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CONSTRUCTIVE APPROXIMATION

Bernstein's Inequality for Algebraic Polynomials on Circular Arcs

Béla Nagy · Vilmos Totik

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Abstract In this paper, we prove a sharp Bernstein-type inequality for algebraic polynomials on circular arcs.

Keywords Polynomial inequalities · Circular arc · Bernstein inequality · Normal derivatives of Green's functions

Mathematics Subject Classification 42A05

1 Results

Inequalities for algebraic or trigonometric polynomials play a fundamental role in various problems ranging from number theory to differential equations. One of the most classical ones is Riesz' inequality [11, Satz I']: if P_n is a polynomial of degree at most n, C_1 denotes the unit circle, and $\|\cdot\|_K$ denotes supremum norm on a set K, then

$$\left|P_n'(z)\right| \le n\|P_n\|_{C_1}, \quad z \in C_1. \tag{1}$$

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B. Nagy · V. Totik (✉)
Bolyai Institute, Analysis and Stochastics Research Group of the Hungarian Academy of Sciences, University of Szeged, Aradi v. tere 1, Szeged 6720, Hungary
e-mail: totik@math.usf.edu

B. Nagy
e-mail: nbela@math.u-szeged.hu

V. Totik
Department of Mathematics and Statistics, University of South Florida, 4202 E. Fowler Ave, PHY 114, Tampa, FL 33620-5700, USA

The corresponding inequality for an interval was proven by Bernstein [1]:

$$\left|P_n'(x)\right| \le \frac{n}{\sqrt{1-x^2}}\|P_n\|_{[-1,1]}, \quad -1 < x < 1, \tag{2}$$

and for the uniform norm of the derivative, we have the so-called Markov inequality [4]

$$\left\|P_n'\right\|_{[-1,1]} \le n^2\|P_n\|_{[-1,1]}.$$

In this paper, we prove the following analog for circular arcs. Let $0 < \omega \le \pi$, and let

$$K_\omega = \left\{e^{i\theta} \mid \theta \in [-\omega, \omega]\right\} \tag{3}$$

be the circular arc on the unit circle of central angle 2ω and with midpoint at 1.

Theorem 1 *If P_n is a polynomial of degree at most n, then*

$$\left|P_n'\left(e^{i\theta}\right)\right| \le \frac{n}{2}\left(1 + \frac{\sqrt{2}\cos(\theta/2)}{\sqrt{\cos\theta - \cos\omega}}\right)\|P_n\|_{K_\omega}, \quad \theta \in (-\omega, \omega). \tag{4}$$

This is sharp:

Theorem 2 *For every $\theta \in (-\omega, \omega)$, there are nonzero polynomials P_n of degree $n = 1, 2, \dots$ such that*

$$\left|P_n'\left(e^{i\theta}\right)\right| \ge \left(1 - o(1)\right)\frac{n}{2}\left(1 + \frac{\sqrt{2}\cos(\theta/2)}{\sqrt{\cos\theta - \cos\omega}}\right)\|P_n\|_{K_\omega}. \tag{5}$$

Of course, the $\omega = \pi$ case is just the original Riesz inequality (1). Also, if we write the consequence for an arc on the circle $RC_1 - R = \{z \mid |z + R| = R\}$:

$$\left|P_n'\left(R\left(e^{i\theta} - 1\right)\right)\right| \le \frac{n}{2R}\left(1 + \frac{\sqrt{2}\cos(\theta/2)}{\sqrt{\cos\theta - \cos\omega}}\right)\|P_n\|_{RK_\omega - R}, \quad \theta \in (-\omega, \omega); \tag{6}$$

apply it with $\omega = 1/R$ and $\theta = x/R$, $x \in [-1, 1]$; and let $R \to \infty$; then we obtain (2) (with a change of variable), since

$$\left|P_n'\left(R\left(e^{i\theta} - 1\right)\right)\right| \to \left|P_n'(ix)\right|, \qquad \|P_n\|_{RK_\omega - R} \to \|P_n\|_{[-i,i]}$$

