Acta Math. Hungar., **136 (3)** (2012), 222–231
DOI: 10.1007/s10474-011-0177-0
First published online November 29, 2011

THE SIZE OF IRREGULAR POINTS FOR A MEASURE

VILMOS TOTIK[1,2,*]

[1]Bolyai Institute, Analysis Research Group of the Hungarian Academy of Sciences, University of Szeged, Szeged, Aradi v. tere 1, 6720, Hungary

[2]Department of Mathematics and Statistics, University of South Florida, 4202 E. Fowler Ave, PHY 114, Tampa, FL 33620-5700, USA
e-mail: totik@mail.usf.edu

(Received June 26, 2011; accepted July 15, 2011)

Abstract. For a measure μ on the complex plane μ-regular points play an important role in various polynomial inequalities. In the present work it is shown that every point in the set $\{\mu' > 0\}$ (actually of a larger set where μ is strong) with the exception of a set of zero logarithmic capacity is a μ-regular point. Here "set of zero logarithmic capacity" cannot be replaced by "β-logarithmic Hausdorff measure 0" with $\beta = 1$ (it can be replaced by "β-logarithmic measure 0" with any $\beta > 1$). On the other hand, for arbitrary μ the set of μ-regular points can be quite small, but never empty.

Let ν be a Borel-measure with compact support $S(\nu)$ on the complex plane. The class **Reg** of measures plays an important role in the theory of orthogonal polynomials since it provides a weak global condition which appears in many results. Let us recall its definition from [5]: if $p_n(z) = \gamma_n z^n + \cdots$ denotes the n-th orthonormal polynomial with respect to ν with the normalization $\gamma_n > 0$, then it is always true that

$$\liminf_{n\to\infty} \gamma_n^{1/n} \geqq \frac{1}{\operatorname{cap}\big(S(\nu)\big)},$$

where $\operatorname{cap}\big(S(\nu)\big)$ denotes the logarithmic capacity of the set $S(\nu)$ (see [3], [4] or [6] for the concepts of logarithmic potential theory used in this work). Now ν is said to be in the **Reg** class if $\operatorname{cap}\big(S(\nu)\big) > 0$, and

$$\lim_{n\to\infty} \gamma_n^{1/n} = \frac{1}{\operatorname{cap}\big(S(\nu)\big)}.$$

* Supported by ERC grant No. 267055.
Key words and phrases: orthogonal polynomial, regularity of measure.
2010 Mathematics Subject Classification: 42C05, 42C90.

In some way, measures in this class behave "normally" from the point of view of orthogonal polynomials. $\nu \in \mathbf{Reg}$ is a fairly weak global condition, see [5] for different equivalent conditions and several regularity criteria.

One characterization of regularity is via the set of ν-regular/irregular points. We say that $z \in S(\nu)$ is a ν-regular point if

$$\limsup_{n\to\infty} \left(\frac{P_n(z)|}{\|P_n\|_{L^2(\nu)}} \right)^{1/n} \leqq 1 \tag{1}$$

for any sequence $\{P_n\}$ of polynomials of corresponding degree at most $n = 1, 2, \ldots$. Otherwise z is called a ν-irregular point. Let $R(\nu)$ resp. $I(\nu)$ be the set of ν-regular resp. ν-irregular points of $S(\nu)$. Thus $R(\nu) \cup I(\nu) = S(\nu)$.

Now [5, Theorem 3.2.1] (see in particular (iii) and (v) in that theorem) claims that $\nu \in \mathbf{Reg}$ if and only if $I(\nu)$ is of zero logarithmic capacity. In other words, $\nu \in \mathbf{Reg}$ means that with the exception of a set of zero capacity, on $S(\nu)$ polynomials cannot be exponentially large compared to their $L^2(\nu)$-norm.

