Lazy clones and essentially minimal groupoids

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***Abstract*—We determine idempotent lazy operations and binary lazy operations, and apply the latter to characterize groupoids with an essentially minimal clone of a certain type.**

I. Introduction

The main goal of this paper is to describe certain classes of lazy clones, and to apply this in the study of essentially minimal clones.

A clone is a composition-closed set of operations that contains the projections. On a finite set, every clone contains a minimal clone, i.e., a clone whose only proper subclone is the trivial clone (the clone of projections). Therefore, understanding minimal clones is fundamental for the study of clones, which is an important topic in multiple-valued logic as well as in universal algebra. However, despite numerous partial results, we are still very far from having a complete description of minimal clones (see [1], [6]).

An analogous notion is obtained by considering not only the clone of projections as a "trivial" clone, but all unary clones: a clone is said to be essentially minimal if all of its proper subclones consist of operations that depend on at most one variable.

Lazy clones are also "small" clones in some sense: a clone is lazy if there is an operation f such that every element of the clone (except possibly the projections) can be obtained from f by identifying and/or permuting variables. Note that this implies that there is a finite bound on the essential arities of operations in the clone. Laziness and essential minimality are independent notions: neither of them implies the other. However, as observed in [4], lazy clones can be used to determine certain types of essentially minimal clones, and this is our main motivation for investigating lazy clones.

After recollecting the required preliminaries (Section II), we determine lazy clones generated by an idempotent operation (Section III); as it turns out, these are analogues of rectangular bands in possibly more than two variables. In Section IV we focus on lazy binary operations, and we describe them in terms of identities. Finally, in Section V we use the latter result together with a theorem from [4] to characterize certain types of binary essentially minimal clones.

II. Preliminaries

Throughout this paper, A denotes a nonempty set, $\mathcal{O}_A^{(n)}$ stands for the set of all n-ary operations on A, and $\mathcal{O}_A$ is the set of all operations on A. We say that the i-th variable of $f \in \mathcal{O}_A^{(n)}$ is *essential* (in other words, f depends on its i-th variable) if there exist $a_1, \ldots, a_i, a_i', \ldots, a_n \in A$ such that

$$f(a_1, \ldots, a_i, \ldots, a_n) \neq f(a_1, \ldots, a_i', \ldots, a_n).$$

If $n \geq 2$ and f depends on at least two variables, then we simply say that f is an *essential operation*, and the algebra $\mathbb{A} = (A; f)$ is also said to be essential in this case.

For $1 \leq i \leq n \in \mathbb{N}$ we define the i-th n-ary projection $e_i^{(n)}$ by $e_i^{(n)}(x_1, \ldots, x_n) = x_i$. The set of all projections on A is denoted by $\mathcal{J}_A$. Observe that $e_1^{(1)} = \mathrm{id}$ is the identity function on A.

A *clone* is a set $\mathcal{C} \subseteq \mathcal{O}_A$ of operations that is closed under composition and contains every projection. If $\mathcal{C}$ contains at least one essential operation then $\mathcal{C}$ is an *essential clone*. The clone generated by a given operation f is the clone $[f]$ containing all operations that can be obtained from f and the projections by composition. Equivalently, $[f]$ is the clone of term functions of the algebra $\mathbb{A} = (A; f)$. If f is a binary operation, then we will use the notation $f(x, y) = x \cdot y = xy$, and then the algebra $\mathbb{A} = (A; f) = (A; \cdot)$ is called a *groupoid*.

The smallest clone on A is $\mathcal{J}_A$, the clone of projections, also called the trivial clone. If $\mathcal{C} \neq \mathcal{J}_A$ and $\mathcal{C}$ has no subclones other than $\mathcal{C}$ and $\mathcal{J}_A$, then $\mathcal{C}$ is a *minimal clone*. Similarly, $\mathcal{C}$ is an *essentially minimal* clone, if $\mathcal{C}$ is essential, but all subclones of $\mathcal{C}$ are nonessential, and $\mathcal{C}$ is not a minimal clone. We also use these terms for algebras: $(A; f)$ is an (essentially) minimal algebra if $[f]$ is an (essentially) minimal clone.

An important class of minimal groupoids is the variety of rectangular bands: $(A; \cdot)$ is a *rectangular band* if it is a semigroup (i.e., the multiplication is associative) and satisfies the identities $xx = x$ (idempotence) and $xyz = xz$. It is well known (and easy to prove) that every rectangular band is isomorphic to a groupoid of the form $(A_1 \times A_2; \cdot)$, where the multiplication is defined by $(a_1, a_2) \cdot (b_1, b_2) = (a_1, b_2)$.

