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Smoothing properties for a Hirota-Satsuma systems

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Abstract

We study local existence and smoothing properties for the initial value problem associated to Hirota-Satsuma systems that describes an interaction of two long waves with different dispersion relations.

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Mathematics Subject Classification: $35Q53, 47J35$

1 Introduction

This paper is concerned with gain in regularity of solutions of the Hirota-Satsuma system

$$u_t - a\,u_{xxx} + 6\,u\,u_x = 2\,b\,v\,v_x \tag{1.1}$$

$$v_t + v_{xxx} + 3\,u\,v_x = 0 \tag{1.2}$$

$$u(x,\,0) = u_0(x) \tag{1.3}$$

$$v(x,\,0) = v_0(x) \tag{1.4}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$ and $u = u(x,\,t)$, $v = v(x,\,t)$ are real unknown functions. a and b are real constants with $b > 0$. In equation (1.1), $2\,b\,v\,v_x$ acts as a force term on the Korteweg-de Vries(KdV) wave system with the linear dispersion relation $\omega = a\,\kappa^3$. This system was introduced by Hirota and Satsuma [19] to describe and interaction of two long waves with different dispersion relations. If there is no effect of one of the long waves on the other, the latter obeys the ordinary KdV equation. They showed that for all values of a an b this system possesses three conservation laws. Indeed

$$\begin{aligned} I_1 &= u, \qquad I_2 = \frac{1}{2}\int_{\mathbb{R}}\left[u^2 + \frac{2}{3}\,b\,v^2\right]dx \\ I_3 &= \int_{\mathbb{R}}\left[\frac{1}{2}\,(1+a)\,u_x^2 + b\,v_x^2 - (1+a)\,u^3 - b\,u\,v^2\right]dx. \end{aligned} \tag{1.5}$$

They further showed that for all values of b, but only $a = \frac{1}{2}$, the system possesses two further conservation laws

$$\begin{aligned} I_4 \;&=\; u^4 - 2\,u\,u_x^2 + \frac{1}{5}\,u_{xx}^2 \\ &\quad + \frac{4}{5}\left(u^2\,v^2 + \frac{2}{3}\,u\,v\,v_{xx} + \frac{8}{3}\,u\,v_x^2 - \frac{13}{18}\,v_{xx}^2\right) + \frac{4}{15}\,b^2\,v^4 \end{aligned} \tag{1.6}$$

$$\begin{aligned} I_5 &= u^5 - 5\,u^2\,u_x^2 + u\,u_{xx}^2 - \frac{1}{14}\,u_{xxx}^2 \\ &\quad + \frac{1}{21}\,b\left(20\,u^3\,v^2 - 10\,u_x^2\,v^2 - +20\,u^2\,v\,v_{xx} + 40\,u^2\,v_x^2 + 4\,u\,v\,v_{xxxx} + 56\,u\,v_x\,v_{xxx} + 12\,u\,v_{xx}^2 + 8\,v_{xxx}^2\right) \\ &\quad + \frac{20}{63}\,b^2(u\,v^4 - 4\,v^2\,v_x^2). \end{aligned} \tag{1.7}$$

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The system (1.1)-(1.4) has been studied by several authors, see [8, 19, 20] and the references there. In 1986, N. Hayashi *et al.* [16] showed that for the nonlinear Schrödinger equation (NLS): $i\,u_t + u_{xx} = \lambda\,|u|^{p-1}\,u,\ (x,\,t) \in \mathbb{R}\times\mathbb{R}$ with initial condition $u(x,\,0) = u_0(x),\ x \in \mathbb{R}$ and a certain assumption on λ and p, all solutions of finite energy are smooth for $t \neq 0$ provided the initial functions in $H^1(\mathbb{R})$(or on $L^2(\mathbb{R})$) decay sufficiently rapidly as $|x| \to \infty$. The main tool is the operator J defined by $Ju = e^{i\,x^2/4\,t}\,(2\,i\,t)\,\partial_x(e^{-\,i\,x^2/4\,t}\,u) = (x + 2\,i\,t\,\partial_x)u$ which has the remarkable property that it commutes with the operator L defined by $L = (i\,\partial_t + \partial_x^2)$, namely $LJ - JL = [L,\,J] = 0$.
For the Korteweg-de Vries type equation (KdV), Saut and Temam [29] remarked that a solution u cannot gain or lose regularity. They showed that if $u(x,\,0) = u_0(x) \in H^s(\mathbb{R})$ for $s \geq 2$, then $u(\,\cdot\,,\,t) \in H^s(\mathbb{R})$ for all $t > 0$. The same result was obtained independently by Bona and Scott [4] though a different method. For the KdV equation on the line, Kato [22] motivated by work of Cohen [9] showed that if $u(x,\,0) = u_0(x) \in L^2_b \equiv H^2(\mathbb{R}) \cap L^2(e^{bx}\,dx)(b > 0)$ then the solution $u(x,\,t)$ of the KdV equation becomes C^∞ for all $t > 0$. A main ingredient in the proof was the fact that formally the semi-group $S(t) = e^{-\,\partial_x^3}$ in $L^2_b(\mathbb{R})$ is equivalent to $S_b(t) = e^{-\,t\,(\partial_x - b)^3}$ in $L^2(\mathbb{R})$ when $t > 0$. One would be inclined to believe that this was a special property of the KdV equation. This is not however the case. The effect is due to the dispersive nature of the linear part of the equation. Kruzkov and Faminskii [26] proved that $u(x,\,0) = u_0(x) \in L^2(\mathbb{R})$ such that $x^\alpha\,u_0(x) \in L^2((0,\,+\infty))$ the weak solution of the KdV equation has l-continuous space derivatives for all $t > 0$ if $l < 2\,\alpha$. The proof of this result is based on the asymptotic behavior of the Airy function and its derivatives, and on the smoothing effect of the KdV equation which was found in [22, 26]. While the proof of Kato appears to depend on special a priori estimates, some of this mystery has been resolved by the result of local gain of finite regularity for various others linear and nonlinear dispersive equation due to Constantin and Saut [13], Ginibre and Velo [15] and others. However, all of them require growth conditions on the nonlinear term. In 1992, W. Craig *et al.* [12] proved for fully nonlinear KdV equation $u_t + f(u_{xxx},\,u_{xx},\,u_x,\,u,\,x,\,t) = 0$ and certain additional assumption over f that C^∞ solutions $u(x,\,t)$ are obtained for all $t > 0$ if the initial data $u_0(x)$ decays faster than polynomially on $\mathbb{R}^+ = \{x \in \mathbb{R} :\ x > 0\}$ and has certain initial Sobolev regularity. Following with this idea, in 2001, O. Vera and G. Perla Menzala [33, 34] proved that the solutions of the initial value problem (P) are locally smooth due to the dispersive of the coupled system of equations of Korteweg - de Vries type

