \tag{5.3}$$

and then let n_{m+1}, ε_{m+1} be the numbers with

$$x^{(\varepsilon_{m+1})}_{n_{m+1},j_0^{\varepsilon_{m+1}}} = e^{-n_{m+1}}$$

that the above procedure gives for $\eta = \varepsilon_m$ and $\eta' = \varepsilon'_m$ (actually, in that procedure n_{m+1} can be any sufficiently large number—just pick any one of them). This completes the definition of the sequences $\{\varepsilon_m\}$, $\{n_m\}$.

Note that (5.2) holds, since, by (5.3) with $\varepsilon = \varepsilon_{m+1}$ (and m replaced by the l in the summation below)

$$\begin{aligned}|x^{(\varepsilon_m)}_{n_k,j_0^{\varepsilon_m}} - e^{-n_k}| &= |x^{(\varepsilon_m)}_{n_k,j_0^{\varepsilon_m}} - x^{(\varepsilon_k)}_{n_k,j_0^{\varepsilon_k}}| \\ &\leq \sum_{l=k}^{m-1} |x^{(\varepsilon_{l+1})}_{n_k,j_0^{\varepsilon_{l+1}}} - x^{(\varepsilon_l)}_{n_k,j_0^{\varepsilon_l}}| \leq \sum_{l=k}^{m-1} e^{-n_k}/l^2 \leq e^{-n_k}/k.\end{aligned}$$

Now if ε is the limit of $\{\varepsilon_m\}$, then it follows that

$$x^{(\varepsilon)}_{n_k,j_0^{\varepsilon}} - e^{-n_k} = e^{-n_k} O(k^{-1}) \tag{5.4}$$

for all k, and this proves (5.1). $\square$

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