For $f \in \mathcal{O}_A^{(n)}$ and $g \in \mathcal{O}_A^{(m)}$, we say that g is an *identification minor* (or simply a minor) of f (notation: $g \preceq f$), if there exists a map $\sigma\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$ such that

$$g(x_1, \ldots, x_m) = f\left(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right).$$

The relation $\preceq$ gives rise to a quasiorder on $\mathcal{O}_A$. The corresponding equivalence relation is defined by $f \equiv g \iff f \preceq g$ and $g \preceq f$, and it is clear that $f \equiv g$ if and only if they differ only in inessential variables and/or in the order of their variables. Note that for any $f \in \mathcal{O}_A$, we have $f \equiv \mathrm{id}$ if and only if $f \in \mathcal{J}_A$. We use the notation $\downarrow f$ for the principal ideal (downset) generated by f in the subfunction quasiorder: $\downarrow f := \{g \in \mathcal{O}_A \colon g \preceq f\}$. Note that the set $\downarrow f$ contains only one unary operation, namely $f^*(x) = f(x, \dots, x)$.

Clearly, $\mathcal{J}_A \cup \downarrow f \subseteq [f]$ holds for every operation f. If $[f] = \mathcal{J}_A \cup \downarrow f$, i.e., f generates only minors of itself and projections, then we say that f is a *lazy operation* and $[f]$ is a *lazy clone*.

For $f \in \mathcal{O}_A^{(n)}$ and $k \in \mathbf{n} := \{1, 2, \dots, n\}$, let $f_k \in \mathcal{O}_A^{(2n-1)}$ denote the function obtained from f by substituting f for its k-th variable; more precisely,

$$f_k(x_1, \dots, x_{2n-1}) := \\ f(x_1, \dots, x_{k-1}, f(x_k, \dots, x_{k+n-1}), x_{k+n}, \dots, x_{2n-1}).$$

Clearly, if the operation $f \in \mathcal{O}_A^{(n)}$ is lazy then $f_k \preceq f$ or $f_k \in \mathcal{J}_A$ for every $k \in \mathbf{n}$.

III. Lazy idempotent operations

As a possible strengthening of the definition of laziness, one could require $[f] = \downarrow f$; we may call such operations f *very lazy*. Since the only unary operation in $\downarrow f$ is the diagonal function $f^*(x) = f(x, \dots, x)$ and $\mathrm{id} \in [f]$, we must have $f^* \equiv \mathrm{id}$ (i.e., f is idempotent) if f is very lazy. On the other hand, if f is an idempotent lazy operation, then we have $[f] = \mathcal{J}_A \cup \downarrow f = \downarrow f$, since $\mathrm{id} = f^* \in \downarrow f$. Thus, the very lazy operations are just the idempotent lazy operations. Prototypical examples of algebras with an idempotent lazy fundamental operation are rectangular bands. A natural n-ary generalization of this notion is the following. By an *n-ary rectangular band* we mean an algebra of the form $(A_1 \times \dots \times A_n; f)$, where $A_1, \dots, A_n$ are nonempty sets and f is defined by

$$f\left(\left(a_1^1, \dots, a_n^1\right), \dots, \left(a_1^n, \dots, a_n^n\right)\right) = \left(a_1^1, \dots, a_n^n\right)$$

for all $a_i^j \in A_i$ $(i, j \in \mathbf{n})$. Note that this algebra is the direct product of the algebras $\left(A_i; e_i^{(n)}\right)$ $(i \in \mathbf{n})$, and conversely, such a direct product of algebras with projection operations is always an n-ary rectangular band.

We prove in this section that up to isomorphism the idempotent lazy algebras are the n-ary rectangular bands. This result appears in [5]; here we give a more direct proof.

Lemma 1: If $f \in \mathcal{O}_A^{(n)}$ is an idempotent lazy operation depending on all of its variables, then f satisfies the following identity for every $k \in \mathbf{n}$:

$$f_k(y_1, \dots, y_{k-1}, x_1, \dots, x_n, y_{k+1}, \dots, y_n) = \\ f(y_1, \dots, y_{k-1}, x_k, y_{k+1}, \dots, y_n). \quad (1)$$