$$(P)\begin{cases} u_t + u_{xxx} + a_3\,v_{xxx} + u\,u_x + a_1\,v\,v_x + a_2\,(u\,v)_x = 0 \qquad x \in \mathbb{R}, \quad t \geq 0 \\ b_1\,v_t + v_{xxx} + b_2\,a_3\,u_{xxx} + v\,v_x + b_2\,a_2\,u\,u_x + b_2\,a_1\,(u\,v)_x = 0 \\ u(x,\,0) = u_0(x) \\ v(x,\,0) = v_0(x) \end{cases}$$

where $u = u(x,\,t),\ v = v(x,\,t)$ are real-valued functions of the variables x and t and $a_1,\,a_2,\,a_3,\,b_1,\,b_2$ are real constants with $b_1 > 0$ and $b_2 > 0$. The original coupled system is

$$(\widehat{P})\begin{cases} u_t + u_{xxx} + a_3\,v_{xxx} + u^p\,u_x + a_1\,v^p\,v_x + a_2\,(u^p\,v)_x = 0 \quad \text{in } -\infty < x < +\infty, \quad t \geq 0 \\ b_1\,v_t + v_{xxx} + b_2\,a_3\,u_{xxx} + v^p\,v_x + b_2\,a_2\,u^p\,u_x + b_2\,a_1\,(u\,v^p)_x = 0 \\ u(x,\,0) = u_0(x) \\ v(x,\,0) = v_0(x) \end{cases}$$

where $u = u(x,\,t),\ v = v(x,\,t)$ are real-valued functions of the variables x and t and $a_1,\,a_2,\,a_3,\,b_1,\,b_2$ are real constants with $b_1 > 0$ and $b_2 > 0$. The power p is an integer larger than or equal to one. The system $(\widehat{P})$ has the structure of a pair of Korteweg - de Vries equations coupled through both dispersive and nonlinear effects. In the case $p = 1$, system $(\widehat{P})$ was derived by Gear and Grimshaw in 1984 [14] as a model to describe the strong interaction of weakly nonlinear, long waves. Mathematical results on the system $(\widehat{P})$ were given by J. Bona et al. [3]. They proved that $(\widehat{P})$ is globally well posed in $H^s(\mathbb{R})\times H^s(\mathbb{R})$ for any $s \geq 1$ provided $|a_3| < 1/\sqrt{b_2}$. The system $(\widehat{P})$ has been intensively studied by several authors. See [2, 3] and the references therein. We have the following conservation laws

$$\phi_1(u) = \int_{\mathbb{R}} u\,dx, \qquad \phi_2(v) = \int_{\mathbb{R}} v\,dx, \qquad \phi_3(u,\,v) = \int_{\mathbb{R}} (b_2u^2 + b_1v^2)dx \tag{1.8}$$

The time-invariance of the functionals ϕ_1 and ϕ_2 expresses the property that the mass of each mode separately is conserved during interaction, while that of ϕ_3 is a expression of the conservation of energy for the system of two models taken as a whole. Solutions of $(\widehat{P})$ satisfy an additional conservation law which is revealed by the time-invariance of the functional

$$\phi_4 = \int_{\mathbb{R}} \left(b_2\, u_x^2 + v_x^2 + 2\, b_2\, a_3\, u_x\, v_x - b_2\, \frac{u^3}{3} - b_2\, a_2\, u^2\, v - b_2\, a_2\, u^2\, v - b_2\, a_1\, u\, v^2 - \frac{v^3}{3} \right) dx \tag{1.9}$$