In the paper [2] the authors considered general measures ν on the real line and showed that the set of ν-irregular points cannot be large on the set $\nu' > 0$ in the sense of β-logarithmic measure. Their setup was the following. Let $h : [0, \infty) \to [0, \infty]$ be an increasing continuous function with $h(0) = 0$. Given $E \subset \mathbf{C}$ the Hausdorff outer measure of E with respect to h is

$$m_h(E) = \inf \left\{ \sum_{j=1}^{\infty} h(r_j) \,\Big|\, E \subset \bigcup_j \Delta_{r_j} \right\},$$

where the infimum is taken for all covers of E by balls Δ_{r_j} of radius r_j. For

$$h_\beta(t) = \begin{cases} \left(\log \dfrac{1}{t}\right)^{-\beta}, & t \in (0,1), \\ \infty, & \text{otherwise} \end{cases}$$

this definition gives β-logarithmic Hausdorff measure. A theorem of Frostman (see [6, Theorem III.19]) says that if E has zero logarithmic capacity then $m_{h_\beta}(E) = 0$ for all $\beta > 1$, and conversely, by a theorem of Erdős and Gillis (see [6, Theorem III.20]) $m_{h_1}(E) < \infty$ implies $\operatorname{cap}(E) = 0$.

For a Borel-measure with compact support let $H(\nu)$ be the set of points $z \in S(\nu)$ such that

$$\liminf_{r\to 0} \nu\big(\Delta_r(z)\big)/r^m > 0 \tag{2}$$

for some m, where

$$\Delta_r(z) = \{\, w \mid |w - z| < r \,\}$$

is the disk of radius r with center at z. In other words,

$$H(\nu)=\left\{z\,\middle|\,\limsup_{r\to 0}\frac{\log 1/\nu(\Delta_r(z))}{\log 1/r}<\infty\right\}. \tag{3}$$

Note that e.g. for $S(\nu)\subset\mathbf{R}$ this set includes all points where the classical derivative of ν with respect to linear Lebesgue-measure is positive.

With these notations E. Levin and D. S. Lubinsky [2] proved that for any ν supported on the real line the set of ν-irregular points in $H(\nu)$ is of zero m_{h_β}-measure for all $\beta>1$. They also wrote ([2, Remark (a)]) "It seems unlikely that the set of irregular points can have zero capacity in $\{\nu'>0\}$". Our first result says that actually the set of ν-irregular points in $H(\nu)$ is always of zero capacity, even if the measure is not supported on the real line.

Theorem 1. *The set of irregular points in $H(\nu)$ is of zero capacity, i.e.* $\operatorname{cap}(I(\nu)\cap H(\nu))=0$.

As a corollary one can derive the main result of [2] without assuming the measure to lie on the real line.

Before giving the proof first we note that the sets $I(\nu)$, $R(\nu)$ are Borel-measurable. Indeed, if $\{Q_{k,n}\}_{k=1}^{\infty}$ is a countable dense set in the space of polynomials of degree at most n equipped with the supremum norm on $S(\nu)$ and if

$$A_{k,m,n}=\left\{z\in S(\nu)\;\middle|\;|Q_{k,n}(z)|>e^{n/m}\|Q_{k,n}\|_{L^2(\nu)}\right\},$$

then clearly

$$I(\nu)=\bigcup_{m=1}^{\infty}\limsup_{n\to\infty}\bigcup_{k=1}^{\infty}A_{k,m,n},$$

so it is a Borel-set, and $R(\nu)$ is just its complement relative to $S(\nu)$ (here

$$\limsup_{n\to\infty}B_n:=\bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty}B_n).$$

Thus, by the capacitability of Borel-sets, we can talk of the logarithmic capacity of $I(\nu)$ and $R(\nu)$ and their cousins that appear below.