Proof: Laziness of f implies that $f_1 \preceq f$, i.e, $f_1(x_1, \dots, x_n, y_2, \dots, y_n) = f(z_1, \dots, z_n)$, where $z_i \in \{x_1, \dots, x_n, y_2, \dots, y_n\}$ for each $i \in \mathbf{n}$. Since f is idempotent, we have

$$\begin{aligned} f_1(x, \dots, x, y_2, \dots, y_n) &= f(f(x, \dots, x), y_2, \dots, y_n) \\ &= f(x, y_2, \dots, y_n), \end{aligned}$$

and this operation depends on all of its variables. Therefore, $z_1, \dots, z_n$ must be pairwise distinct variables, and at most one (and hence exactly one) of $x_1, \dots, x_n$ can appear among them: $\{z_1, \dots, z_n\} = \{x_{i_1}, y_2, \dots, y_n\}$ for some $i_1 \in \mathbf{n}$. Thus $f_1(x_1, \dots, x_n, y_2, \dots, y_n)$ does not depend on x_j whenever $j \neq i_1$, hence

$$\begin{aligned} f_1(x_1, \dots, x_n, y_2, \dots, y_n) &= f_1(x_{i_1}, \dots, x_{i_1}, y_2, \dots, y_n) \\ &= f(f(x_{i_1}, \dots, x_{i_1}), y_2, \dots, y_n) \\ &= f(x_{i_1}, y_2, \dots, y_n) \end{aligned}$$

by the idempotence of f.

A similar argument shows that for every $k \in \mathbf{n}$ there is $i_k \in \mathbf{n}$ such that f satisfies the identity

$$f(y_1, \dots, y_{k-1}, f(x_1, \dots, x_n), y_{k+1}, \dots, y_n) = \\ f(y_1, \dots, y_{k-1}, x_{i_k}, y_{k+1}, \dots, y_n). \quad (2)$$

Repeatedly applying this identity with $k = 1, \dots, n$, we obtain

$$f\left(f\left(x_1^1, \dots, x_n^1\right), \dots, f\left(x_1^n, \dots, x_n^n\right)\right) = f\left(x_{i_1}^1, \dots, x_{i_n}^n\right).$$

Let us replace each x_i^j by x_i:

$$f(f(x_1, \dots, x_n), \dots, f(x_1, \dots, x_n)) = f(x_{i_1}, \dots, x_{i_n}).$$

The left hand side of the above identity equals $f(x_1, \dots, x_n)$, as f is idempotent. Therefore, we can conclude that $f(x_1, \dots, x_n) = f(x_{i_1}, \dots, x_{i_n})$. This means that the map $k \mapsto i_k$ is a permutation of $\mathbf{n}$ and f is invariant under this permutation. Permuting the variables of the inner function $f(x_1, \dots, x_n)$ on the left hand side of (2) by this permutation, we get the identity

$$f(y_1, \dots, y_{k-1}, f(x_{i_1}, \dots, x_{i_n}), y_{k+1}, \dots, y_n) = \\ f(y_1, \dots, y_{k-1}, x_{i_k}, y_{k+1}, \dots, y_n),$$

which is equivalent to the statement of the lemma. ■

The identities (1) actually characterize idempotent lazy operations, as we shall see in the following theorem. Let us note that algebras corresponding to item (iii) below were called *diagonal algebras* in [5], and the equivalence of (i) and (iii) was proved in [5].

Theorem 2: For every idempotent operation $f \in \mathcal{O}_A^{(n)}$ the following three conditions are equivalent:

(i) f is lazy;
(ii) f satisfies the identities (1) for every $k \in \mathbf{n}$;
(iii) f satisfies the following identity:

$$f\left(f\left(x_1^1, \dots, x_n^1\right), \dots, f\left(x_1^n, \dots, x_n^n\right)\right) = \\ f\left(x_1^1, \dots, x_n^n\right). \quad (3)$$

Proof: Lemma 1 shows (i) $\implies$ (ii). For (ii) $\implies$ (i), let us consider an arbitrary operation $g \in [f]$. Then g is a term function of the algebra $\mathbb{A} = (A; f)$ corresponding to a term t, and we may assume that t is of minimum length. If t is just a variable, then $g \equiv \mathrm{id}$, while if the operation symbol f appears in t exactly once, then $g \preceq f$. Now assume that f appears at least twice in t. Then t is of the form $t = f(t_1, \ldots, t_{k-1}, f(s_1, \ldots, s_n), t_{k+1}, \ldots, t_n)$ with appropriate subterms t_i, s_j. Using (1), we see that t is equivalent to $f(t_1, \ldots, t_{k-1}, s_k, t_{k+1}, \ldots, t_n)$, which is shorter than t, contradicting the minimality of the length of t.