The functional ϕ_4 is a Hamiltonian for the system $(\widehat{P})$ and if $b_2 a_3^2 < 1$, ϕ_4 will be seen to provide an a priori estimate for the solutions $(u,\, v)$ of $(\widehat{P})$ in the space $H^1(\mathbb{R}) \times H^1(\mathbb{R})$. Furthermore, the linearization of $(\widehat{P})$ about the rest state can be reduced to two, linear Korteweg - de Vries equations by a process of diagonalization. Using this remark and the smoothing properties (in both the temporal and spatial variables) for the linear Korteweg - de Vries derived by Kato [22, 23], Kenig, Ponce and Vega [24, 25] it will be shown that $(\widehat{P})$ is locally well-posed in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for any $s \geq 1$ whenever $\sqrt{b_2} a_3 \neq 1$. Indeed, all this appears in the following Theorem:

Theorem 1.1 *(See [3]). Let $s \geq 1$ and $(u_0(x),\, v_0(x)) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$. Consider the system $(\widehat{P})$ together with these initial conditions. Let $p \geq 1$, p be an integer and a_j, b_k (real) constants $\sqrt{b_2}\, a_3 < 1$, $b_1 > 0$, $b_2 > 0$ $(j = 1,\, 2,\, 3\,;\, k = 1,\, 2)$. Then, there exists $T_0 = T_0(\|(u_0(\cdot),\, v_0(\cdot))\|_{Y_m},\, p) > 0$ and a unique solution $(u(x,\, t),\, v(x,\, t)) \in X_s(T_0) \times X_s(T_0)$, of $(\widehat{P})$ with initial data $(\varphi(x),\, \psi(x))$ where $X_s(T_0) = C(0,\, T_0 : H^s(\mathbb{R})) \cap C^1(0,\, T_0 : H^{s-3}(\mathbb{R}))$. Moreover, the pair $(u,\, v)$ depends continuously on $(u_0(x),\, v_0(x))$ in the sense that the map $(u_0(x),\, v_0(x)) \to (u,\, v)$ is continuous from Y_s into the space $X_s(T_0) \times X_s(T_0)$, where s is a real number, $Y_s = H^s(\mathbb{R}) \times H^s(\mathbb{R})$ with the norm $\|(u,\, v)\|_{Y_s}^2 = \|u\|_{H^s(\mathbb{R})}^2 + \|v\|_{H^s(\mathbb{R})}^2$.*

This result was improved by J. Marshall et al.[1] They proved that the system $(\widehat{P})$(with $p = 1$), is globally well-posed in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ provided that $|a_3| \neq 1/\sqrt{b_2}$. This kind of dispersive problem exhibits the interesting phenomenon of dispersive smoothing, that is, If the initial data belong to a certain Sobolev space and has a good behavior as $|x| \to +\infty$, then the solutions in any time $t \neq 0$ are smoother than the initial data.

Our aim in this paper, is to study gain in regularity for the equation (1.1)-(1.4). Specifically, we prove conditions on (1.1)-(1.4) for which initial data (u_0, v_0) possessing sufficient decay at infinity and minimal amount of regularity will lead to a unique solution $(u(t),\, v(t)) \in C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R})$ for $0 < t < T$, where T is the existence time of the solution. This paper is organized as follows: Section 2 outlines briefly the notation and terminology to be used subsequently. Section 3 we prove the main inequality. Section 4 we prove an important a priori estimate. Section 5 we prove a basic-local-in-time existence and uniqueness theorem. Section 6 we develop a series of estimates for solutions of equations (1.1)-(1.4) in weighted Sobolev norms. These provide a starting point for the a priori gain of regularity. In section 7 we prove the following theorem:

Theorem 1.2 *(Main Theorem) Let $T > 0$, $a < 0$ and $(u,\, v)$ be a solution of (1.1)-(1.4) in the region $\mathbb{R} \times [0,\, T]$ such that*

$$(u,\, v) \in L^\infty([0,\, T] : H^3(W_{0\, L\, 0})) \times L^\infty([0,\, T] : H^3(W_{0\, L\, 0})) \tag{1.10}$$

for some $L \geq 2$. Then

$$u \in L^\infty([0,\, T] : H^{3+l}(W_{\sigma,\, L-l,\, l})) \cap L^2([0,\, T] : H^{4+l}(W_{\sigma,\, L-l-1,\, l}))$$
$$v \in L^\infty([0,\, T] : H^{3+l}(W_{\sigma,\, L-l,\, l})) \cap L^2([0,\, T] : H^{4+l}(W_{\sigma,\, L-l-1,\, l}))$$

for all $0 \leq l \leq L-1$ and all $\sigma > 0$ where the weight classes will be defined in Section 2.

2 Preliminaries

We consider the initial value problem

$$u_t - a\,u_{xxx} + 6\,u\,u_x = 2\,b\,v\,v_x \tag{2.1}$$
$$v_t + v_{xxx} + 3\,u\,v_x = 0 \tag{2.2}$$
$$u(x,\,0) = u_0(x) \tag{2.3}$$
$$v(x,\,0) = v_0(x) \tag{2.4}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$ and $u = u(x,\,t)$, $v = v(x,\,t)$ are real unknown functions. b and a are real constants with $b > 0$.

Notation 2.1 *We write* $\partial = \partial/\partial x, \quad \partial_t = \partial/\partial t$ *and we abbreviate* $u_j = \partial^j u$.