Proof of Theorem 1. Suppose to the contrary, that the claim is not true. If H_m is the set of points z for which

$$\nu(\Delta_r(z))\geqq\frac{r^m}{m},\qquad\text{for all}\quad 0<r\leqq 1/m, \tag{4}$$

then $H(\nu) = \cup_{m=1}^{\infty} H_m$. Similarly, if I_θ is the set

(5)
$$I_\theta = \Big\{ z \in S(\nu) \mid \big|P_n(z)\big| > e^{\theta n} \|P_n\|_{L^2(\nu)} \text{ for infinitely many } P_n,\ n \to \infty \Big\},$$

then $I(\nu) = \cup_{k=1}^{\infty} I_{1/k}$. Hence, by our contrapositive assumption, for some k, m the set $I_{1/k} \cap H_m$ must be of positive capacity. Fix such a k and m, and select a compact subset K of $I_{1/k} \cap H_m$ of positive capacity. By Ancona's theorem [1] we may assume that K is regular with respect to the Dirichlet problem in the unbounded component of $\overline{\mathbf{C}} \setminus K$, i.e. the Green's function $g_{\overline{\mathbf{C}} \setminus K}(z, \infty)$ of this unbounded component with pole at infinity is continuous (and hence 0) on K. Thus, for every $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that for $\operatorname{dist}(z, K) \leqq \delta_\varepsilon$ we have $g_{\overline{\mathbf{C}} \setminus K}(z, \infty) < \varepsilon$.

Fix a $z_0 \in K$. Let P_n be an arbitrary polynomial of degree at most n, let $M = \|P_n\|_K \geqq \big|P_n(z_0)\big|$ be its supremum norm on K, and let $z_n \in K$ be a point where this supremum norm is attained. For $\operatorname{dist}(u, K) \leqq \delta_\varepsilon$ we have by the Bernstein–Walsh lemma [7, p. 77]

$$\big|P_n(u)\big| \leqq \|P_n\|_K \exp\big(n g_{\overline{\mathbf{C}} \setminus K}(z, \infty)\big) \leqq M e^{n\varepsilon},$$

and then we obtain from Cauchy's formula for the derivative of analytic functions that for $\operatorname{dist}(v, K) \leqq \delta_\varepsilon/2$ the estimate $\big|P_n'(v)\big| \leqq 2Me^{n\varepsilon}/\delta_\varepsilon$ holds. This implies that for $|z - z_n| \leqq r_n$ with $r_n = \delta_\varepsilon e^{-n\varepsilon}/4$ we have

$$\big|P_n(z)\big| \geqq \big|P_n(z_n)\big| - \big(2Me^{n\varepsilon}/\delta_\varepsilon\big)\,|z - z_n| \geqq M - (M/2) = M/2.$$

In other words, in the disk $\Delta_{r_n}(z_n)$ we have $|P_n| \geqq M/2$. Since $z_n \in H_m$, for $r \leqq 1/m$ we have $\nu\big(\Delta_r(z_0)\big) \geqq r^m/m$, and this gives with $r = r_n$ (for large n)

$$\int |P_n|^2 \, d\nu \geqq \int_{\Delta_{r_n}(z_n)} |P_n|^2 \, d\nu \geqq (M/2)^2 \nu\big(\Delta_{r_n}(z_n)\big) \geqq \frac{M^2 \delta_\varepsilon^m e^{-nm\varepsilon}}{m 4^{1+m}}.$$

In view of $M \geqq \big|P_n(z_0)\big|$ this shows that

$$\limsup_{n \to \infty} \Big(\big|P_n(z_0)\big| / \|P_n\|_{L^2(\nu)}\Big)^{1/n} \leqq e^{m\varepsilon/2},$$

which is impossible for $m\varepsilon/2 < 1/k$ by the choice of $I_{1/k}$ because $z_0 \in I_{1/k}$. Since $\varepsilon > 0$ is arbitrary, we can make $m\varepsilon/2$ smaller than $1/k$, and then the contradiction obtained proves the theorem. $\square$

Next, with the ideas used in Theorem 1, we derive a criterion for regularity. First we prove

Theorem 2. *For a measure ν with support of positive capacity the following are pairwise equivalent*:

(a) *ν is in the* **Reg** *class*,

(b) *the set of ν-irregular points is of zero capacity*,

(c) *the set of ν-regular points is of full capacity* (*i.e.* $\operatorname{cap}(R(\nu)) = \operatorname{cap}(S(\nu))$.

