Asymptotically sharp Markov and Schur inequalities on general sets

Sergei Kalmykov, Béla Nagy and Vilmos Totik

Abstract. Markov's inequality for algebraic polynomials on $[-1,1]$ goes back to more than a century and it is widely used in approximation theory. Its asymptotically sharp form for unions of finitely many intervals has been found only in 2001 by the third author. In this paper we extend this asymptotic form to arbitrary compact subsets of the real line satisfying an interval condition. With the same method a sharp local version of Schur's inequality is given for such sets.

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1. Results

Markov's inequality is one of the most fundamental results in approximation theory, it states that if P_n is an algebraic polynomial of degree n, then

$$\|P_n'\|_{[-1,1]} \le n^2 \|P_n\|_{[-1,1]}, \tag{1}$$

where $\|\cdot\|_{[-1,1]}$ is the sup-norm over $[-1,1]$. It is sharp, for the classical Chebyshev polynomials equality holds.

When one considers the analogue of (1) on a set K consisting of several intervals, a new feature emerges: if a_j, $j=1,\ldots,2m$ are the endpoints of the intervals that make up K, then around each a_j there is a local Markov inequality

$$\|P_n'\|_{[a_j-\varepsilon,a_j+\varepsilon]\cap K} \le (1+o(1))M(K,a_j)n^2 \|P_n\|_K,$$

with some best constants $M(K,a_j)$, where $o(1)$ tends to 0 uniformly in P_n as the degree n tends to infinity. In general, these local Markov factors $M(K,a_j)$ are different, they depend on the location of a_j in the set K. The paper [9] gave an explicit expression for them. The asymptotically sharp global Markov inequality

$$\|P_n'\|_K \le (1+o(1))\Big(\max_j M(K,a_j)\Big)n^2\,\|P_n\|_K$$

is then an immediate consequence.

The aim of this paper is to prove a sharp local version of (1) for general compact subsets of $\mathbf{R}$ rather than for $[-1,1]$ or for sets consisting of finitely many intervals. To this end we call the point a a right-endpoint of the compact set $K\subset\mathbf{R}$ if there is a $\rho>0$ such that

$$[a-2\rho,a]\subset K \quad\text{and}\quad (a,a+2\rho)\cap K=\emptyset. \tag{2}$$

We shall refer to (2) as the interval condition. The numbers a,ρ will be fixed for the whole paper, and we shall always assume that $K\subset\mathbf{R}$ satisfies this condition.

We introduce the (asymptotic) Markov factor for K at the endpoint a as

$$M(K,a) := \limsup_{n\to\infty}\ \sup\left\{\frac{\|P_n'\|_{[a-\rho,a]}}{n^2\,\|P_n\|_K}\ \middle|\ \deg(P_n)\le n\right\}. \tag{3}$$

Without changing the value of $M(K,a)$, the norm in the numerator could have been taken instead of $[a-\rho,a]$ on any interval $[a-\eta,a]$ so long as $[a-\eta-\varepsilon,a]\subset K$ for some $\varepsilon>0$. This is because on compact subsets of the interior $\mathrm{Int}(K)$ of K the norm of P_n' is at most $Cn\|P_n\|_K$ with some C (depending on the compact subset), see (12) below. In a similar manner, $M(K,a)$ would not change if we used $|P_n'(a)|$ in (3) instead of $\|P_n'\|_{[a-\rho,a]}$. This is not absolutely trivial, but it will follow from the considerations below.

To formulate the results, we need some potential theory and we refer to the books [2], [4], [7] or [8] for an introduction. In particular, we denote the equilibrium measure of $K\subset\mathbf{R}$ of positive logarithmic capacity $\mathrm{cap}(K)>0$ by ν_K. This is absolutely continuous on the (one dimensional) interior of K, and we denote its density with respect to the Lebesgue-measure by ω_K: $\frac{d\nu_K(t)}{dt}=\omega_K(t)$. If K satisfies the interval condition (2) then, around a, the density ω_K behaves like $1/\sqrt{|t-a|}$. The Markov factor $M(K,a)$ is related to the quantity

$$\Omega(K,a) := \lim_{t\to a-0}\omega_K(t)\,|t-a|^{1/2}.$$

It will be proven in the next section that this limit exists, positive and finite, and with it we can state

Theorem 1.1. *If $K\subset\mathbf{R}$ satisfies the interval condition (2) at a point $a\in K$, then*

$$M(K,a)=2\pi^2\Omega(K,a)^2. \tag{4}$$

There is another problem which can be solved via the quantity $\Omega(K,a)$, namely Schur's inequality on general sets. The original Schur inequality (see e.g. [5, Theorem 6.1.2.8]) claims that if P_n is a polynomial of degree at most n for which

$$|P_n(x)|\le\frac{1}{\sqrt{1-x^2}},\qquad x\in(-1,1), \tag{5}$$

then

$$\|P_n\|_{[-1,1]} \le n+1. \tag{6}$$

The next theorem gives an asymptotically optimal local version of this for general subsets of $\mathbf{R}$ rather than $[-1,1]$.

Theorem 1.2. *Let K be a compact subset of $\mathbf{R}$ with the interval condition (2) at a point $a \in K$. Suppose that for polynomials P_n of degree $n = 1, 2, \ldots$ we have*

$$|P_n(x)| \le \frac{h(x)}{\sqrt{a-x}}, \qquad x \in [a-\rho, a), \tag{7}$$

with some positive and continuous function h on $[a-\rho, a]$, and assume also the global condition

$$\limsup_{n\to\infty} \|P_n\|_K^{1/n} \le 1. \tag{8}$$

Then

$$\|P_n\|_{[a-\rho,a]} \le n(1+o(1))2\pi h(a)\Omega(K,a). \tag{9}$$

This estimate is sharp for any h, for there are polynomials $P_n \not\equiv 0$ satisfying (7) and (8) for which

$$|P_n(a)| \ge n(1-o(1))2\pi h(a)\Omega(K,a). \tag{10}$$

The $o(1)$ in (9) depends only on the function h and on the speed of convergence in (8).