and

$$2R^2\left(\cos(x/R) - \cos(1/R)\right) \to 1 - x^2.$$

The inequality in Theorem 1 can be written in alternative forms using the equilibrium measure ν_{K_ω} of K_ω and Green's function $g(z) = g_{\overline{\mathbf{C}}\setminus K_\omega}(z, \infty)$ with pole at infinity of the complement of K_ω (see [3, 9] or [10] for these concepts). In fact, if $d\nu_{K_\omega}(z)/ds$ is the density (Radon-Nikodym derivative) of the equilibrium measure ν_{K_ω} with respect to arc length on C_1, then (4) is the same as

$$\left|P_n'(\zeta)\right| \le \frac{n}{2}\left(1 + 2\pi\frac{d\nu_{K_\omega}(\zeta)}{ds}\right)\|P_n\|_{K_\omega}, \quad \zeta \in K_\omega, \tag{7}$$

and if $g'_{\pm}(\zeta)$ denote the normal derivatives of Green's function in the direction of the two normals to K_ω, then another equivalent form is

$$\left|P_n'(\zeta)\right| \le n \max\left\{g'_-(\zeta), g'_+(\zeta)\right\}\|P_n\|_{K_\omega}, \quad \zeta \in K_\omega. \tag{8}$$

For the reformulations of Theorem 1 given in (7) and (8), see the proof below, in particular Proposition 3 and Corollaries 4 and 5. We believe that the form given in (8) is (with a possible $(1+o(1))$ factor) the correct form of the Bernstein inequality on smooth Jordan curves. Our proof for Theorem 2 shows that if K_ω is replaced by any C^2 Jordan curve or Jordan arc, or even by a family of these, then an estimate better than (8) cannot be given; i.e., the asymptotic Bernstein factor is at least as large as $n \max\{g'_-(\zeta), g'_+(\zeta)\}$.

We also mention the Markov-type inequality: if P_n is a polynomial of degree at most n, then

$$\left\|P_n'\right\|_{K_\omega} \le \left(1+o(1)\right)\frac{n^2}{2}\cot\left(\frac{\omega}{2}\right)\|P_n\|_{K_\omega}. \tag{9}$$

This is sharp again: for some nonzero polynomials P_n, we have

$$\left\|P_n'\right\|_{K_\omega} \ge \left(1-o(1)\right)\frac{n^2}{2}\cot\left(\frac{\omega}{2}\right)\|P_n\|_{K_\omega}. \tag{10}$$

These are immediate consequences of [2], p. 243, see Sect. 2.

For even n, Theorem 1 is an easy consequence of the classical Videnskii inequality on trigonometric polynomials, and for odd n, it also follows from a related inequality of Videnskii for a trigonometric expression in which the frequencies of cosine and sine are an integer plus one half. This derivation will be done in the next section. The proof of Theorem 2 in Sect. 5 will be based on a theorem in [5] for Bernstein-type inequalities on a Jordan curve (homeomorphic image of the unit circle). In the process, we shall need to calculate the normal derivatives of Green's function of the complement of $\overline{\mathbf{C}} \setminus K_\omega$, which will be done in Sect. 3. Once this is done, we give in Sect. 4 an alternative proof for Theorem 1 using a result of Borwein and Erdélyi.

The authors are thankful to Paul Nevai, who called their attention to the fact that inequality (1) was first published by M. Riesz and was later rediscovered by S.N. Bernstein.

2 Theorem 1 and Videnskii's Inequalities

Let

$$V(\theta) = V(\omega;\theta) = \frac{\sqrt{2}\cos(\theta/2)}{\sqrt{\cos\theta - \cos\omega}} = \frac{\cos(\theta/2)}{\sqrt{\sin^2\left(\frac{\omega}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}}. \tag{11}$$

The classical Bernstein inequality for trigonometric polynomials was extended by Videnskii (see, e.g., [2], Chap. 5, E.19, p. 242 or [12]): let $Q_m(t)$ be a trigonometric polynomial with real coefficients of degree at most m, and let $\omega \in (0, \pi)$. Then for any $\theta \in (-\omega, \omega)$, we have

$$\left|Q_m'(\theta)\right| \le mV(\omega;\theta)\|Q_m\|_{[-\omega,\omega]}. \tag{12}$$

There is an extension to half-integer trigonometric polynomials [13]: let

$$Q_{m+1/2}(t) = \sum_{j=0}^{m} a_j \cos\left(\left(j+\frac{1}{2}\right)t\right) + b_j \sin\left(\left(j+\frac{1}{2}\right)t\right), \quad a_j, b_j \in \mathbf{R}.$$