To prove (ii) $\Longrightarrow$ (iii), one just needs to apply (1) repeatedly for $k = 1, \ldots, n$ to the left hand side of (3). Finally, to verify (iii) $\Longrightarrow$ (ii), we set $x_1^j = \cdots = x_n^j = y_j$ for every $j \in \mathbf{n} \setminus \{k\}$ and $x_i^k = x_i$ for every $i \in \mathbf{n}$ in the left hand side of (3), and then use idempotence of f to obtain (1). ■

Theorem 3 ([5]): An idempotent operation $f \in \mathcal{O}_A^{(n)}$ is lazy if and only if $(A; f)$ is isomorphic to an n-ary rectangular band.

Proof: It is easy to verify that n-ary rectangular bands satisfy the identities (1), thus they are lazy by Theorem 2. Now assume that $f \in \mathcal{O}_A^{(n)}$ is an idempotent lazy operation. Let 0 denote an arbitrary fixed element of A, and let $A_k = \{f(0, \ldots 0, a, 0, \ldots, 0) : a \in A\}$ (with a appearing in the k-th position) for every $k \in \mathbf{n}$. From (3) it follows that the restriction of f to A_k is (the restriction of) the k-th projection, hence the direct product $\prod_{k \in \mathbf{n}} (A_k; f|_{A_k})$ is an n-ary rectangular band.

We prove that the map $\varphi\colon (A; f) \to \prod_{k \in \mathbf{n}} (A_k; f|_{A_k})$ defined by

$$\varphi(a) = \big(f(a, 0, \ldots, 0, 0), f(0, a, \ldots, 0, 0), \ldots, f(0, 0, \ldots, 0, a)\big) \tag{4}$$

is an isomorphism. Using (1) one can verify that φ is indeed a homomorphism:

$$\begin{aligned}\varphi(f(a_1, \ldots, a_n)) = \\ (f(a_1, 0, \ldots, 0, 0), f(0, a_2, \ldots, 0, 0), \ldots, f(0, 0, \ldots, 0, a_n)) \\ = f(\varphi(a_1), \ldots, \varphi(a_n)).\end{aligned}$$

It is also straightforward to verify with the help of (3) and the idempotence of f that the inverse of φ is the map $\prod_{k \in \mathbf{n}} (A_k; f|_{A_k}) \to (A; f)$, $(a_1, \ldots, a_n) \mapsto f(a_1, \ldots, a_n)$. ■

IV. Lazy binary operations

Throughout this section, let $f(x, y) = xy$ denote a binary operation on the set A depending on both of its variables. The dual of the groupoid $\mathbb{A} = (A; f)$ is $\mathbb{A}^d = (A; g)$, where $g(x, y) = yx$, and the dual of a groupoid variety V is $V^d = \{\mathbb{A}^d \colon \mathbb{A} \in V\}$. Clearly, a groupoid is lazy if and only if its dual is lazy. Now we have $f_1(x, y, z) = (xy)z$ and $f_2(x, y, z) = x(yz)$. If f is a lazy operation then $f_1, f_2 \in \mathcal{J}_A \cup \downarrow f$, hence $\mathbb{A}$ satisfies the identities $(xy)z = t_1$ and $x(yz) = t_2$ for some choice of the terms $t_1, t_2 \in \{x, y, z, x^2, y^2, z^2, xy, yx, yz, zy, xz, zx\}$. This gives us 144 possibilities; we will prove in this section that only 20 of these are possible. Examining these cases, we will find that lazy groupoids belong to 13 varieties, each being defined by two identities. To obtain a more concrete description of lazy groupoids, we will also describe the free algebras in these varieties; all lazy groupoids are quotients of these free lazy groupoids.

Lemma 4: If the binary operation $\cdot$ satisfies the identity $(xy)z = t_1$ for some $t_1 \in \{x, y, z, zy, zx, yx\}$, or it satisfies $x(yz) = t_2$ for some $t_2 \in \{z, y, x, yx, zx, zy\}$, then $\cdot$ is essentially at most unary.

Proof: The identity $(xy)z = t_1(x, y, z)$ implies $t_1(xy, z, u) = ((xy)z)u = t_1(x, y, z) \cdot u$. If $t_1 = x$, then we obtain $xy = xu$, which shows that xy does not depend on y. Similarly, for $t_1 = y$ we obtain $z = yu$, and for $t_1 = z$ we get $u = zu$. If $t_1 = zy$, then we get $uz = (zy)u$, which yields $uz = uy$ after applying $(zy)u = t_1(z, y, u)$. For $t_1 = zx$ we can argue similarly: $u(xy) = (zx)u = t_1(z, x, u) = uz$, showing that uz does not depend on z.