The equivalence of (a) and (b) was proven in [5, Theorem 3.2.1] (see in particular (iii) and (v) in that theorem), but it is interesting to know that they are also equivalent to (c) which is seemingly much weaker than (b). Thus, the complementary sets $I(\nu)$, $R(\nu)$ behave in a rather unexpected way: if $I(\nu)$ is of positive capacity, then necessarily $R(\nu)$ has smaller capacity than $S(\nu)$ has (in general, complementary sets may both have full capacities).

This theorem combined with Theorem 1 gives (see (3) for the definition of $H(\nu)$)

Corollary 3. *If* $\operatorname{cap}(H(\nu)) = \operatorname{cap}(S(\nu))$, *then ν is in the* **Reg** *class.*

This was Criterion Λ in [5, Section 4.2].

Proof of Theorem 2. As we have already mentioned, the equivalence of (a) and (b) was proven in [5, Theorem 3.2.1], and (b) clearly implies (c) since $R(\nu) = S(\nu) \setminus I(\nu)$. Thus, it is left to show that (c) implies (b).

Suppose to the contrary that $\operatorname{cap}(R(\nu)) = \operatorname{cap}(S(\nu))$ and at the same time $\operatorname{cap}(I(\nu)) > 0$. Then for some $\theta > 0$ the set I_θ in (5) is of positive capacity. Fix such a $\theta > 0$.

If

$$R_N = \Big\{ z \in S(\nu) \;\big|\; |P_n(z)| \leqq e^{\theta n/3} \|P_n\|_{L^2(\nu)} \text{ for } n \geqq N \text{ and all } P_n,\ \deg(P_n) \leqq n \Big\}, \tag{6}$$

then $\cup_{N=1}^{\infty} R_N$ contains the set $R(\nu)$, hence it is of full capacity. Therefore, as $N \to \infty$, we have $\operatorname{cap}(R_N) \to \operatorname{cap}(S(\nu))$, and we can select increasing compact sets $K_N \subset R_N$ such that $\operatorname{cap}(K_N) \to \operatorname{cap}(S(\nu))$. We claim that then the equilibrium measures μ_{K_N} converge in the weak*-topology to the equilibrium measure $\mu_{S(\nu)}$ of the support $S(\nu)$. In fact, since from any subsequence of $\{\mu_{K_N}\}_N$ we can select a weak*-convergent subsequence (Helly's theorem), it is enough to show that if σ is a weak*-limit of some subsequence of $\{\mu_{K_{N_j}}\}_j$, then $\sigma = \mu_{S(\nu)}$. But this is clear: if

$$\mathcal{I}(\rho) = \int\!\!\int \log \frac{1}{|u-t|} d\rho(u) d\rho(t)$$

is the logarithmic energy of a measure ρ, then, by the principle of descent (see [4, Theorem I.6.8, (6.16)]), we have

$$\mathcal{I}(\sigma) \leqq \liminf_{j\to\infty} \mathcal{I}(\mu_{K_{N_j}}).$$

However,

$$\mathcal{I}(\mu_{K_{N_j}}) = \log \frac{1}{\operatorname{cap}(K_{N_j})},$$

and the right-hand side tends to $\log 1/\operatorname{cap}\big(S(\nu)\big) = \mathcal{I}\big(\mu_{S(\mu)}\big)$ as $j \to \infty$. Thus, we get $\mathcal{I}(\sigma) \leqq \mathcal{I}\big(\mu_{S(\nu)}\big)$, and at the same time σ is a unit Borel-measure supported on $S(\nu)$. By the minimality and unicity of the equilibrium measure this implies $\sigma = \mu_{S(\nu)}$, as was claimed.

Thus, $\mu_{K_N} \to \mu_{S(\nu)}$ in the weak*-topology, and then it follows from the lower envelope theorem [4, Theorem I.6.9] that for the logarithmic potentials

$$U^{\mu_{K_N}}(z) = \int \log \frac{1}{|z-t|}\, d\mu_{K_N}(t)$$

we have

$$\liminf_{n\to\infty} U^{\mu_{K_N}}(z) = U^{\mu_{S(\nu)}}(z) \tag{7}$$

for quasi-every $z \in \mathbf{C}$, i.e. for all $z \in \mathbf{C}$ with the exception of a set of capacity zero.