Definition 2.2 *A function* $\xi = \xi(x,\,t)$ *belongs to the weight class* $W_{\sigma\;i\;k}$ *if it is a positive* C^∞ *function on* $\mathbb{R} \times [0,\,T]$, $\partial\xi > 0$ *and there is a constant* C_j, $0 \le j \le 5$ *such that*

$$0 < C_1 \le t^{-\,k}\,e^{-\,\sigma\,x}\,\xi(x,\,t) \le C_2 \qquad \forall\,x < -1, \quad 0 < t < T. \tag{2.5}$$
$$0 < C_3 \le t^{-\,k}\,x^{-\,i}\,\xi(x,\,t) \le C_4 \qquad \forall\,x > 1, \quad 0 < t < T. \tag{2.6}$$
$$\left(t\,|\partial_t\xi| + |\partial^j\xi|\right)/\xi \le C_5 \quad \forall\,(x,\,t) \in \mathbb{R}\times[0,\,T],\ \forall\,j \in \mathbb{N}. \tag{2.7}$$

Remark 2.3 *We shall always take* $\sigma \ge 0$, $i \ge 1$ *and* $k \ge 0$.

Example 2.4 *Let*

$$\xi(x) = \begin{cases} 1 + e^{-\,1/x} & \text{for } x > 0 \\ 1 & \text{for } x \le 0 \end{cases}$$

then $\xi \in W_{0\;i\;0}$.

Definition 2.5 *Let* N *be a positive integer. By* $H^N(W_{\sigma,\,i,\,k})$ *we denote the Sobolev space on* $\mathbb{R}$ *with a weight; that is, with the norm*

$$\|v\|^2_{H^N(W_{\sigma,\,i,\,k})} = \sum_{j=0}^{N}\int_{\mathbb{R}} |\partial^j v(x)|^2\,\xi(x,\,t)\,dx < +\infty$$

for any $\xi \in W_{\sigma\;i\;k}$ *and* $0 < t < T$. *Even though the norm depends on* ξ, *all such choices lead to equivalent norms.*

Remark 2.6 $H^s(W_{\sigma\;i\;k})$ *depends on* t *(because* $\xi = \xi(x,\,t)$*).*

Lemma 2.7 *(See [7]). For* $\xi \in W_{\sigma\;i\;0}$ *and* $\sigma \ge 0$, $i \ge 0$, *there exists a constant* $C > 0$ *such that, for* $u \in H^1(W_{\sigma\;i\;0})$

$$\sup_{x\in\mathbb{R}} |\xi\,u^2| \le C\int_{-\infty}^{+\infty}\left(|u|^2 + |\partial u|^2\right)\xi\,dx.$$

Lemma 2.8 *(The Gagliardo-Nirenberg inequality). Let* q, r *be any real numbers satisfying* $1 \le q,\,r \le \infty$ *and let* j *and* m *be a nonnegative integers such that* $j \le m$. *Then*

$$\|\partial^j u\|_{L^p(\mathbb{R})} \le C\,\|\partial^m u\|^a_{L^r(\mathbb{R})}\,\|u\|^{1-a}_{L^q(\mathbb{R})}$$

where $\frac{1}{p} = j + a\left(\frac{1}{r} - m\right) + \frac{(1-a)}{q}$ *for all* a *in the interval* $\frac{j}{m} \le a \le 1$, *and* M *is a positive constant depending only* m, j, q, r *and* a.

Definition 2.9 *By* $L^2([0,\,T] : H^N(W_{\sigma\;i\;k}))$ *we denote the space of functions* $v(x,\,t)$ *with the norm(*N *positive integer)*

$$\|v\|^2_{L^2([0,\,T]:\,H^N(W_{\sigma\;i\;k}))} = \int_0^T \|v(\cdot\,,\,t)\|^2_{H^N(W_{\sigma\;i\;k})}\,dt < +\infty.$$

Remark 2.10 *The usual Sobolev space is* $H^N(\mathbb{R}) = H^N(W_{0\,0\,0})$ *without a weight.*

Remark 2.11 *We shall derive the a priori estimates assuming that the solution is* C^∞, *bounded as* $x \to -\infty$, *and rapidly decreasing as* $x \to +\infty$, *together with all of its derivatives.*

Considering the above notation, the Hirota-Satsuma system can be written as

$$u_t - a\,u_3 + 6\,u\,u_1 = 2\,b\,v\,v_1 \tag{2.8}$$

$$v_t + v_3 + 3\,u\,v_1 = 0 \tag{2.9}$$

$$u(x,\,0) = u_0(x) \tag{2.10}$$

$$v(x,\,0) = v_0(x) \tag{2.11}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$ and $u = u(x,\,t)$, $v = v(x,\,t)$ are real unknown functions. b and a are real constants with $b > 0$.

Throughout this paper C is a generic constant, not necessarily the same at each occasion(it will change from line to line), which depend in an increasing way on the indicated quantities. In this part we only consider the case $t > 0$. The case $t < 0$ can be treated analogously.