Then for any $\theta \in (-\omega, \omega)$, we have

$$\left|Q_{m+1/2}'(\theta)\right| \le \left(m+\frac{1}{2}\right)V(\omega;\theta)\|Q_{m+1/2}\|_{[-\omega,\omega]}. \tag{13}$$

A standard trick leads to the same inequalities with complex coefficients: for example, if $\tilde{Q}_m$ is a trigonometric polynomial with complex coefficients and $\theta \in (-\omega, \omega)$, then let $|\tau| = 1$ be such that $\tau \tilde{Q}_m'(\theta) = |\tilde{Q}_m'(\theta)|$. Now if we apply (12) to the real trigonometric polynomial $Q_m(t) = \Re(\tau \tilde{Q}_m(t))$, then we get (12) for $\tilde{Q}_m$.

Proof of Theorem 1 Let P_n be an algebraic polynomial of degree at most n, and set

$$Q_{n/2}(t) := e^{-i\frac{n}{2}t} P_n\left(e^{it}\right). \tag{14}$$

For this,

$$\|Q_{n/2}\|_{[-\omega,\omega]} = \|P_n\|_K,$$

and

$$Q_{n/2}'(\theta) = e^{-i\frac{n}{2}\theta}(-in/2)P_n\left(e^{i\theta}\right) + e^{-i\frac{n}{2}\theta}P_n'\left(e^{i\theta}\right)e^{i\theta}i. \tag{15}$$

So

$$\left|P_n'\left(e^{i\theta}\right)\right| \le \left|Q_{n/2}'(\theta)\right| + \frac{n}{2}\left|P_n\left(e^{i\theta}\right)\right|, \quad \theta \in (-\omega, \omega),$$

and (4) is an immediate consequence of (12) (in the case when n is even) and (13) (when n is odd) with $m = n/2$, because the second term on the right is $\le \|P_n\|_{K_\omega}$.

Since (15) gives, for $t \in (-\omega, \omega)$,

$$\left|\left|Q_{n/2}'(t)\right| - \left|P_n'(e^{it})e^{-i\frac{n}{2}t}\right|\right| \le \|P_n\|_{K_\omega}\frac{n}{2}, \tag{16}$$

(9) follows from the following inequality of Videnskii (see, e.g., [2], p. 243): if $Q_m(t)$ is a trigonometric polynomial of degree m, then for $2m \ge (3\tan^2(\frac{\omega}{2})+1)^{1/2}$,

$$\left\|Q_m'\right\|_{[-\omega,\omega]} \le 2m^2 \cot\frac{\omega}{2}\|Q_m\|_{[-\omega,\omega]}. \tag{17}$$

Indeed, we may assume that n is even (if it is odd, consider P_n as a polynomial of degree at most $n+1$), and then we can apply (17) to the $Q_{n/2}$ in (14) (note that now the term on the right of (16) is $o(n^2)$).

Since (17) is sharp (see [2], p. 243), (10) also follows.

3 The Normal Derivatives of Green's Function

Let $K = K_\omega$. Denote Green's function of $\overline{\mathbf{C}} \setminus K$ with pole at infinity by $g(\zeta)$, $g(\zeta) = g_{\overline{\mathbf{C}}\setminus K}(\zeta, \infty)$. There are two normals to K; the "outer" normal is pointing into the exterior of the unit circle, and the "inner" normal is pointing towards its interior. We need to compute the normal derivatives g'_+, g'_- of g with respect to both normals.

Denote the equilibrium measure of K by ν. It is known that ν is absolutely continuous with respect to arc length, see [10], p. 209, Theorem 2.1. We write $d\nu/ds$ for the density of ν with respect to arc measure. Recall also in the next proposition the definition of V from (11).