Now let us consider the case $t_1 = yx$. Then we have

$$\begin{aligned} z(xy) &= t_1(xy, z, u) = ((xy)z)u \\ &= t_1(x, y, z) \cdot u = (yx)u = t_1(y, x, u) = xy, \end{aligned}$$

which immediately implies $(uv)(xy) = xy$. On the other hand, $(uv)(xy) = t_1(u, v, xy) = vu$. Thus, we have $xy = vu$, i.e., f is a constant operation.

The identites $x(yz) = t_2$ are the duals of the above ones. ■

Now we are left with 36 pairs (t_1, t_2); these possibilities are summarized in Table I. We will prove in the next lemma that the entries marked by '$-$' contradict the essentiality of f, while the other cases give rise to 7 varieties $L_1, \ldots, L_7$ of lazy groupoids together with their duals (note that L_7 is selfdual).

Lemma 5: Let $(A; f) = (A; \cdot)$ be a groupoid and assume that the operation f depends on both of its variables. If $(A; f)$ is a lazy groupoid then it belongs to one of the 13 varieties $L_1, \ldots, L_7, L_1^d, \ldots, L_6^d$, which are defined by the following identities:

$$\begin{aligned}
L_1 &: \quad (xy)z = x^2, \quad x(yz) = x^2; \\
L_2 &: \quad (xy)z = x^2, \quad x(yz) = xy; \\
L_3 &: \quad (xy)z = xy, \quad x(yz) = x^2; \\
L_4 &: \quad (xy)z = xz, \quad x(yz) = x^2; \\
L_5 &: \quad (xy)z = xy, \quad x(yz) = xy; \\
L_6 &: \quad (xy)z = xz, \quad x(yz) = xy; \\
L_7 &: \quad (xy)z = xz, \quad x(yz) = xz.
\end{aligned}$$

Proof: We can derive the following three identites from $(xy)z = t_1$ and $x(yz) = t_2$:

$$t_1(x, y, zu) = (xy)(zu) = t_2(xy, z, u); \tag{5a}$$
$$t_1(x, yz, u) = (x(yz))u = t_2(x, y, z) \cdot u; \tag{5b}$$
$$x \cdot t_1(y, z, u) = x((yz)u) = t_2(x, yz, u). \tag{5c}$$

In all the 16 cases marked by '$-$' in Table I, at least one of the above three identities contradicts the essentiality of the operation f. We work out the details only for $t_1 = y^2, t_2 = xy$ (here we will need the identity (5c)); the other cases are similar

	x^2	y^2	z^2	xy	yz	xz	t_2
x^2	L_1	$L_1 \cap L_1^d$	$L_1 \cap L_1^d$	L_2	–	–	
y^2	$L_1 \cap L_1^d$	$L_1 \cap L_1^d$	$L_1 \cap L_1^d$	–	–	–	
z^2	$L_1 \cap L_1^d$ (!)	$L_1 \cap L_1^d$	L_1^d	–	L_3^d	L_4^d	
xy	L_3	–	–	L_5	–	–	
yz	–	–	L_2^d	–	L_5^d	L_6^d	
xz	L_4	–	–	L_6	–	L_7	
t_1							

TABLE I

	$x \cdot y$	$(x,y) \cdot z$	$x \cdot (y,z)$	$(x,y) \cdot (z,u)$
L_1	(x,y)	(x,x)	(x,x)	(x,x)
L_2	(x,y)	(x,x)	(x,y)	(x,x)
L_3	(x,y)	(x,y)	(x,x)	(x,y)
L_4	(x,y)	(x,z)	(x,x)	(x,x)
L_5	(x,y)	(x,y)	(x,y)	(x,y)
L_6	(x,y)	(x,z)	(x,y)	(x,z)
L_7	(x,y)	(x,z)	(x,z)	(x,u)

TABLE II

or even simpler:

$$\begin{aligned} xz^2 &= x \cdot t_1 (y,z,u) = x((yz)u) \\ &= t_2 (x, yz, u) = x(yz) = xy. \end{aligned}$$

Now it only remains to verify the entries marked by $L_1 \cap L_1^d$ in Table I. These can be handled with the help of the identites (5); again, we provide details only for one case, namely for $t_1 = y^2, t_2 = x^2$. Note that the variety $L_1 \cap L_1^d$ is axiomatized by the identites $(xy)z = x(yz) = x^2 = z^2$. This means that a groupoid $\mathbb{A}$ belongs to $L_1 \cap L_1^d$ if and only if $\mathbb{A}$ is a semigroup and there is a constant $c \in A$ such that $xyz = x^2 = c$. It is clear that such semigroups satisfy $(xy)z = y^2$ and $x(yz) = x^2$. Conversely, assume now that $(xy)z = y^2$ and $x(yz) = x^2$ hold in a groupoid $\mathbb{A}$. Let us write out (5b):

$$(yz)^2 = t_1 (x, yz, u) = (x(yz))u = t_2 (x,y,z) \cdot u = x^2 u.$$

We can conclude that $(yz)^2$ depends neither on y nor on z, hence there is a constant $c \in A$ such that $(yz)^2 = c$. Now let us use (5a):

$$y^2 = t_1 (x, y, zu) = (xy)(zu) = t_2 (xy, z, u) = (xy)^2.$$

This implies that y^2 is constant c, hence $\mathbb{A}$ satisfies $(xy)z = x(yz) = x^2 = c$; therefore, $\mathbb{A} \in L_1 \cap L_1^d$.