The condition (8) is a very weak one, but something like that is needed, for the polynomials P_n cannot be arbitrary outside $[a-\rho, a]$: just set $K = [-2, 1]$, $a = 1$, and with the classical Chebyshev polynomials $\mathcal{T}_n(x) = \cos(n \arccos x)$ consider $P_n(x) = \mathcal{T}'_{n+1}(x)/(n+1)$. In this case (7) is true with $\rho = 1$, $h(x) = 1/\sqrt{1+x}$ (apply Bernstein's inequality (11) below), but (9) does not hold because $P_n(1) = n+1$ and $\Omega(K,1) = 1/\pi\sqrt{3}$.

Schur's inequality (6) is one way to deduce Markov's inequality (1) from Bernstein's inequality

$$|P'_n(x)| \le \frac{n}{\sqrt{1-x^2}}\|P_n\|_{[-1,1]}, \qquad x \in (-1,1). \tag{11}$$

The same happens with (9) and the estimate $M(K,a) \le 2\pi^2\Omega(K,a)^2$ in Theorem 1.1. In fact, if $K \subset \mathbf{R}$ is a regular compact set (regular with respect to the Dirichlet problem in $\overline{\mathbf{C}} \setminus K$), then the Bernstein-Walsh lemma (see e.g. [11, p. 77] or [7, Theorem 5.5.7]) and Cauchy's formula for the derivative of an analytic function easily give that

$$\|P'_n\|_K = e^{o(n)}\|P_n\|_K.$$

This implies (8) for the polynomial $P'_n(x)/n\|P_n\|_K$. On the other hand, [9, Theorem 3.1] (see also [3]) claims that on the interior of K we have

$$|P'_n(x)| \le n\pi\omega_K(x)\|P_n\|_K, \tag{12}$$

hence

$$\left|\frac{P'_n(x)}{n\|P_n\|_K}\right| \le \frac{h(x)}{\sqrt{a-x}}, \qquad x \in [a-\rho, a], \tag{13}$$

where $h(x)=\sqrt{a-x}\pi\omega_K(x)$ on $[a-\rho,a]$ (and extend this h to an arbitrary continuous and positive function from there). This is condition (7) for the polynomial $P_n'(x)/n\|P_n\|_K$. Furthermore, here $h(a)=\pi\Omega(K,a)$. Therefore, we can apply Theorem 1.2 to conclude that

$$\left\|\frac{P_n'}{n\|P_n\|_K}\right\|_{[a-\rho,a]} \le n(1+o(1))2\pi\Big(\pi\Omega(K,a)\Big)\Omega(K,a),$$

which implies

$$\|P_n'\|_{[a-\rho,a]} \le n^2(1+o(1))2\pi^2\Omega(K,a)^2\|P_n\|_K, \tag{14}$$

which is precisely the inequality $M(K,a)\le 2\pi^2\Omega(K,a)^2$ in Theorem 1.1. When K is not regular, in the reasoning above, instead of K, just use the sets $K_m^-\subset K$ from (23) to be introduced in Section 2, make the conclusion

$$\begin{aligned}\|P_n'\|_{[a-\rho,a]} &\le n^2(1+o(1))2\pi^2\Omega(K_m^-,a)^2\|P_n\|_{K_m^-}\\ &\le n^2(1+o(1))2\pi^2\Omega(K_m^-,a)^2\|P_n\|_K,\end{aligned}$$

instead of (14), and use the fact that, by Proposition 2.3 below, the quantity $\Omega(K_m^-,a)$ is as close to $\Omega(K,a)$ as we wish if m is sufficiently large.

The quantity $\Omega(K,a)$ has been formulated in terms of the equilibrium density, but we can give a direct formulation as follows. $\mathbf{R}\setminus K$ is the union of countably many open intervals: $\mathbf{R}\setminus K=\cup_{j=0}^{\infty}I_j$, where, say, I_0 and I_1 are the two unbounded complementary intervals (if K itself consists of finitely many intervals, then the preceding union should be replaced by finite one). We may also assume that the numbering is such that I_2 contains $(a,a+2\rho)$ (if $I_0\cup I_1$ does not do so). For $m\ge 2$ consider the set

$$K_m^+=\mathbf{R}\setminus\left(\bigcup_{j=0}^{m}I_j\right). \tag{15}$$

This contains K, it satisfies the interval condition (2), and it consists of m disjoint closed intervals: $K_m^+=\cup_{j=1}^{m}[a_{j,m},b_{j,m}]$ with $a_{1,m}\le b_{2,m}<a_{2,m}\le b_{2,m}<\cdots<a_{m,m}\le b_{m,m}$. When $a_{j,m}=b_{j,m}$ for some j, then the corresponding interval is degenerated, and we can drop it from the consideration below, so we may assume $a_{j,m}<b_{j,m}$ for all $j=1,\ldots,m$. The equilibrium density of K_m^+ is (see e.g. [9, Lemma 2.3])

$$\omega_{K_m^+}(x)=\frac{\prod_{j=1}^{m-1}|x-\lambda_{j,m}|}{\pi\sqrt{\prod_{j=1}^{m}|x-a_{j,m}||x-b_{j,m}|}},\qquad x\in\mathrm{Int}(K_m^+), \tag{16}$$

where $\lambda_{j,m}$ are chosen so that

$$\int_{b_{k,m}}^{a_{k+1,m}}\frac{\prod_{j=1}^{m-1}(t-\lambda_{j,m})}{\sqrt{\prod_{j=1}^{m}|t-a_{j,m}||t-b_{j,m}|}}dt=0 \tag{17}$$

for all $k = 1, \dots, m-1$. It can be easily shown that these $\lambda_{j,m}$'s are uniquely determined and there is one $\lambda_{j,m}$ on every contiguous interval $(b_{k,m}, a_{k+1,m})$. Now a is one of the $b_{j,m}$'s, say $a = b_{j_0,m}$, and then clearly

$$\Omega(K_m^+, a) = \frac{\prod_{j=1}^{m-1} |a - \lambda_{j,m}|}{\pi \sqrt{\prod_{j=1}^{m} |a - a_{j,m}|} \sqrt{\prod_{j=1,\ j\neq j_0}^{m} |a - b_{j,m}|}}. \tag{18}$$