Proposition 3 *Let $\zeta_0 = e^{i\theta_0}$ be an inner point of K and g'_+, g'_- the normal derivatives of Green's function in the direction of the outer and inner normals, respectively. Then*

$$g'_+(\zeta_0) = \frac{1}{2}\left(1 + \frac{\sqrt{2}\cos\theta_0/2}{\sqrt{\cos\theta_0 - \cos\omega}}\right) \tag{18}$$

and

$$g'_-(\zeta_0) = \frac{1}{2}\left(-1 + \frac{\sqrt{2}\cos\theta_0/2}{\sqrt{\cos\theta_0 - \cos\omega}}\right). \tag{19}$$

Corollary 4 *We have*

$$g'_+(\zeta_0) + g'_-(\zeta_0) = 2\pi\frac{d\nu(\zeta_0)}{ds}, \tag{20}$$

$$g'_+(\zeta_0) + g'_-(\zeta_0) = V(\theta_0), \tag{21}$$

and

$$g'_+(\zeta_0) - g'_-(\zeta_0) = 1. \tag{22}$$

Proof Fix $\zeta_0 = e^{i\theta_0} \in K$, where $\theta_0 \in (-\omega, \omega)$.

Let $\tilde{g}$ be the analytic conjugate of g with the normalization $\tilde{g}(e^{-i\omega}) = \lim_{\zeta\to e^{-i\omega}} \tilde{g}(\zeta) = 0$, and let $G(z) = g(z) + i\tilde{g}(z)$ be the complex Green's function. Then, using the properties of Green's functions, it is easy to see that $\Psi(z) = \exp(G(z))$ maps $\mathbf{C} \setminus K$ conformally onto the exterior of the unit circle.

Set

$$R(z) = -\big(z - e^{i\omega}\big)\big(z - e^{-i\omega}\big)$$

and $S(z) = (z+1)$. We cut the plane along the arc K and take the branch of the square root $\sqrt{R(z)}$ which takes the value i at 0 (note that $R(0) = -1$). Then $\sqrt{R(0)} = i = iS(0)$.

With these notations, it was proved in [7], formula (5.12), that

$$G(z) = \frac{1}{2}\int_{e^{-i\omega}}^{z} \frac{1}{\zeta}\left(1 - \frac{iS(\zeta)}{\sqrt{R(\zeta)}}\right) d\zeta, \tag{23}$$

where the integration is along a path from $e^{-i\omega}$ to z that does not intersect K. This can also be seen directly from the facts that

- $\sqrt{R(z)}/i$ is positive on $(-\infty, 1)$ and negative on $(1, \infty)$, the integrand in (23) behaves as $2/\zeta$ for large ζ, and therefore $\Re G(z) \sim \log|z|$, and
- $iS(\zeta)/\sqrt{R(\zeta)}$ is real (see the calculation below) on both sides of the arc K, hence G is purely imaginary on both sides of the cut.

Note also that the integration in (23) is independent of the path of integration because

$$\oint_K \frac{1}{\zeta}\left(1 - \frac{iS(\zeta)}{\sqrt{R(\zeta)}}\right) d\zeta = 0,$$

see (26) below.

Now if $\zeta_0 = e^{i\theta_0}$ is an inner point of K, then

$$g'_+(\zeta_0) = \Re \frac{\partial G(\zeta_0)}{\partial n_+} = \Re\, \zeta_0 G'(\zeta_0+) = \Re \frac{1}{2}\left(1 - \frac{iS(\zeta_0)}{\sqrt{R}(\zeta_0+)}\right), \tag{24}$$

where ζ_0+ indicates that the appropriate value is taken on the outer side of K (which is the side that lies outside the unit disk), while

$$g'_-(\zeta_0) = \Re \frac{\partial G(\zeta_0)}{\partial n_-} = -\Re\, \zeta_0 G'(\zeta_0-) = \Re \frac{1}{2}\left(\frac{iS(\zeta_0)}{\sqrt{R}(\zeta_0-)} - 1\right), \tag{25}$$

and here ζ_0- indicates that the appropriate value is taken on the inner side of K. Here, for $\zeta_0 = e^{i\theta_0}$ lying in the inner side of K, we have

$$\begin{aligned} R\big(e^{i\theta_0}\big) &= -\big(e^{i\theta_0} - e^{i\omega}\big)\big(e^{i\theta_0} - e^{-i\omega}\big) = -e^{i\theta_0}\big(e^{i\theta_0} - 2\cos\omega + e^{-i\theta_0}\big) \\ &= -2e^{i\theta_0}(\cos\theta_0 - \cos\omega), \end{aligned}$$