Let $g_{\overline{\mathbf{C}}\setminus K_N}(z,\infty)$ be the Green's function of the unbounded component of $\mathbf{C} \setminus K_N$ with pole at ∞. Now $\operatorname{cap}(K_N) \to \operatorname{cap}\big(S(\nu)\big)$ and (7) give, in view of the formula (see e.g. [4, (I.4.8)])

$$g_{\overline{\mathbf{C}}\setminus K_N}(z,\infty) = \log \frac{1}{\operatorname{cap}(K_N)} - U^{\mu_{K_N}}(z),$$

that

$$\limsup_{N\to\infty} g_{\overline{\mathbf{C}}\setminus K_N}(z,\infty) = g_{\overline{\mathbf{C}}\setminus S(\nu)}(z,\infty)$$

for quasi-every $z \in \mathbf{C}$, and note also that the right-hand side is 0 for quasi-every $z \in S(\nu)$ by Frostman's theorem (see e.g. [3, Theorem 3.3.4 and Section 4.4]). Hence, for quasi-every $z_0 \in S(\nu)$ we have

$$\lim_{N\to\infty} g_{\overline{\mathbf{C}}\setminus K_N}(z_0,\infty) = 0. \tag{8}$$

In particular, there is a point $z_0 \in I_\theta$ where (8) holds (note that I_θ has positive capacity by our assumption). Therefore, for large N, say for $N \geqq N_\theta$, we have $g_{\overline{\mathbf{C}}\setminus K_N}(z_0,\infty) \leqq \theta/3$.

Now let P_n be a polynomial of degree at most n and let $n > N > N_\theta$. Then, by the definition of the set R_N in (6) and by $K_N \subseteq R_N$, we have $\|P_n\|_{K_N} \leqq e^{n\theta/3}\|P_n\|_{L^2(\nu)}$, and hence, by the Bernstein–Walsh lemma [7, p. 77],

$$\left|P_n(z_0)\right| \leqq \exp\left(ng_{\overline{\mathbf{C}}\setminus K_N}(z_0,\infty)\right)\|P_n\|_{K_N} \leqq \exp(n\theta/3)e^{n\theta/3}\|P_n\|_{L^2(\nu)}$$

$$= e^{2n\theta/3}\|P_n\|_{L^2(\nu)}.$$

This shows that z_0 cannot lie in the set I_θ (see (5)). But z_0 was chosen to be an element of I_θ, and this contradiction proves the theorem. □

Since zero logarithmic capacity implies zero β-logarithmic measure ([6, Theorem III.19]), Theorem 1 gives that $I(\nu) \cap H(\nu)$ is always of zero h_β-measure for $\beta > 1$ (when $S(\nu) \subset \mathbf{R}$ this is Theorem 1.1 in [2]). However, it need not be of zero h_1-measure as the following example shows.

EXAMPLE 4. There exists a compact set $K \subset \mathbf{R}$ of positive h_1-measure and a Borel-measure ν on K with support equal to K such that

$$\liminf_{d\to 0+0} \nu\big([x-d, x+d]\big)/d > 0 \tag{9}$$

for every $x \in K$ and all but countably many points of K are ν-irregular points.

Note that in view of Theorem 1 such a K is necessarily of zero logarithmic capacity.

PROOF. Starting from $K_0 = [0,1]$ do the Cantor construction (in each step removing a middle portion of the remaining intervals) in such a way that at level k we have a set K_k consisting of 2^k intervals each of length $2^{-2^{k+1}}$, and set $K = \cap_{k=1}^\infty K_k$. Note that the complementary intervals at level k, i.e. the intervals of $[0,1] \setminus K_k$ all have length $\geqq (7/8)2^{-2^k}$.