3 Main Inequality

Lemma 3.1 *Let* $(u,\,v)$ *be a solution to* (2.8)-(2.9) *with enough Sobolev regularity(for instance,* $(u,\,v) \in H^N(\mathbb{R}) \times H^N(\mathbb{R})$, $N \geq 3$*), then*

$$\begin{aligned}
&\frac{1}{4\,b}\,\partial_t \int_{\mathbb{R}} \xi\,u_\alpha^2\,dx + \frac{1}{6}\,\partial_t \int_{\mathbb{R}} \xi\,v_\alpha^2\,dx + \int_{\mathbb{R}} \mu_1\,u_{\alpha+1}^2\,dx + \int_{\mathbb{R}} \mu_2\,v_{\alpha+1}^2\,dx \\
&\int_{\mathbb{R}} \theta_1\,u_\alpha^2\,dx + \int_{\mathbb{R}} \theta_2\,v_\alpha^2\,dx + \int_{\mathbb{R}} R_\alpha\,dx = 0
\end{aligned} \tag{3.1}$$

where

$$\mu_1 = -\frac{3\,a}{4\,b}\,\partial\xi \qquad for \quad a < 0 \tag{3.2}$$

$$\mu_2 = \frac{3}{2}\,\partial\xi \tag{3.3}$$

$$\theta_1 = -\frac{1}{4\,b}\,[\,\xi_t - a\,\partial^3\xi + 6\,\partial(\xi\,u)\,] \tag{3.4}$$

$$\theta_2 = -\frac{1}{6}\,[\,\xi_t + \partial^3\xi\,] \tag{3.5}$$

$$\begin{aligned}
R_\alpha = {} & \frac{1}{3\,b}\sum_{\beta=1}^{\alpha} \frac{\alpha!}{\beta!\,(\alpha-\beta)!}\,\xi\,u_\alpha\,u_\beta\,u_{\alpha+1-\beta} - \sum_{\beta=0}^{\alpha-1} \frac{\alpha!}{\beta!\,(\alpha-\beta)!}\,\xi\,u_\alpha\,v_\beta\,v_{\alpha+1-\beta} \\
& + \sum_{\beta=0}^{\alpha-1} \frac{\alpha!}{\beta!\,(\alpha-\beta)!}\,\xi\,v_\alpha\,u_\beta\,v_{\alpha+1-\beta}
\end{aligned} \tag{3.6}$$

Proof. Differentiating (2.8) α-times(for $\alpha \geq 0$) over $x \in \mathbb{R}$ leads to

$$\partial_t u_\alpha - a\,u_{\alpha+3} + 6\,(u\,u_1)_\alpha = 2\,b\,(v\,v_1)_\alpha \tag{3.7}$$

Let $\xi = \xi(x,\,t)$, then multiplying (3.7) by $2\,\xi\,u_\alpha$ we have

$$2\int_{\mathbb{R}} \xi\,u_\alpha\,\partial_t u_\alpha \; - 2\,a \int_{\mathbb{R}} \xi\,u_\alpha\,u_{\alpha+3}\,dx + 12 \int_{\mathbb{R}} \xi\,u_\alpha\,(u\,u_1)_\alpha\,dx = 4\,b \int_{\mathbb{R}} \xi\,u_\alpha\,(v\,v_1)_\alpha\,dx. \tag{3.8}$$

Each term is calculated separately, integrating by parts

$$2\int_{\mathbb{R}} \xi\,u_\alpha\,\partial_t u_\alpha\,dx = \frac{d}{dt}\int_{\mathbb{R}} \xi\,u_\alpha^2\,dx - \int_{\mathbb{R}} \xi_t\,u_\alpha^2\,dx$$

$$- 2\,a \int_{\mathbb{R}} \xi\, u_\alpha\, u_{\alpha+3}\, dx = a \int_{\mathbb{R}} \partial^3 \xi\, u_\alpha^2\, dx - 3\,a \int_{\mathbb{R}} \partial\xi\, u_{\alpha+1}^2\, dx.$$

Using Leibniz's Formula, we have

$$\begin{aligned}
(u\,u_1)_\alpha &= \sum_{\beta=0}^{\alpha} \frac{\alpha!}{\beta!\,(\alpha-\beta)!}\, u_\beta\, u_{\alpha+1-\beta} = u\, u_{\alpha+1} + \sum_{\beta=1}^{\alpha} \frac{\alpha!}{\beta!\,(\alpha-\beta)!}\, u_\beta\, u_{\alpha+1-\beta} \\
(v\,v_1)_\alpha &= \sum_{\beta=0}^{\alpha} \frac{\alpha!}{\beta!\,(\alpha-\beta)!}\, v_\beta\, v_{\alpha+1-\beta}
\end{aligned}$$

then

$$\begin{aligned}
12 \int_{\mathbb{R}} \xi\, u_\alpha\, (u\,u_1)_\alpha\, dx &= 12 \int_{\mathbb{R}} \xi\, u\, u_\alpha\, u_{\alpha+1}\, dx + 12 \sum_{\beta=1}^{\alpha} \frac{\alpha!}{\beta!\,(\alpha-\beta)!} \int_{\mathbb{R}} \xi\, u_\alpha\, u_\beta\, u_{\alpha+1-\beta}\, dx \\
&= -6 \int_{\mathbb{R}} \partial(\xi\, u)\, u_\alpha^2\, dx + 12 \sum_{\beta=1}^{\alpha} \frac{\alpha!}{\beta!\,(\alpha-\beta)!} \int_{\mathbb{R}} \xi\, u_\alpha\, u_\beta\, u_{\alpha+1-\beta}\, dx,
\end{aligned}$$

$$4\,b \int_{\mathbb{R}} \xi\, u_\alpha\, (v\,v_1)_\alpha\, dx = 4\,b \sum_{\beta=0}^{\alpha} \frac{\alpha!}{\beta!\,(\alpha-\beta)!} \int_{\mathbb{R}} \xi\, u_\alpha\, v_\beta\, v_{\alpha+1-\beta}\, dx$$