and hence

$$\frac{iS(\zeta_0)}{\sqrt{R}(\zeta_0-)} = \frac{1 + e^{i\theta_0}}{e^{i\theta_0/2}\sqrt{2(\cos\theta_0 - \cos\omega)}} = \frac{\sqrt{2}\cos\theta_0/2}{\sqrt{\cos\theta_0 - \cos\omega}}$$

is real and positive. In a similar vein, for $\zeta_0 = e^{i\theta_0}$ lying in the outer side of K, we have

$$\frac{iS(\zeta_0)}{\sqrt{R}(\zeta_0+)} = -\frac{iS(\zeta_0)}{\sqrt{R}(\zeta_0-)} = -\frac{\sqrt{2}\cos\theta_0/2}{\sqrt{\cos\theta_0 - \cos\omega}}. \tag{26}$$

Plugging these into (24)–(25) we get (18) and (19).

From these formulae (21) and (22) immediately follow. Formula (20) is known, see, e.g., [10], Theorem 2.3, p. 211. □

Corollary 5 *Let Ψ be a conformal map from $\mathbf{C} \setminus K$ onto the exterior of the unit disk. Then for $\zeta_0 = e^{i\theta_0}$ lying in the interior of K, we have*

$$g'_+(\zeta_0) = \frac{V(\theta_0) + 1}{2} = \big|\Psi'(\zeta_0+)\big|. \tag{27}$$

The derivative on the right-hand side is understood from the outside of Δ (by the Kellogg-Warschawski theorem $\Psi'(\zeta_0+)$ exists on the boundary in the sense that $\Psi'(\zeta)$ has a limit as $\zeta \to \zeta_0$ from the outside, see [8], Theorems 3.5, 3.6; furthermore this limit is nonzero).

Note also that different Ψ's differ by a multiplicative constant of modulus 1, so it does not matter which one we take.

Proof The first equality was verified in (18).

In the proof of Proposition 3, we have also seen that

$$g_+'(\zeta_0) = \Re\, \zeta_0 G'(\zeta_0+) = \Re \frac{\Psi'(\zeta_0+)\zeta_0}{\Psi(\zeta_0+)}.$$

Now at ζ_0, the direction of the outer normal to K is ζ_0, so (using the conformality of Ψ) $\Psi'(\zeta_0+)\zeta_0/|\Psi'(\zeta_0+)|$ is the direction of the outer normal to C_1 at the point $z = \Psi(\zeta_0+)$, but this direction is again the same as $z = \Psi(\zeta_0+)$. As a consequence, $\Psi'(\zeta_0+)\zeta_0/\Psi(\zeta_0+)$ is positive, and hence we have the formula

$$g_+'(\zeta_0) = \Re \frac{\Psi'(\zeta_0+)\zeta_0}{\Psi(\zeta_0+)} = \left|\Re \frac{\Psi'(\zeta_0+)\zeta_0}{\Psi(\zeta_0+)}\right| = \left|\frac{\Psi'(\zeta_0+)\zeta_0}{\Psi(\zeta_0+)}\right| = \left|\Psi'(\zeta_0+)\right|. \tag{28}$$

□

4 An Alternative Proof for Theorem 1

In this section, we prove Theorem 1 using the following result of P. Borwein and T. Erdélyi (see [2], p. 324, Theorem 7.1.7). Recall that we denote the unit disk by Δ and the unit circle by C_1. Let $a_k \in \mathbf{C} \setminus C_1$, $k = 1, \dots, m$; set

$$B_m^+(z) := \sum_{k:|a_k|>1} \frac{|a_k|^2 - 1}{|a_k - z|^2}, \qquad B_m^-(z) := \sum_{k:|a_k|<1} \frac{1 - |a_k|^2}{|a_k - z|^2};$$

and let

$$B_m(z) := \max\bigl(B_m^+(z), B_m^-(z)\bigr).$$

Then, for every rational function $r(z)$ of the form $r(z) = Q(z)/\prod_{k=1}^m (z - a_k)$, where Q is a polynomial of degree at most m, we have

$$\left|r'(z)\right| \le B_m(z)\|r\|_{C_1} \qquad z \in C_1. \tag{29}$$

We shall need the function

$$\zeta = \Phi(z) = z\frac{1 + z\sin(\omega/2)}{z + \sin(\omega/2)}. \tag{30}$$

Simple computation gives, as, e.g., in [6], Eq. (4), that Φ is a conformal map from the complement of the unit disk onto $\mathbf{C} \setminus K$, so $\Psi = \Phi^{-1}$ is one of the Ψ's in Corollary 5.