The entry marked by $L_1 \cap L_1^d$ (!) in Table I is special in the sense that the identities $(xy)z = z^2, x(yz) = x^2$ do not guarantee laziness. Here (5a) yields $(zu)^2 = (xy)^2$, i.e., $(xy)^2$ is constant. Since the only constant in $\mathcal{J}_A \cup \downarrow f$ is the diagonal operation $f^*(x) = x^2$, we see that x^2 must be constant. Then we have $(xy)z = x(yz) = x^2 = z^2$, hence $\mathbb{A} \in L_1 \cap L_1^d$. ∎

In order to complete the description of lazy groupoids, we still need to verify that every groupoid belonging to the varieties $L_1, \ldots, L_7, L_1^d, \ldots, L_6^d$ defined in Lemma 5 is indeed lazy. We shall also show that none of the 13 cases is trivial in the sense that there exists groupoids with essentially binary multiplication in each of these varieties. We can achieve both goals by describing free algebras in the varieties $L_1, \ldots, L_7$ (we omit $L_1^d, \ldots, L_6^d$). In the following, whenever we use one of the two defining identities for any one of our varieties, we write "$\overset{1.}{=}$" or "$\overset{2.}{=}$" to indicate whether we have used the first or the second indentity (as listed in Lemma 5).

Lemma 6: Let V be one of the varieties $L_1, \ldots, L_7$ and let X be a set of variables. Then the free algebra $\mathbb{F}_X(V)$ of V freely generated by X is isomorphic to the groupoid $(X \cup (X \times X); \cdot)$ whose multiplication is given by Table II. If X has at least two elements, then this multiplication is essentially binary and $\mathbb{F}_X(V)$ is a lazy groupoid, hence every essential member of V is lazy.

Proof: We prove the lemma only for L_2; the other varieties can be dealt with in an analogous way. Let X be a set with at least two elements, and let us consider the groupoid $\mathbb{A} = (X \cup (X \times X); \cdot)$, where the multiplication $\cdot$ is defined by the row corresponding to L_2 in Table II. It is straightforward to verify that this operation is essential and $\mathbb{A} \in L_2$.

We prove by term induction that every term of L_2 is equivalent to x or xy for some $x, y \in X$. Let t be a term of L_2 over the set X of variables that contains at least two multiplications (i.e., at least three, not necessarily distinct variables). Then $t = s_1 \cdot s_2$, where the terms s_1 and s_2 are shorter than t, hence, by the induction hypothesis, they are equivalent to a variable or to a product of two variables. Therefore, we have the following three possibilities with some (not necessarily distinct) variables $x, y, z, u \in X$:

$$\begin{aligned} s_1 &= xy, & s_2 &= z & &\Longrightarrow s_1 s_2 = (xy)z \overset{1.}{=} x^2; \\ s_1 &= x, & s_2 &= yz & &\Longrightarrow s_1 s_2 = x(yz) \overset{2.}{=} xy; \\ s_1 &= xy, & s_2 &= zu & &\Longrightarrow s_1 s_2 = (xy)(zu) \overset{1.}{=} x^2. \end{aligned} \tag{6}$$

Thus, every term over L_2 is indeed equivalent to a variable or a product of two variables, showing that every essential member of L_2 is a lazy groupoid.

Let us note that the terms $x \ (x \in X)$ and $xy \ (x, y \in X)$ are pairwise inequivalent over L_2; in fact, it is easy to see that they are pairwise inequivalent already over $\mathbb{A}$. This fact together with (6) shows that $\mathbb{F}_X(L_2)$ is isomorphic to $\mathbb{A}$. ∎

Theorem 7: A groupoid is lazy if and only if it belongs to one of the 13 varieties $L_1, \ldots, L_7, L_1^d, \ldots, L_6^d$.

Proof: The "only if" part is covered by Lemma 5, while the "if" part follows from Lemma 6 and its dual. ∎

Remark 1: In order to describe lazy groupoids more explicitly (i.e., up to isomorphism) one could determine congruences and the corresponding quotients of the free algebras given in Lemma 6. We leave this task for future work.