Hence replacing in (3.8) and performing straightforward calculus we have

$$\begin{aligned}
&\frac{1}{4\,b}\,\frac{d}{dt} \int_{\mathbb{R}} \xi\, u_\alpha^2\, dx - \frac{3\,a}{4\,b} \int_{\mathbb{R}} \partial\xi\, u_{\alpha+1}^2\, dx + \frac{1}{4\,b} \int_{\mathbb{R}} [\, -\xi_t + a\,\partial^3\xi - 6\,\partial(\xi\,u)\,]\, u_\alpha^2\, dx \\
&\frac{1}{3\,b} \sum_{\beta=1}^{\alpha} \frac{\alpha!}{\beta!\,(\alpha-\beta)!} \int_{\mathbb{R}} \xi\, u_\alpha\, u_\beta\, u_{\alpha+1-\beta}\, dx = \sum_{\beta=0}^{\alpha} \frac{\alpha!}{\beta!\,(\alpha-\beta)!} \int_{\mathbb{R}} \xi\, u_\alpha\, v_\beta\, v_{\alpha+1-\beta}\, dx. \qquad (3.9)
\end{aligned}$$

Differentiating (2.9) α-times of (for $\alpha \geq 0$) over $x \in \mathbb{R}$ leads to

$$\partial_t v_\alpha + v_{\alpha+3} = -\,3\,(u\,v_1)_\alpha \qquad (3.10)$$

Multiply this equation by $2\,\xi\,v_\alpha$ and integrate over $x \in \mathbb{R}$ we have

$$2 \int_{\mathbb{R}} \xi\, v_\alpha\, \partial_t v_\alpha\, dx + 2 \int_{\mathbb{R}} \xi\, v_\alpha\, v_{\alpha+3}\, dx = -\,6 \int_{\mathbb{R}} \xi\, v_\alpha\, (u\,v_1)_\alpha\, dx \qquad (3.11)$$

Performing straightforward calculations as above we obtain

$$\begin{aligned}
&\frac{1}{6}\,\frac{d}{dt} \int_{\mathbb{R}} \xi\, v_\alpha^2\, dx + \frac{3}{2} \int_{\mathbb{R}} \partial\xi\, v_{\alpha+1}^2\, dx + \frac{1}{6} \int_{\mathbb{R}} [\, -\,\xi_t - \partial^3\xi\,]\, v_\alpha^2\, dx \\
&= -\sum_{\beta=0}^{\alpha} \frac{\alpha!}{\beta!\,(\alpha-\beta)!} \int_{\mathbb{R}} \xi\, v_\alpha\, u_\beta\, v_{\alpha+1-\beta}\, dx. \qquad (3.12)
\end{aligned}$$

Adding (3.9) and (3.12) we have

$$\begin{aligned}
&\frac{1}{4\,b}\,\frac{d}{dt} \int_{\mathbb{R}} \xi\, u_\alpha^2\, dx + \frac{1}{6}\,\frac{d}{dt} \int_{\mathbb{R}} \xi\, v_\alpha^2\, dx - \frac{3\,a}{4\,b} \int_{\mathbb{R}} \partial\xi\, u_{\alpha+1}^2\, dx + \frac{3}{2} \int_{\mathbb{R}} \partial\xi\, v_{\alpha+1}^2\, dx \\
&+ \frac{1}{4\,b} \int_{\mathbb{R}} [-\xi_t + a\,\partial^3\xi - 6\,\partial(\xi\,u)]\, u_\alpha^2\, dx + \frac{1}{6} \int_{\mathbb{R}} [-\,\xi_t - \partial^3\xi]\, v_\alpha^2\, dx \\
&\frac{1}{3\,b} \sum_{\beta=1}^{\alpha} \frac{\alpha!}{\beta!\,(\alpha-\beta)!} \int_{\mathbb{R}} \xi\, u_\alpha\, u_\beta\, u_{\alpha+1-\beta}\, dx \\
&= \sum_{\beta=0}^{\alpha} \frac{\alpha!}{\beta!\,(\alpha-\beta)!} \int_{\mathbb{R}} \xi\, u_\alpha\, v_\beta\, v_{\alpha+1-\beta}\, dx - \sum_{\beta=0}^{\alpha} \frac{\alpha!}{\beta!\,(\alpha-\beta)!} \int_{\mathbb{R}} \xi\, v_\alpha\, u_\beta\, v_{\alpha+1-\beta}\, dx. \qquad (3.13)
\end{aligned}$$