It is also easy to see that if $\Re z > -\sin(\omega/2)$, then $\zeta = \Phi(z)$ lies on the outer side of the arc K (i.e., then $\zeta = \zeta+$), while if $\Re z < -\sin(\omega/2)$, then $\zeta = \Phi(z)$ lies in the inner side of the arc K (i.e., in this case $\zeta = \zeta-$).

Without loss of generality, we may assume that the polynomial in Theorem 1 is of the form $P_n(\zeta) = (\zeta - \alpha_1)\dots(\zeta - \alpha_n)$ (i.e., it has leading coefficient 1) and define

$$r(z) := P_n\left(\frac{1}{\Phi(z)}\right), \tag{31}$$

where Φ is the function from (30). Then

$$\|r\|_{C_1} = \|P_n\|_K$$

and (see (30))

$$\begin{aligned} r(z) &= \prod_{j=1}^{n}\left(\frac{z+\sin(\omega/2)}{z(1+z\sin(\omega/2))} - \alpha_j\right) \\ &= \frac{\prod_{j=1}^{n}(-\alpha_j \sin(\omega/2)z^2 + (1-\alpha_j)z + \sin(\omega/2))}{z^n(z\sin(\omega/2)+1)^n}. \end{aligned}$$

So, to use (29), we set $m = 2n$, $a_1 = \cdots = a_n = 0$, and $a_{n+1} = \cdots = a_{2n} = -1/\sin(\omega/2)$. For $z = e^{it}$, we see that

$$B_{2n}^{-}(z) = n \quad \text{and} \quad B_{2n}^{+}(z) = n\frac{\left|\frac{-1}{\sin(\omega/2)}\right|^2 - 1}{\left|\frac{-1}{\sin(\omega/2)} - e^{it}\right|^2},$$

and here the second term is

$$\begin{aligned} B_{2n}^{+}(z) &= n\frac{\cos^2(\omega/2)}{|1+\sin(\omega/2)\cos t + i\sin(\omega/2)\sin t|^2} \\ &= n\frac{\cos^2(\omega/2)}{1+\sin^2(\omega/2)+2\sin(\omega/2)\cos t}. \end{aligned}$$

Taking maximum, we get

$$B_{2n}(z) = \begin{cases} n & \text{if } \Re z = \cos t \geq -\sin(\omega/2), \\ n\frac{\cos^2(\omega/2)}{1+\sin^2(\omega/2)+2\sin(\omega/2)\cos t} & \text{if } \Re z = \cos t \leq -\sin(\omega/2). \end{cases} \tag{32}$$

It is important to observe that $B_{2n}(z) = B_{2n}^{-}(z) = n$ (first line) if $\zeta = \Phi(z)$ is "from the outer side" of K. Hence, the Borwein–Erdélyi inequality implies that

$$\left|P_n'\left(\frac{1}{\Phi(z)}\right)\frac{\Phi'(z)}{\Phi^2(z)}\right| \leq B_{2n}(z)\|P_n\|_K,$$

and since here $|\Phi(z)| = 1$ for $z \in C_1$, we get for $\Phi(z) = \zeta =: e^{i\theta}$, $\theta \in (-\omega, \omega)$,

$$\left|P_n'\left(e^{-i\theta}\right)\right| \leq \frac{B_{2n}(z)}{|\Phi'(z)|}\|P_n\|_K. \tag{33}$$

For each $\theta \in (-\omega, \omega)$, there are two $z \in C_1$ such that $\Phi(z) = e^{i\theta}$, one on the arc in the half-plane $\{z \mid \Re z \geq -\sin(\omega/2)\}$, and one on the complementary arc of C_1. We choose the former one in (33) which corresponds to the first line in (32), and get

$$\left|P_n'\left(e^{-i\theta}\right)\right| \leq \frac{n}{|\Phi'(z)|}\|P_n\|_K. \tag{34}$$