Let ν_k be the measure that puts mass 2^{-2^k} to each left endpoint of the intervals at level k and let $\nu = \sum_{k\geqq 1}\nu_k$. With this choice condition (9) is satisfied at every point x of K: if $2^{-2^{k+1}} \leqq d < 2^{-2^k}$, then

$$\nu\big([x-d, x+d]\big) \geqq \nu_k\big([x-2^{2^{k+1}}, x+2^{-2^{k+1}}]\big) = 2^{-2^k} \geqq d.$$

Let P_{2^n} be the monic polynomial of degree 2^n with zeros at the left endpoints of the intervals at the n-th level. For an $x \in K$ let $J_k(x) \subset K_k$ be the interval at the k-th level that contains x. Then there is one zero closer (= not farther) to x than $2^{-2^{n+1}}$ in $J_n(x)$, another zero closer than 2^{-2^n} in $J_{n-1}(x)$, and in general for each $k = 0, 1, \dots, n-2$ there are 2^k zeros closer than $2^{-2^{n-k}}$ in $J_{n-k-1}(x)$ not accounted for before. Finally, there are 2^{n-1} zeros closer than 1 in $J_0=[0,1]$. Therefore,

$$|P_{2^n}(x)| \leqq 2^{-2^{n+1}} \prod_{k=0}^{n-2} \left(2^{-2^{n-k}}\right)^{2^k} = 2^{-(n+1)2^n},$$

and hence, since P_{2^n} is zero on the support of $\nu_1,\dots,\nu_n$, the $L^2(\nu)$-norm of P_{2^n} is at most

(10)

$$\|P_{2^n}\|_{L^2(\nu)} \leqq \|P_n\|_{L^\infty(K)} \biggl(\sum_{i=n+1}^{\infty} \nu_k(\mathbf{C})\biggr)^{1/2} \leqq 2^{-(n+1)2^n}\left(2^{n+2}2^{-2^n}\right)^{1/2}.$$

On the other hand, if x belongs to the right interval of $J_n(x)\cap K_{n+1}$ (note that this set consists of two intervals), then its distance from either of the 2^k zeros in J_{n-k-1} considered before is at least

$$2^{-2^{n-k}} - 2\cdot 2^{-2^{n+1-k}} = 2^{-2^{n-k}}\left(1-2^{-2^{n-k}+1}\right) \geqq 2^{-2^{n-k}}(7/8)$$

for $k=0,\dots,n-2$, this distance is $\geqq 7/8$ for $k=n-1$ and it is $\geqq (7/8)\cdot 2^{-2^{n+1}}$ for the only zero in $J_n(x)$. These give that

$$|P_{2^n}(x)| \geqq (7/8)^{2^n} 2^{-2^{n+1}} \prod_{k=0}^{n-2} \left(2^{-2^{n-k}}\right)^{2^k} = (7/8)^{2^n} 2^{-(n+1)2^n},$$

and so, in view of (10),

$$|P_{2^n}(x)|/\|P_{2^n}\|_{L^2(\nu)} \geqq (14/8)^{2^n}/2^{(n+2)/2}.$$

This shows that if x lies in infinitely many “right” intervals, then it is ν-irregular. But the only points in K that lie in only finitely many “right” intervals are the left endpoints of intervals in different levels, so unless x is a left endpoint of an interval at some level, then x is ν-irregular.

Now we show that K, and hence also the set of ν-irregular points, has positive h_1-measure. In fact, let $K\subset\cup_{j=1}^{l} I_j$ be a cover of K by open intervals I_j, $j=1,2,\dots$. By compactness we may assume that their number l is finite. Now K is the intersection of the compact sets K_m constructed on the individual levels $m=1,2,\dots$, hence the open cover $\cup_{j=1}^{l} I_j$ contains all the intervals on some level, say $K_M\subset\cup_{i=j}^{l} I_j$. Let I_j contain k_j subintervals $J_{1,j},\dots,J_{k_j,j}$ of K_M (each of length $2^{-2^{M+1}}$). Then for $k_j>1$, say for $2^s<k_j\leqq 2^{s+1}$, $s\geqq 0$, the interval I_j must contain at least one complementary interval at level $M-s$ (for otherwise all the subintervals $J_{1,j},\dots,J_{k_j,j}$