When K consists of finitely many intervals, i.e. $K = K_m^+$ for some m, then this gives an explicit expression for $\Omega(K, a)$. In the general case, since $K_{m+1}^+ \subset K_m^+$, the equilibrium measure $\nu_{K_{m+1}^+}$ is the balayage of $\nu_{K_m^+}$ onto K_{m+1}^+ (see [8, Theorem IV.1.6,e]), hence $\omega_{K_{m+1}^+}(t) \geq \omega_{K_m^+}(t)$ for all $t \in \mathrm{Int}(K_{m+1}^+)$. As a consequence, the sequence $\{\Omega(K_m^+, a)\}_{m=2}^{\infty}$ is increasing, and we shall verify in the next section that

$$\Omega(K, a) = \lim_{m\to\infty} \Omega(K_m^+, a).$$

The just used monotonicity argument will be used later: if $K \subset S$ both satisfy the interval condition (2), then

$$\Omega(S, a) \leq \Omega(K, a). \tag{19}$$

2. Properties of $\Omega(K, a)$

First, we show that the limit $\Omega\,(K, a)$ exists in a uniform way.

Let

$$\mathcal{E} := \Big\{ K \subset \mathbf{R} \,\Big|\, K \text{ compact, satisfies (2)} \Big\}.$$

Lemma 2.1. *For all $K \in \mathcal{E}$ there exists $L_K \in (0, \infty)$ such that*

$$\lim_{t\to a-0} \omega_K\,(t)\,|t-a|^{1/2} = L_K.$$

Moreover, this convergence is uniform in $K \in \mathcal{E}$: for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $K \in \mathcal{E}$, $t \in (a-\delta, a)$ we have

$$\Big| \omega_K\,(t)\,|t-a|^{1/2} - L_K \Big| < \varepsilon.$$

Proof. Let δ_x denote the Dirac measure at x and let $\mathrm{Bal}\,(\delta_x, [b, a]\,; t)$ denote the density at t of the balayage of δ_x onto $[b, a]$, $b < a$. Sometimes we also use the same notation for the measure: $\mathrm{Bal}\,(\delta_x, [b, a]\,; H)$ denotes the balayage measure of the Borel-set H. We use [8, (4.47), Ch.II]:

$$\mathrm{Bal}\,(\delta_x, [b, a]\,; t) = \frac{1}{\pi} \frac{\sqrt{|x-b|\,|x-a|}}{|t-x|\,\sqrt{|t-a|\,|t-b|}}. \tag{20}$$

Thus, in this case clearly

$$\lim_{t\to a-0} \mathrm{Bal}\,(\delta_x, [b, a]\,; t)\,|t-a|^{1/2} = \frac{1}{\pi} \frac{\sqrt{|x-b|}}{\sqrt{|x-a|\,|a-b|}} =: L_x. \tag{21}$$

Below we set $b = a - \rho$. The family

$$\left\{\mathrm{Bal}\left(\delta_x, [a-\rho, a]; t\right) \,\middle|\, x \in \mathbf{R} \setminus [a-2\rho, a+\rho]\right\}$$

is uniform in the sense that for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in \mathbf{R} \setminus [a-2\rho, a+\rho]$ and for all $t \in (a-\delta, a)$ we have

$$\left|\sqrt{|a-t|} \cdot \mathrm{Bal}\left(\delta_x, [a-\rho, a]; t\right) - \frac{1}{\pi}\frac{\sqrt{|x-b|}}{\sqrt{|x-a|\,|a-b|}}\right|$$
$$= \left|\frac{1}{\pi}\frac{\sqrt{|x-b|\,|x-a|}}{|t-x|\,\sqrt{|t-b|}} - \frac{1}{\pi}\frac{\sqrt{|x-b|}}{\sqrt{|x-a|\,|a-b|}}\right| < \varepsilon, \qquad b = a-\rho. \quad (22)$$

This is a simple calculus exercise, we skip it.

If μ is any positive Borel-measure with compact support and $\mathrm{supp}\,(\mu) \subset \mathbf{R} \setminus [a-2\rho, a+\rho]$ and $\mu\,(\mathbf{R}) \le 1$, then on $(a-\rho, a)$ the density function of the measure

$$\mu^*(\cdot) := \int \mathrm{Bal}\left(\delta_x, [a-\rho, a]\,; \cdot\right)\, d\mu\,(x)$$

has the form

$$\frac{d\mu^*\,(t)}{dt} = \int \mathrm{Bal}\left(\delta_x, [a-\rho, a]\,;\, t\right)\, d\mu\,(x)\,.$$

This follows from (20): if we fix $t \in (a-\rho, a)$ then this is away from the support of μ^*, so

$$\mathrm{Bal}\left(\delta_x, [a-\rho, a]\,; [t, t+u]\right)/u, \qquad 0 < u < \frac{a-t}{2}$$

is bounded and we can use Lebesgue's dominated convergence theorem when taking the limit for $u \to 0$.

We can rewrite (22) as

$$\left|\sqrt{|a-t|} \cdot \mathrm{Bal}\left(\delta_x, [a-\rho, a]\,;\, t\right) - L_x\right| < \varepsilon, \qquad t \in (a-\delta, a).$$

Now if this inequality is integrated with respect to μ we obtain that the limit

$$\lim_{t \to a-0} \sqrt{|t-a|} \cdot \frac{d\mu^*\,(t)}{dt}$$

exists uniformly in the measure μ^*.