We take $\beta = \alpha$ in (3.14) we obtain

$$
\begin{aligned}
&\frac{1}{4\,b}\,\frac{d}{dt}\int_{\mathbb{R}} \xi\, u_\alpha^2\, dx + \frac{1}{6}\,\frac{d}{dt}\int_{\mathbb{R}} \xi\, v_\alpha^2\, dx - \frac{3\,a}{4\,b}\int_{\mathbb{R}} \partial\xi\, u_{\alpha+1}^2\, dx + \frac{3}{2}\int_{\mathbb{R}} \partial\xi\, v_{\alpha+1}^2\, dx \\
&+ \frac{1}{4\,b}\int_{\mathbb{R}} [-\xi_t + a\,\partial^3\xi - 6\,\partial(\xi\, u)]\, u_\alpha^2\, dx + \frac{1}{6}\int_{\mathbb{R}} [-\,\xi_t - \partial^3\xi]\, v_\alpha^2\, dx \\
&\frac{1}{3\,b}\sum_{\beta=1}^{\alpha} \frac{\alpha!}{\beta!\,(\alpha-\beta)!}\int_{\mathbb{R}} \xi\, u_\alpha\, u_\beta\, u_{\alpha+1-\beta}\, dx \\
&= \sum_{\beta=0}^{\alpha-1} \frac{\alpha!}{\beta!\,(\alpha-\beta)!}\int_{\mathbb{R}} \xi\, u_\alpha\, v_\beta\, v_{\alpha+1-\beta}\, dx - \sum_{\beta=0}^{\alpha-1} \frac{\alpha!}{\beta!\,(\alpha-\beta)!}\int_{\mathbb{R}} \xi\, v_\alpha\, u_\beta\, v_{\alpha+1-\beta}\, dx
\end{aligned}
\tag{3.14}
$$

the lemma follows.

Lemma 3.2 *For $\mu_1,\ \mu_2 \in W_{\sigma\ i\ k}$ an arbitrary weight functions and $a < 0$, there exist $\xi_1,\ \xi_2 \in W_{\sigma\ i+1\ k}$ respectively such that*

$$
\mu_1 = -\,\frac{3\,a}{4\,b}\,\partial\xi_1 \qquad \text{and} \qquad \mu_2 = -\,\frac{3}{2}\,\partial\xi_2. \tag{3.15}
$$

Indeed, we have

$$
\xi_1 = -\frac{4\,b}{3\,a}\int_{-\infty}^{x} \mu_1(y,\,t)\, dy \qquad \text{and} \qquad \xi_2 = \frac{2}{3}\int_{-\infty}^{x} \mu_2(y,\,t)\, dy. \tag{3.16}
$$

Lemma 3.3 *The expression R_α in the inequality of Lemma 3.1 is a sum of terms of the form*

$$
\xi\, u_{\nu_1}\, u_{\nu_2}\, u_\alpha, \qquad \xi\, v_{\nu_1}\, v_{\nu_2}\, u_\alpha, \qquad \xi\, u_{\nu_1}\, v_{\nu_2}\, v_\alpha \tag{3.17}
$$

where $1 \le \nu_1 \le \nu_2 \le \alpha$ and

$$
\nu_1 + \nu_2 = \alpha + 1. \tag{3.18}
$$

Proof. The result follows using (3.6).

4 An a priori estimate

We show now a fundamental a priori estimate used for a basic local-in-time existence theorem. We construct a mapping $\mathcal{Z} : L^\infty([0,\,T] :\ H^s(\mathbb{R})) \longmapsto L^\infty([0,\,T] :\ H^s(\mathbb{R}))$ with the following property: Given $u^{(n)} = \mathcal{Z}(u^{(n-1)})$ and $\operatorname{esssup}_{t\in[0,\,T]}\|u^{(n-1)}\|_s \le C_0$ then $\operatorname{esssup}_{t\in[0,\,T]}\|u^{(n)}\|_s \le C_0$, where s and $C_0 > 0$ are constants. This property tells us that $\mathcal{Z} : \mathbb{B}_{C_0}(0) \longmapsto \mathbb{B}_{C_0}(0)$ where $\mathbb{B}_{C_0}(0) = \{v(x,\,t) :\ \|v(\cdot,\,t)\|_s \le C_0\}$ is a ball in $L^\infty([0,\,T] :\ H^s(\mathbb{R}))$. To guarantee this property, we will appeal to an a priori estimate which is the main object of this section.

Differentiating (2.8) and (2.9) respectively two times leads to

$$
\partial_t u_2 - a\, u_5 + 6\, u\, u_3 + 18\, u_1\, u_2 = 2\, b\, v\, v_3 + 6\, b\, v_1\, v_2 \tag{4.1}
$$

$$
\partial_t v_2 + v_5 + 3\, u\, v_3 + 6\, u_1\, v_2 + 3\, u_2\, v_1 = 0. \tag{4.2}
$$

Let $u = \wedge w$ and $v = \wedge z$ where $\wedge = (I - \partial^2)^{-1}$. Then $u = (I - \partial^2)^{-1} w$ then $u - u_2 = w$ where $\partial_t u_2 = -\,w_t + u_t$, and in a similar way, $\partial_t v_2 = -\,z_t + v_t$. Replacing on (4.1) and (4.2) respectively we obtain

$$
\begin{aligned}
&-\,w_t - a\wedge w_5 + 6\wedge w\wedge w_3 + 18\wedge w_1\wedge w_2 - [-\,a\wedge w_3 + 6\wedge w\wedge w_1] + 2\,b\wedge z\wedge z_1 \\
&= 2\,b\wedge z\wedge z_3 + 6\,b\wedge z_1\wedge z_2
\end{aligned}
\tag{4.3}
$$

$$- z_t + \wedge z_5 + 3\,\wedge w \wedge z_3 + 6\wedge w_1 \wedge z_2 + 3\wedge w_2 \wedge z_1 - [\wedge z_3 + 3\wedge w \wedge z_1] = 0 \tag{4.4}$$