would belong to the same interval at level $M-s$, which is not possible, since an interval at level $M-s$ has at most 2^s subintervals at level M). But all subintervals of $[0,1]\setminus K_{M-s}$ are of length $\geqq (7/8)2^{-2^{M-s}}$, and so

$$\frac{1}{\log 1/|I_j|}\geqq\frac{1}{\log 1/\big((7/8)2^{-2^{M-s}}\big)}\geqq\frac{2^{s-M}}{2\log 2}\geqq k_j2^{-M}/4.$$

If $k_j=1$ then we just use

$$\frac{1}{\log 1/|I_j|}\geqq\frac{1}{\log 1/|J_{1,j}|}=\frac{2^{-M-1}}{\log 2}\geqq k_j2^{-M}/4.$$

Hence

$$\sum_{j=1}^{l}\frac{1}{\log 1/|I_j|}\geqq\sum_{j=1}^{l}k_j2^{-M}/4=2^M2^{-M}/4=\frac{1}{4},$$

which proves the claim. □

Example 4 is rather extreme: it exhibits a measure ν such that its support $S(\nu)$ is relatively large (has positive h_1-measure), but the set $R(\nu)$ of regular points is small (countable). This raises the question if $R(\nu)$ can even be smaller, for example can it be empty? Our last result claims that the answer is no:

THEOREM 5. *For any measure ν, ν-almost all points are regular, i.e.* $\nu\big(\mathbf{C}\setminus R(\nu)\big)=0$.

PROOF. Define the Christoffel functions associated with ν as

$$\lambda_n(z)=\inf_{P_n(z)=1}\int|P_n|^2\,d\mu.$$

With this the definition of ν-regularity in (1) clearly takes the form

$$\limsup_{n\to\infty}\frac{1}{\lambda_n(z)^{1/n}}=1.$$

It is well known that if p_k are the orthonormal polynomials associated with ν then

$$\frac{1}{\lambda_n(z)}=\sum_{k=0}^{n}\big|p_k(z)\big|^2.$$

Now for a $q>1$ the set

$$G_{n,q}=\big\{\,z\in S(\nu)\mid 1/\lambda_n(z)>q^n\big\}$$

is of measure at most $(n+1)/q^n$ since

$$q^n \nu(G_{n,q}) \leqq \int_{G_{n,q}} \frac{1}{\lambda_n(z)}\, d\nu(z) \leqq \int \left(\sum_{k=0}^{n} |p_k|^2 \right) d\nu = n+1.$$

Therefore, the set

$$\limsup_{n\to\infty} G_{n,q} := \bigcap_{N\to\infty} \bigcup_{n=N}^{\infty} G_{n,q}$$

is of zero ν-measure. Now this proves the claim, since the set of ν-irregular points is

$$\bigcup_{q>1} \limsup_{n\to\infty} G_{n,q} = \lim_{q\searrow 1} \limsup_{n\to\infty} G_{n,q},$$

and hence it has zero ν-measure. □

References

[1] A. Ancona, Démonstration d'une conjecture sur la capacité et l'effilement, *C. R. Acad. Sci. Paris*, **297** (1983), 393–395.

[2] E. Levin and D. S. Lubinsky, The size of the set of μ-irregular points of a measure μ, *Acta Math. Hungar.*, **133** (2011), 242–250.

[3] T. Ransford, *Potential Theory in the Complex plane*, Cambridge University Press (Cambridge, 1995).

[4] E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Grundlehren der mathematischen Wissenschaften, **316**, Springer-Verlag (New York/Berlin, 1997).

[5] H. Stahl and V. Totik, *General Orthogonal Polynomials*, Encyclopedia of Mathematics and its Applications, **43**, Cambridge University Press (Cambridge, 1992).

[6] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen (Tokyo, 1959).

[7] J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, 3rd ed., Amer. Math. Soc. Colloquium Publications, **XX**, Amer. Math. Soc. (Providence, 1960).