Finally, we use that $\nu_{[a-\rho,a]}$ is the balayage of ν_K onto $[a-\rho, a]$ (see [8, Theorem IV.1.6,e]). When forming this balayage measure the part of ν_K that is on $[a-\rho, a]$ is left unchanged, and the rest of ν_K is swept onto $[a-\rho, a]$, and this latter balayage measure is

$$\mathrm{Bal}(\nu_K|_{K\setminus[a-\rho,a]}, [a-\rho, a]; H) = \int_{K\setminus[a-\rho,a]} \mathrm{Bal}\left(\delta_x, [a-\rho, a]\,; H\right) d\nu_K(x)$$

where H is arbitrary (Borel-) set. Thus, we have the formula

$$\nu_{[a-\rho,a]}\,(H) = \nu_K|_{[a-\rho,a]}\,(H) + \int_{K\setminus[a-\rho,a]} \mathrm{Bal}\left(\delta_x, [a-\rho, a]\,; H\right)\, d\nu_K\,(x)\,.$$

Rewriting this for the densities, we have for $t \in (a-\rho, a)$

$$\omega_K(t) = \omega_{[a-\rho,a]}(t) - \int_{K\setminus[a-\rho,a]} \mathrm{Bal}\left(\delta_x, [a-\rho,a]\,; t\right) \, d\nu_K(x) \, .$$

Now if σ is either of the terms on the right-hand side, then, as we have just seen, the limit

$$\lim_{t\to a-0} \sqrt{|t-a|} \cdot \sigma(t)$$

exists, and this limit is uniform in the set $K \in \mathcal{E}$. This proves the claim in the lemma. □

We shall need to use a theorem of Ancona [1]: Let $K \subset \mathbf{R}$ be a compact set of positive logarithmic capacity. Then, for every m, there exists a regular compact set (regular with respect to the solution of the Dirichlet problem in its complement relative to $\overline{\mathbf{C}}$) $K_m^- \subset K$ such that

$$\mathrm{cap}\,(K) \le \mathrm{cap}(K_m^-) + 1/m. \tag{23}$$

Since the union of two regular compact sets is regular and K satisfies the interval condition (2), we may assume that K_m^- also satisfies that condition (if not, just unite it with $[a-2\rho, a]$), and also that $K_m^- \subseteq K_{m+1}^-$.

Lemma 2.2. *For the sets $K_m^\pm$ from (15) and (23) we have $\nu_{K_m^\pm} \to \nu_K$ in weak* sense as $m \to \infty$.*

Proof. From the monotone convergence of $K_m^\pm$ to K it follows that $\mathrm{cap}(K_m^\pm) \to \mathrm{cap}(K)$ as $m \to \infty$, see [7, Theorem 5.1.3].

Let $\nu_{K_m^\pm} \to \nu$, $m \to \infty$, $m \in \mathcal{N}$ (for some subsequence $\mathcal{N}$ of the natural numbers), be a weak*-limit of the sequence $\{\nu_{K_m^\pm}\}$. Clearly, ν is supported on K and it has total mass 1. Let

$$I(\mu) = \int\int \log \frac{1}{|z-t|} d\mu(t) d\mu(z)$$

be the logarithmic energy of a measure μ. The equilibrium measure ν_K minimizes this energy among all probablity measures on K, and with it we have the formula $\mathrm{cap}(K) = \exp(-I(\nu_K))$ (see [7, Definition 5.1.1]). Now it follows from $\mathrm{cap}(K_m^\pm) \to \mathrm{cap}(K)$ and from the principle of descent (see e.g. [8, Theorem I.6.8]), that

$$I(\nu_K) = \lim_{m\to\infty,\ m\in\mathcal{N}} I\left(\nu_{K_m^\pm}\right) = \liminf_{m\to\infty,\ m\in\mathcal{N}} I\left(\nu_{K_m^\pm}\right) \ge I(\nu) \ge I(\nu_K).$$

But the equilibrium measure ν_K is the unique measure to minimize the logarithmic energy among all unit Borel-measures with support on K, hence ν must be equal to ν_K. Since this is true for all weak*-convergent subsequences of $\{\nu_{K_m^\pm}\}$, the claim in the lemma follows. □

Finally, we verify

Proposition 2.3. *For the sets $K_m^\pm$ from (15) and (23) we have that $\Omega(K_m^\pm, a) \to \Omega(K, a)$ as $m \to \infty$.*

Proof. Denote Green's function of $\overline{\mathbf{C}} \setminus K$ with pole at infinity by $g_K(z)$. It has the form

$$g_K(z) = \int \log|z-t| d\nu_K(t) - \log \operatorname{cap}(K), \tag{24}$$

see [7, Sec. 4.4] or formula [8, (I.4.8)]. Consider the function

$$u_m(z) = g_{K_m^\pm}(z) - g_K(z).$$

This is harmonic in $\overline{\mathbf{C}} \setminus (K \cup K_m^\pm)$. In view of Lemma 2.2, of $\operatorname{cap}(K_m^\pm) \to \operatorname{cap}(K)$ and of (24), the functions $u_m(z)$ tend to 0 uniformly on compact subsets of $\overline{\mathbf{C}} \setminus K$ as $m \to \infty$. Since $[a-2\rho, a]$ is part of all the sets $K_m^\pm$, we have $g_{K_m^\pm}(z),\ g_K(z) \le g_{[a-2\rho,a]}(z)$, so $u_m(z) \to 0$ as $z \to a-\rho$, and this convergence is uniform in m. Thus, $u_m(z) \to 0$ uniformly on the circle $C_\rho(a) := \{z \,|\, |z-a| = \rho\}$. Let $D_\rho(a)$ be the interior of that circle.