Since $\Psi(\Phi(z)) = z$, we have $\Psi'(\Phi(z))\Phi'(z) = 1$; i.e., $|\Psi'(\zeta)| = 1/|\Phi'(z)|$. If we substitute this into (34) and use Corollary 5, we obtain (4) (note also that $V(-\theta) = V(\theta)$). □

5 Proof of Theorem 2

It was proved in [5], Theorems 1.3, 1.4, that if Γ is a C^2 smooth Jordan curve (homeomorphic image of the unit circle), Ω is the unbounded component of its complement, and $g_\Omega(z, \infty)$ is Green's function in Ω with pole at infinity, then

$$\left|P_n'(\zeta)\right| \leq \left(1 + o(1)\right) n \frac{\partial g_\Omega(\zeta, \infty)}{\partial \mathbf{n}} \|P_n\|_\Gamma, \quad \zeta \in \Gamma,$$

where $\mathbf{n}$ is the inner normal to Γ with respect to Ω. Furthermore, this is sharp, for if $\zeta \in \Gamma$ is given, then there are nonzero polynomials P_n with

$$\left|P_n'(\zeta)\right| \geq \left(1 - o(1)\right) n \frac{\partial g_\Omega(\zeta, \infty)}{\partial \mathbf{n}} \|P_n\|_\Gamma. \tag{35}$$

Now consider K and a point ζ on K which is not one of the endpoints of K. We augment K to a C^2 smooth Jordan curve Γ by attaching a small domain (as the interior of Γ) to K that lies in the unit disk (see Fig. 1).

We can do that in such a way that if $\varepsilon > 0$ is given, then

$$\frac{\partial g_\Omega(\zeta, \infty)}{\partial \mathbf{n}} \geq (1 - \varepsilon) \frac{\partial g_{\mathbf{C} \setminus K}(\zeta, \infty)}{\partial \mathbf{n}} = (1 - \varepsilon) g_+'(\zeta). \tag{36}$$

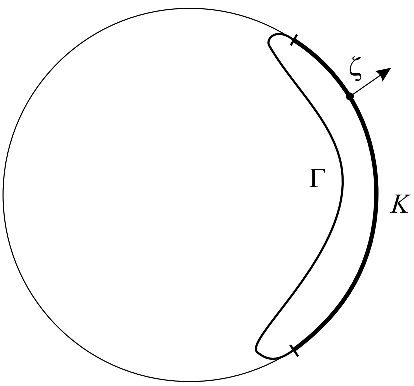

Fig. 1 The domain attached to K

In fact, since K is part of Γ, we have $g_\Omega(\zeta,\infty) \le g_{\mathbf{C}\setminus K}(\zeta,\infty)$, and at infinity the difference $g_{\mathbf{C}\setminus K}(\zeta,\infty) - g_\Omega(\zeta,\infty)$ coincides with $\log(\mathrm{cap}(\Gamma)/\mathrm{cap}(K))$ (see [9], Theorem 5.2.1), where $\mathrm{cap}(\cdot)$ denotes logarithmic capacity. As we shrink Γ to K, the capacity of Γ tends to the capacity of K, and so the nonnegative harmonic function $g_{\mathbf{C}\setminus K}(\zeta,\infty) - g_\Omega(\zeta,\infty)$ tends to zero at infinity (this difference is also harmonic there). Now we get from Harnack's theorem ([9], Theorems 1.3.1 and 1.3.3) that this difference tends to 0 uniformly on compact subsets of $\overline{\mathbf{C}} \setminus K$, and then (36) will be true if Γ is sufficiently close to K by [5], Lemma 7.1.

Now apply (35) to this Γ. For the corresponding polynomials P_n we can write, in view of $\|P_n\|_K \le \|P_n\|_\Gamma$,

$$\left|P_n'(\zeta)\right| \ge \bigl(1-o(1)\bigr)n\frac{\partial g_\Omega(\zeta,\infty)}{\partial \mathbf{n}}\|P_n\|_\Gamma \ge \bigl(1-o(1)\bigr)n(1-\varepsilon)g_+'(\zeta)\|P_n\|_K.$$

Since here $\varepsilon > 0$ is arbitrary, and by Corollary 5 the last factor on the right-hand side is $(1+V(\theta))/2$ with $\zeta = e^{i\theta}$, the proof is complete. □

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