V. Application to essentially minimal groupoids

As the following lemma shows, for binary operations there is a strong relationship between laziness and essential minimality.

Lemma 8: Let $\mathbb{A} = (A; \cdot)$ be an essential lazy groupoid, and let $\mathcal{C}$ denote the clone of term functions of $\mathbb{A}$. If $\mathbb{A}$ is idempotent, then $\mathcal{C}$ is a minimal clone, whereas if $\mathbb{A}$ is not idempotent then $\mathcal{C}$ is an essentially minimal clone.

Proof: Laziness of $\mathbb{A}$ implies that every operation in $\mathcal{C}$ is equivalent to one of the operations xy, x^2 and x. If $x^2 = x$ holds in $\mathbb{A}$, then $\mathcal{C}$ has exactly two subclones, namely $\mathcal{C} = [xy]$ and $\mathcal{J}_A = [x]$, hence $\mathcal{C}$ is a minimal clone. Otherwise, $\mathcal{C}$ has exactly three subclones, namely $\mathcal{C} = [xy]$, $\left[x^2\right]$ and $\mathcal{J}_A = [x]$, hence $\mathcal{C}$ is an essentially minimal clone. ■

Theorem 9: Every essential groupoid in the varieties $L_1, \ldots, L_6, L_1^d, \ldots, L_6^d$ has an essentially minimal clone. The variety L_7 contains groupoids with essentially minimal clones as well as groupoids with minimal clones (the latter are exactly the rectangular bands).

Proof: If $\mathbb{A}$ is an idempotent groupoid in one of the varieties L_1, L_2, L_3, L_5, then $\mathbb{A}$ satisfies the identity $xz = (xx)\,z \overset{1.}{=} x^2$, hence $\mathbb{A}$ is not essential. Similarly, if $\mathbb{A} \in L_4$ is idempotent, then $\mathbb{A}$ is not essential, since it satisfies $xy = x\,(yy) \overset{2.}{=} x^2$. For L_6 we can draw the same conclusion from $xy = (xy)\,(xy) \overset{1.}{=} x\,(xy) \overset{2.}{=} x^2$. Therefore, by Lemma 8, we have that every essential groupoid in the varieties $L_1, \ldots, L_6, L_1^d, \ldots, L_6^d$ has an essentially minimal clone.

The variety L_7 consists of semigroups satisfying $xyz = xz$, thus the idempotent members of L_7 are just the rectangular bands, which are known to have minimal clones (this also follows from Lemma 8). However, L_7 also contains nonidempotent semigroups (for example $\mathbb{F}_X\,(L_7)$; see Lemma 6), and these have essentially minimal clones, again by Lemma 8. ■

Remark 2: We have seen in the above theorem that L_7 contains a proper essential subvariety, namely the variety of rectangular bands. One can verify that L_1 and L_1^d have two proper essential subvarieties, namely the semigroup varieties defined by $xyz = x^2 = y^2$ (this is just $L_1 \cap L_1^d$) and $xyz = x^2, xy = yx$, respectively. On the other hand, the varieties $L_2, \ldots, L_6, L_2^d, \ldots, L_6^d$ have no proper essential subvarieties.

Our main motivation for studying lazy clones is Proposition 4.2 of [4], which reduces the description of certain types of essentially minimal clones to the description of lazy clones. In order to recall this result, we need to introduce some notation and terminology. Let $f\,(x,y) = xy$ be a binary operation on a nonempty set A, and, as before, let f^* denote the corresponding diagonal operation: $f^*\,(x) = f\,(x,x) = x^2$. Let us denote by $\Gamma\,(f)$ the set of periodic points of f^*:

$$\Gamma\,(f) = \{a \in A\colon\ (f^*)^n\,(a) = a \text{ for some } n \in \mathbb{N}\}\,.$$

Two types of essentially minimal clones are defined in [4]; here we can state this definition in a slightly simpler form, as we are dealing only with lazy binary operations. Let f be a non-idempotent lazy binary operation (then $[f]$ is an essentially minimal clone by Lemma 8). If the restriction of f to $\Gamma\,(f)$ is essential, then we say that the operation f (or the clone $[f]$) is of type A, otherwise it is of type B. According to Proposition 4.2 of [4], essentially minimal clones of type A are closely related to lazy clones. Here we formulate this result for binary operations.

Theorem 10 ([4]): Let $f\,(x,y) = xy$ be a binary operation on A such that the restriction of f to $\Gamma\,(f)$ is essential, and let $\mathbb{A} = (A; \cdot)$ be the corresponding groupoid. Then $[f]$ is an essentially minimal clone if and only if the following are satisfied.