In what follows we shall use the main branch of the square root function. $\zeta = i\sqrt{z-a}$ maps $D_\rho(a) \setminus [a-\rho, a]$ onto the upper half-disk $\{\zeta \,|\, |\zeta| < \sqrt{\rho},\ \Im\zeta > 0\}$, so $v_m(\zeta) = u_m(a-\zeta^2)$ is a harmonic function there, which vanishes on $(-\sqrt{\rho}, \sqrt{\rho})$. By the reflection principle we can extend it to a harmonic function in the disk $D_{\sqrt{\rho}}(0)$. From the fact that $u_m \to 0$ uniformly on $C_\rho(a)$ it is immediate that $v_m \to 0$ uniformly on $C_{\sqrt{\rho}}(0)$, so its partial derivative in the i-direction tends to 0 uniformly on compact subsets of $D_{\sqrt{\rho}}(0)$. For $x \in (a-\rho, a)$ we have

$$\frac{\partial}{\partial \mathbf{n}_+} v_m(w) = \frac{\partial}{\partial \mathbf{n}_+} u_m(x) \cdot 2\sqrt{a-x}, \qquad w = -\sqrt{a-x},$$

where $\partial u_m(x)/\partial \mathbf{n}^+$ denotes the derivative of u_m with respect to the upper normal to $\mathbf{R}$ at x. As we have just mentioned, the left-hand side tends to 0 uniformly on compact subsets of $D_{\sqrt{\rho}}(0) \cap \mathbf{R}$. It follows that $\sqrt{a-x} \cdot (\partial u_m(x)/\partial \mathbf{n}^+)$ tends uniformly to 0 on $[a-\rho/2, a]$. Since

$$\frac{\partial}{\partial \mathbf{n}_+} u_m(z) = \frac{\partial}{\partial \mathbf{n}_+} g_{K_m^\pm}(z) - \frac{\partial}{\partial \mathbf{n}_+} g_K(z) = \pi\omega_{K_m^\pm}(z) - \pi\omega_K(z)$$

(see e.g. [6, II.(4.1)]), this proves the Proposition. □

3. Proof of Theorem 1.1

When K consists of finitely many intervals like the sets K_m^+ in (15), the theorem follows from [9, Theorem 4.1] and from (18).

Proof of $M(K,a) \le 2\pi^2\Omega(K,a)^2$. First we prove this inequality when K is regular with respect to the Dirichlet problem in $\overline{\mathbf{C}} \setminus K$, and later we remove the regularity condition.

So assume that K is regular and satisfies the interval condition (2). Fix $\varepsilon > 0$, and let $K^+ := [\min K, \max K]$. There exist (see e.g. [8, Corollary VI.3.6]) $0 < \tau < 1$ and polynomial $Q_{n\varepsilon}$ of $\deg(Q_{n\varepsilon}) \le n\varepsilon$ such that

a) $1 - e^{-n\tau} \le Q_{n\varepsilon}(x) \le 1$ if $x \in [a-\rho, a]$,
b) $0 \le Q_{n\varepsilon}(x) \le 1$ if $x \in [a-3\rho/2, a-\rho) \cup (a, a+3\rho/2]$,

c) $|Q_{n\varepsilon}(x)| \le e^{-n\tau}$ if $x \in K^+ \setminus [a - 3\rho/2, a + 3\rho/2]$.

In particular, $\|Q_{n\varepsilon}\|_{K^+} \le 1$. Let g_K denote Green's function of $\overline{\mathbf{C}} \setminus K$ with pole at infinity. The regularity of K implies that g_K is continuous and vanishes on K. Hence, there exists $0 < \theta < 1$, $\theta = \theta(\tau)$, such that

$$\text{if } x \in \mathbf{R},\ \operatorname{dist}(x, K) \le \theta,\ \text{then } g_K(x) \le \tau^2. \tag{25}$$

Choose a large m such that for the sets K_m^+ from (15) we have $\operatorname{dist}(x, K) < \theta$ for all $x \in K_m^+$. We are going to apply [9, Theorem 4.1] for the polynomial $P_n Q_{n\varepsilon}$ on K_m^+ where P_n is an arbitrary polynomial with degree n. Then $P_n Q_{n\varepsilon}$ is a polynomial of degree at most $(1+\varepsilon)n$ and we estimate its sup-norm on K_m^+ as follows. First, if $x \in K$, then $|P_n(x) Q_{n\varepsilon}(x)| \le \|P_n\|_K$, see properties a)–c) above. Second, if $x \in K_m^+ \setminus K$, then we apply the Bernstein-Walsh lemma (see e.g. [11, p. 77] or [7, Theorem 5.5.7]) for P_n and property c) for $Q_{n\varepsilon}$, as well as (25) to obtain

$$\begin{aligned}|P_n(x) Q_{n\varepsilon}(x)| &\le \|P_n\|_K \exp(n\, g_K(x)) \exp(-n\tau)\\ &\le \|P_n\|_K \exp\left(n\tau^2\right) \exp(-n\tau) \le \|P_n\|_K.\end{aligned}$$

Hence,

$$\|P_n Q_{n\varepsilon}\|_{K_m^+} \le \|P_n\|_K. \tag{26}$$

Next, for $x \in [a-\rho, a]$

$$\begin{aligned}\left|(P_n Q_{n\varepsilon})'(x)\right| &\ge |P_n'(x) Q_{\varepsilon n}(x)| - |P_n(x) Q_{n\varepsilon}'(x)|\\ &\ge |P_n'(x)| \left(1 - e^{-n\tau}\right) - |P_n(x) Q_{n\varepsilon}'(x)|\end{aligned}$$

and here we can use the (transformed form of the) Markov inequality (1) to conclude

$$\|Q_{n\varepsilon}'\|_{K^+} \le C_1 \varepsilon^2 n^2$$

with some constant C_1. Therefore, for $x \in [a-\rho, a]$

$$\begin{aligned}|P_n'(x)| \left(1 - e^{-n\tau}\right) &\le \left|(P_n Q_{n\varepsilon})'(x)\right| + |P_n(x) Q_{n\varepsilon}'(x)|\\ &\le \left|(P_n Q_{n\varepsilon})'(x)\right| + \|P_n\|_K C_1 \varepsilon^2 n^2.\end{aligned}$$

Now we use that, as has already been mentioned, the theorem is true for the set K_m^+ since it consists of finitely many intervals. Hence, we can continue the preceding estimate as

$$\begin{aligned}&\le ((1+\varepsilon)n)^2 \left(1 + o_{K_m^+}(1)\right) 2\pi^2 \Omega\left(K_m^+, a\right)^2 \|P_n Q_{n\varepsilon}\|_{K_m^+} + \|P_n\|_K C_1 \varepsilon^2 n^2\\ &\le n^2 \|P_n\|_K \left(\left(1 + o_{K_m^+}(1)\right)(1+\varepsilon)^2 2\pi^2 \Omega\left(K_m^+, a\right)^2 + C_1 \varepsilon^2\right),\end{aligned}$$

where we also used (26). On applying the monotonicity (19) of $\Omega(., a)$ we can continue the preceding estimates as

$$\le n^2 \|P_n\|_K \left((1 + o_K(1))(1+\varepsilon)^2 2\pi^2 \Omega(K, a)^2 + C_1 \varepsilon^2\right).$$

Since here $\varepsilon > 0$ is arbitrary, the inequality $M(K, a) \le 2\pi^2 \Omega(K, a)^2$ follows for regular K from the just given chain of inequalities.

To remove the regularity condition consider the sets K_m^- from (23). These are regular sets satisfying the interval condition (2), so we can apply the just proven estimate to them:

$$\|P_n'\|_{[a-\rho,a]} \le n^2 \|P_n\|_{K_m^-} \left(1 + o_{K_m^-}(1)\right) 2\pi^2 \Omega\left(K_m^-, a\right)^2$$
$$\le n^2 \|P_n\|_K \left(1 + o_{K_m^-}(1)\right) 2\pi^2 \Omega\left(K_m^-, a\right)^2,$$

where we used $K_m^- \subset K$, and hence $\|P_n\|_{K_m^-} \le \|P_n\|_K$. Since on the right $\Omega(K_m^-, a)$ can be made arbitrarily close to $\Omega(K, a)$ by choosing a large m (see Proposition 2.3), the inequality $M(K,a) \le 2\pi^2\Omega(K,a)^2$ follows in the general case. □

Proof of $M(K,a) \ge 2\pi^2\Omega(K,a)^2$. We construct a sequence of polynomials $\{P_n\}_{n=1}^\infty$, $\deg(P_n) = n$, such that

$$\frac{|P_n'(a)|}{n^2 \|P_n\|_K} \to 2\pi^2 \Omega(K,a)^2 \qquad \text{as } n \to \infty. \tag{27}$$

Consider K_m^+ from (15) for some integer m. It is the union of finitely many intervals, but some of them may be degenerated, i.e. some of them may be a singleton. Replace each such singletons in K_m^+ by an interval of length $< 1/m$ (alternatively, for $m > 1/\rho$ we could set K_m^+ as the set $\{x \mid \operatorname{dist}(x,K) \le 1/m\} \setminus (a, a+2\rho)$). The resulting set, which we continue to denote by K_m^+, consists of non-degenerated intervals, so we can apply the sharpness result in [9], formula (4.7) on page 155, according to which there is a sequence $\{P_{m,n}\}_{n=1}^\infty$, $\deg(P_{m,n}) = n$, of polynomials such that

$$\left|P_{m,n}'(a)\right| \ge \left(1 - o_{K_m^+}(1)\right) 2\pi^2 \Omega\left(K_m^+, a\right)^2 n^2 \|P_{m,n}\|_{K_m^+},$$

where $o_{K_m^+}(1)$ depends on K_m^+ and it tends to 0 as $n \to \infty$ for each fixed m. Since $K \subset K_m^+$, we have $\|P_{m,n}\|_{K_m^+} \ge \|P_{m,n}\|_K$, so

$$\left|P_{m,n}'(a)\right| \ge \left(1 - o_{K_m^+}(1)\right) 2\pi^2 \Omega\left(K_m^+, a\right)^2 n^2 \|P_{m,n}\|_K.$$

Since here $\Omega(K_m^+, a)$ can be made arbitrarily close to $\Omega(K,a)$ by selecting a sufficiently large m (see Proposition 2.3, which holds true also for these modified sets K_m^+), the relation (27) follows for $P_n = P_{m_n,n}$ if m_n tends to infinity sufficiently slowly as $n \to \infty$. □

4. Proof of Theorem 1.2

First we need to verify Theorem 1.2 in the special case when K consists of finitely many intervals. In this case we use the polynomial inverse image technique of [10], and deduce the theorem from Schur's inequality (6).

Proof of Theorem 1.2 *when* K *consists of finitely many intervals.* First we deal with the estimate (9).

Let $K = \cup_{j=1}^{l}[a_{2j-1}, a_{2j}]$. For any $\varepsilon > 0$ there is a set $K^* = \cup_1^l[a_{2j-1}^*, a_{2j}]$ such that

$$a_{2j-1} - \varepsilon < a_{2j-1}^* < a_{2j-1} \qquad \text{for all } j, \tag{28}$$

and K^* is the complete inverse image of $[-1,1]$ under a polynomial T_N of some degree N: $K^* = T_N^{-1}[-1,1]$, see [9, Theorem 2.1] (cf. also the history of this density theorem in [10]). This T_N then has N zeros on K^*, and $T_N(x)$ runs through the interval $[-1,1]$ precisely N times as x runs through K^*. Thus, there are intervals $E_1, \ldots, E_N \subset K^*$, $K^* = \cup_{k=1}^N E_k$, that are disjoint except perhaps for their endpoints and T_N is a bijection from each E_k onto $[-1,1]$. The point a is the right-endpoint of one of these intervals, say of E_1 (the numbering of the E_k's is arbitrary). The equilibrium density of K^* has the form (see [9, (3.8)])

$$\omega_{K^*}(t) = \frac{|T_N'(t)|}{N\pi\sqrt{1-T_N^2(t)}},$$

which easily implies that

$$\Omega(K^*, a) = \frac{|T_N'(a)|^{1/2}}{\sqrt{2}\pi N}. \tag{29}$$

For an $\eta > 0$ choose $\delta > 0$ so that for all $t \in [a-\delta, a]$

$$\frac{\sqrt{2|T_N'(a)|}}{(1+\eta)\sqrt{1-T_N(t)^2}} \le \frac{1}{\sqrt{a-t}} \le (1+\eta)\frac{\sqrt{2|T_N'(a)|}}{\sqrt{1-T_N(t)^2}} \tag{30}$$