(i) $\emptyset \neq \Gamma\,(f) \subset A$
(ii) $\mathbb{A}$ satisfies the identity $xy = x^2y^2$.
(iii) Either
  (a) $\Gamma\,(f)$ is closed under multiplication, and the groupoid $(\Gamma\,(f)\,; \cdot)$ has a minimal clone, or
  (b) $\Gamma\,(f)$ is not closed under multiplication, the operation $(xy)^2$ is not essential, and $[f]$ is minimal among essential lazy clones.

This theorem divides essentially minimal clones into two subtypes, which we will call subtype $\mathrm{A_a}$ and $\mathrm{A_b}$, according to whether condition (a) or (b) holds in (iii). It is widely accepted that the full description of minimal clones is a very difficult problem that is beyond reach even for binary operations, and therefore it does not seem a feasible task to determine essentially minimal groupoids of subtype $\mathrm{A_a}$. However, from Theorem 7 we can easily deduce the following description of essentially minimal groupoids of subtype $\mathrm{A_b}$.

Theorem 11: An essential groupoid has an essentially minimal clone of subtype $\mathrm{A_b}$ if and only if it belongs to the variety L_6 or L_6^d.

Proof: Let $f\,(x,y) = xy$ be a binary operation on A such that $[f]$ is an essentially minimal groupoid of subtype $\mathrm{A_b}$, and let $\mathbb{A} = (A; \cdot)$ be the corresponding groupoid. Then $[f]$ is a lazy clone, hence, by Theorem 7, we can assume (up to duality) that $\mathbb{A}$ belongs to one of the varieties $L_1, \ldots, L_7$. Each of these varieties satisfies $\left(x^2\right)^2 = x^2$, hence $(f^*)^2 = f^*$. This means that the set of periodic points equals the range of f^* (moreover, it coincides with the set of fixed points of f^*): $\Gamma\,(f) = \left\{a^2\colon a \in A\right\}$. Thus $[f]$ is of type A if and only if x^2y^2 is essential. The varieties $L_1, \ldots, L_5$ satisfy $x^2y^2 = x^2$, hence these are of type B, whereas L_6 and L_7 satisfy $x^2y^2 = xy$, hence essential groupoids in these varieties are of type A.

If $\mathbb{A} \in L_7$, then $\Gamma\,(f)$ is closed under multiplication: $x^2y^2 = xy = (xy)^2$ follows from the identity $xyz = xz$, which holds in L_7. Moreover, the multiplication is idempotent on $\Gamma\,(f)$, thus $(\Gamma\,(f)\,; \cdot)$ is a rectangular band, and therefore it has a minimal clone. This shows that every essentially minimal (i.e., non-idempotent, cf. Lemma 8) groupoid in L_7 is of type $\mathrm{A_a}$

Finally, we prove that every essential groupoid $\mathbb{A} \in L_6$ is of subtype $\mathrm{A_b}$. Suppose for contradiction that $\Gamma\,(f)$ is closed under multiplication, i.e., for every $a, b \in A$ there exists $c \in A$ such that $a^2 b^2 = c^2$. Then we have

$$ab \overset{*}{=} a^2 b^2 = c^2 \overset{*}{=} c^2 c^2 = (ab)(ab) \overset{*}{=} a^2. \tag{7}$$

(At the steps indicated by $\overset{*}{=}$ we used the identity $(xy)(zu) = xz$, which can be deduced from the defining identites of L_6 as follows: $(xy)(zu) \overset{1.}{=} x(zu) \overset{2.}{=} xz$.) Since (7) holds for all $a, b \in A$, we can conclude that $\mathbb{A}$ satisfies the identity $xy = x^2$, contradicting the essentiality of the multiplication. The other two conditions of (iii)/(b) are easily verified: $(xy)^2 \overset{*}{=} x^2$ shows that $(xy)^2$ is not essential, and we have seen in the proof of Lemma 8 that $[f]$ has only two proper subclones, and both of them are essentially unary. ■

Remark 3: In [2] several varieties V_1, V_2, V_3, V_4, V_{5p} (p is a prime) of essentially minimal groupoids were given. One can verify that L_5 is a subvariety of V_2 (consequently $L_5^d \subseteq V_2^d$) and L_3 is a subvariety of V_4 (consequently $L_3^d \subseteq V_4^d$). Binary essentially minimal clones on the three-element set were determined in [3]; there are 16 such clones up to conjugacy. Only one of them is lazy, namely the clone of the groupoid A3; this groupoid belongs to the variety L_6^d.

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