(this is possible since $T_N^2(a) = 1$ and $T_N'(a) \neq 0$), and

$$\frac{h(a)}{1+\eta} \le h(t) \le (1+\eta)h(a) \tag{31}$$

are true. We may also suppose that δ is smaller than ρ (see (2)) and smaller than the quarter-length of E_1. For an n choose (see e.g. [8, Corollary VI.3.6]) polynomials $Q_{\varepsilon n}$ of degree at most εn such that with some $0 < q < 1$ we have

(i) $|Q_{\varepsilon n}(t) - 1| < q^n$ if $t \in [a-\delta, a]$,
(ii) $0 \le Q_{\varepsilon n}(t) \le 1$ on $[a-2\delta, a-\delta]$, and
(iii) $0 \le Q_{\varepsilon n}(t) < q^n$ if $t \in K^* \setminus [a-2\delta, a]$.

For $t \in E_1$ let $t_k = t_k(t) \in E_k$ be the point with $T_N(t) = T_N(t_k)$. Now if P_n is a polynomial as in the theorem, then we set for $t \in E_1$

$$S_n(t) = \sum_{k=1}^{N}(P_n Q_{\varepsilon n})(t_k).$$

Actually, $S_n(t)$ is a polynomial with degree at most $n + \varepsilon n$, see [9] formula (3.13). Note that all t_k, $k \ge 2$ are outside the interval $[a-2\delta, a]$, hence, in view of (8) and (iii), with any $0 < q < q_1 < 1$ we have the relation

$$S_n(x) = P_n(x) + O(q_1^n), \qquad x \in E_1, \tag{32}$$

furthermore for $x \in E_1 \setminus [a-2\delta, a]$ we even have $S_n(x) = O(q_1^n)$. Thus, in view of the assumption (7), for $x \in [a-\delta, a]$ we get from (30) and (31) that

$$|S_n(x)| \le \frac{h(x)}{\sqrt{a-x}} + O(q_1^n) \le (1+\eta)^2 h(a) \frac{\sqrt{2|T_N'(a)|}}{\sqrt{1-T_N^2(x)}} + O(q_1^n),$$

which gives

$$|S_n(x)| \le (1+\eta)^3 h(a) \frac{\sqrt{2|T_N'(a)|}}{\sqrt{1-T_N^2(x)}} \tag{33}$$

for all large n.

In a similar manner, if $x \in K \setminus [a-2\delta, a]$, then

$$|S_n(x)| = O(q_1^n) \le (1+\eta)^3 h(a) \frac{\sqrt{2|T_N'(a)|}}{\sqrt{1-T_N^2(x)}}$$

for all large n, i.e. (33) is true for all $x \in E_1$.

Now $S_n(x) = V_m(T_N(x))$ with some polynomial V_m of degree at most $\deg(P_n Q_{\varepsilon n})/N \le (1+\varepsilon)n/N$ (see [10, Section 5]), and then (33) can be written in the form

$$V_m(w) \le (1+\eta)^3 h(a) \frac{\sqrt{2|T_N'(a)|}}{\sqrt{1-w^2}}, \qquad w \in (-1,1).$$

Upon applying the Schur inequality (5)–(6) we obtain from $(m+1) \le (1+2\varepsilon)n/N$ (which certainly holds for large n)

$$\|V_m\|_{[-1,1]} \le (1+\eta)^3 h(a) \sqrt{2|T_N'(a)|}(m+1) \le (1+\eta)^3 (1+2\varepsilon) h(a) \sqrt{2|T_N'(a)|} n/N.$$

Using (29), (32) and $S_n(x) = V_m(T_N(x))$, we can conclude that

$$|P_n(x) + O(q_1^n)| \le (1+\eta)^3 (1+2\varepsilon) h(a) 2\pi \Omega(K^*, a) n, \qquad x \in E_1,$$

which gives

$$|P_n(x)| \le (1+\eta)^4 (1+2\varepsilon) h(a) 2\pi \Omega(K^*, a) n, \qquad x \in E_1, \tag{34}$$

for sufficiently large n. Finally, using the monotonicity property (19) of Ω it follows from $K \subset K^*$ that

$$|P_n(x)| \le (1+\eta)^4 (1+2\varepsilon) h(a) 2\pi \Omega(K, a) n, \qquad x \in E_1. \tag{35}$$

This is the desired estimate on E_1. On $[a-\rho, a] \setminus E_1$ the polynomials P_n are bounded by the assumption (7), hence (35) is true on all $[a-\rho, a]$ if n is large. Since $\varepsilon, \eta > 0$ in (35) are also arbitrarily small, the inequality (9) follows.

We still need to prove (10) in the case considered, i.e. when $K = \cup_1^l [a_{2j-1}, a_{2j}]$. We use the notations from the preceding proof.

The estimate of Schur in (6) is sharp: if $\mathcal{T}_m(x) = \cos(m \arccos x)$ are the classical Chebyshev polynomials and $H_m(x) = \mathcal{T}_{m+1}'(x)/(m+1)$, then (5) is true (use Bernstein's inequality (11)) and

$$|H_m(\pm 1)| = m+1.$$

Set now

$$P_n(x) = h(a)H_m(T_N(x))U_{\sqrt{n}}(x)\frac{\sqrt{2|T_N'(a)|}}{(1+\eta)^2}$$

where $m = [(n-\sqrt{n})/N]$ (the integral part of $(n-\sqrt{n})/N$) and $U_{\sqrt{n}}(x)$ is a polynomial of degree smaller than $\sqrt{n}$ for which $U_{\sqrt{n}}(a) = 1$ and $U_{\sqrt{n}}(x) \to 0$ uniformly on compact subsets of $K^* \setminus \{a\}$. This is a polynomial of degree at most n, and for it we have

$$|P_n(x)| \le h(a)\frac{1}{\sqrt{1-T_N^2(x)}}U_{\sqrt{n}}(x)\frac{\sqrt{2|T_N'(a)|}}{(1+\eta)^2}.$$

Using (30)–(31) and the properties of $U_{\sqrt{n}}$ it follows that for large n we have

$$|P_n(x)| \le \frac{h(x)}{\sqrt{|a-x|}}, \qquad x \in K^*.$$

At the same time, for large n,

$$\begin{aligned}|P_n(a)| = h(a)|H_m(\pm 1)|\frac{\sqrt{2|T_N'(a)|}}{(1+\eta)^2} &= h(a)(m+1)\frac{\sqrt{2|T_N'(a)|}}{(1+\eta)^2}\\ &\ge h(a)\frac{n}{(1+\eta)N}\frac{\sqrt{2|T_N'(a)|}}{(1+\eta)^2},\end{aligned}$$

which gives, in view of (29), the inequality

$$|P_n(a)| \ge h(a)\frac{n}{(1+\eta)^3}2\pi\Omega(K^*, a).$$

This estimate contains $\Omega(K^*, a)$, and here K^* is a set close to K, but it depends on $\varepsilon > 0$ (see (28)). With the same argument that was used in Proposition 2.3 we obtain that on the right-hand side $\Omega(K^*, a)$ is as close to $\Omega(K, a)$ as we wish if $\varepsilon > 0$ is sufficiently small. Since $\eta > 0$ is also arbitrary, (10) follows. □

Proof of Theorem 1.2 *for regular sets.* Let now $K \subset \mathbf{R}$ be a regular compact set with the interval condition (2), and consider the sets K_m^+ from (15). Let P_n be as in the theorem, and with the $Q_{\varepsilon n}$ satisfying properties a)–c) in the proof of Theorem 1.1 apply the finite interval case of Theorem 1.2 to the set K_m^+ and to the polynomial $P_nQ_{\varepsilon n}$ of degree at most $(1+\varepsilon)n$. Since $\|P_nQ_{\varepsilon n}\|_{K_m^+} \le \|P_n\|_K$ (see (26)) and $|P_n(x)Q_{\varepsilon n}(x)| \le |P_n(x)|$ for $x \in [a-\rho, a]$, we can conclude that

$$\begin{aligned}\|P_nQ_{\varepsilon n}\|_{[a-\rho,a]} &\le n(1+\varepsilon)(1+o(1))2\pi h(a)\Omega(K_m^+, a)\\ &\le n(1+\varepsilon)(1+o(1))2\pi h(a)\Omega(K, a),\end{aligned}$$

where we have also used the monotonicity property (19). Since $Q_{\varepsilon n}(x) = 1 - o(1)$ on $[a-\rho, a]$ and $\varepsilon > 0$ is arbitrary, we can conclude (9).

As for (10), we can choose a sequence $\{P_{m,n}\}_{n=1}^{\infty}$ for the set K_m^+ as in (10), i.e. the polynomials $P_{m,n}$ satisfy (7)–(8) with P_n replaced by $P_{m,n}$, and

$$|P_{m,n}(a)| \ge n(1-o(1))2\pi h(a)\Omega(K_m^+, a), \qquad n = 1, 2, \ldots \tag{36}$$

By Proposition 2.3 on the right-hand side the factor $\Omega(K_m^+, a)$ can be as close as we wish to $\Omega(K, a)$. Hence, if $m = m_n$ tends to infinity with n but sufficiently slowly, then the polynomials $P_n = P_{m_n,n}$ satisfy (7)–(8) and (10). □

Proof of Theorem 1.2 *for arbitrary sets.* Let now $K \subset \mathbf{R}$ be an arbitrary compact set with the interval condition (2), and consider the sets K_m^- from (23). These are regular sets satisfying the same interval condition, and if P_n are as in the theorem then clearly P_n satisfy the same conditions on K_m^- instead of K. Thus, according to what we have just proven,

$$\|P_n\|_{[a-\rho,a]} \leq n(1 + o(1))2\pi h(a)\Omega(K_m^-, a). \tag{37}$$

On the right-hand side $\Omega(K_m^-, a)$ converges to $\Omega(K, a)$ as $m \to \infty$ (see Proposition 2.3), hence (9) can be concluded from (37).

As for (10), just repeat the argument given in the preceding proof (with the modification of K_m^+ as at the end of the proof of Theorem 1.1 when K_m^+ contains singletons). □

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Sergei Kalmykov
Bolyai Institute, University of Szeged, Szeged, Aradi v. tere 1, 6720, Hungary;
Far Eastern Federal University, 8 Sukhanova Street, Vladivostok, 690950, Russia; and
Institute of Applied Mathematics, FEBRAS, 7 Radio Street, Vladivostok, 690041, Russia
e-mail: `sergeykalmykov@inbox.ru`

Béla Nagy
MTA-SZTE Analysis and Stochastics Research Group, Bolyai Institute, University of Szeged, Szeged, Aradi v. tere 1, 6720, Hungary
e-mail: `nbela@math.u-szeged.hu`

Vilmos Totik
Bolyai Institute, University of Szeged, Szeged, Aradi v. tere 1, 6720, Hungary; and Department of Mathematics and Statistics, University of South Florida, 4202 E. Fowler Ave. CMC342, Tampa, FL 33620, U.S.A.
e-mail: `totik@mail.usf.edu`