The equations (4.3), (4.4) are linearized equations by substituting a new variable θ and ϕ in each coefficient:

$$\begin{aligned} w_t &= -\,a\wedge w_5 + 6\wedge\theta\wedge w_3 + 18\wedge\theta_1\wedge w_2 - [-\,a\wedge w_3 + 6\wedge\theta\wedge w_1] + 2\,b\wedge\phi\wedge z_1 \\ &\quad + 2\,b\wedge\phi\wedge z_3 + 6\,b\wedge\phi_1\wedge z_2 \end{aligned} \tag{4.5}$$

$$z_t = \wedge z_5 + 3\wedge\theta\wedge z_3 + 6\wedge\theta_1\wedge z_2 + 3\wedge\theta_2\wedge z_1 - [\wedge z_3 + 3\wedge\theta\wedge z_1] \tag{4.6}$$

Equations (4.5) and (4.6) are linear equations at each iteration which can be solved in any interval of time in which the coefficients are defined. These equations have the form

$$\partial_t w = -\,a\wedge w_5 + h^{(2)}\wedge w_3 + h^{(1)}\wedge z_3 + h^{(0)} \tag{4.7}$$

$$\partial_t z = \wedge z_5 + k^{(2)}\wedge w_3 + k^{(1)}\wedge z_3 + k^{(0)} \tag{4.8}$$

We consider the following Lemma to help us to set up the iteration scheme.

Lemma 4.1. Given initial data $(u_0(x),\, v_0(x)) \in \bigcap_{k\geq 0} H^k(W_{0\,i\,0}) \times H^k(W_{0\,i\,0})$ and $a < 0$. Then there exists a unique solution of (4.7), (4.8) where $h^{(2)} = h^{(2)}(\wedge\theta)$, $h^{(1)} = h^{(1)}(\wedge\phi)$, $h^{(0)} = h^{(0)}(\wedge\theta_2,\, \wedge\theta_1,\, \wedge\theta,\, \wedge\phi_2,\, \wedge\phi_1,\, \wedge\phi)$, and $k^{(2)} = k^{(2)}(\wedge\theta)$, $k^{(1)} = k^{(1)}(\wedge\phi)$, $k^{(0)} = k^{(0)}(\wedge\theta_2,\, \wedge\theta_1,\, \wedge\theta,\, \wedge\phi_2,\, \wedge\phi_1,\, \wedge\phi)$. The solution is defined in any time interval in which the coefficients are defined.

Proof. From equations (4.7)-(4.8) we have

$$W_t = A\wedge W_5 + B^{1,2}\wedge W_3 + C^{(0)} \tag{4.9}$$

where

$$A = \begin{bmatrix} -\,a & 0 \\ 0 & 1 \end{bmatrix}, \qquad B^{1,2} = \begin{bmatrix} h^{(2)} & h^{(1)} \\ k^{(2)} & k^{(1)} \end{bmatrix}, \qquad C^{(0)} = \begin{bmatrix} h^{(0)} \\ k^{(0)} \end{bmatrix}, \qquad W = \begin{bmatrix} w \\ z \end{bmatrix}.$$

Let $T > 0$ be arbitrary and $M > 0$ a constant. Define $\mathcal{L} = 2\,\xi\,(\partial_t - A\wedge\partial^5 - B^{1,2}\wedge\partial^3)$. Then in (4.9) we have $\mathcal{L}W = 2\,\xi\, C^{(0)}$. We consider the bilinear form

$$\begin{aligned} B\colon \mathcal{D}\times\mathcal{D} &\longrightarrow \mathbb{R} \\ B(U_1,\, U_2) &= \langle U_1,\, U_2\rangle = \int_0^T \int_{\mathbb{R}} e^{-M\,t}\,(u_1\,u_2 + v_1\,v_2)\,dx\,dt \end{aligned}$$

where $\mathcal{D} = \{U = (u,\, v) \in C_0^\infty(\mathbb{R}\times[0,\, T]) \times C_0^\infty(\mathbb{R}\times[0,\, T]) : u(x,\, 0) = 0 \text{ and } v(x,\, 0) = 0\}$ and

$$U_1 = \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}, \qquad U_2 = \begin{bmatrix} u_2 \\ v_2 \end{bmatrix}.$$

We have

$$\begin{aligned} \int_{\mathbb{R}} \mathcal{L}U\cdot U\,dx &= \int_{\mathbb{R}} 2\,\xi\,[w\,w_t + a\,w\wedge w_5 - h^{(2)}\,w\wedge w_3 - h^{(1)}\,w\wedge z_3 \\ &\quad + z\,z_t - z\wedge z_5 - k^{(2)}\,z\wedge w_3 - k^{(1)}\,z\wedge z_3]\,dx. \end{aligned}$$

Each term is treated separately integrating by parts. The first two terms we have

$$2\int_{\mathbb{R}} \xi\,w\,w_t\,dx = \partial_t \int_{\mathbb{R}} \xi\,w^2\,dx - \int_{\mathbb{R}} \xi_t\,w^2\,dx.$$

$$
\begin{aligned}
2\,a \int_{\mathbb{R}} \xi\, w \wedge w_5\, dx &= 2\,a \int_{\mathbb{R}} \xi \wedge (I-\partial^2) w \wedge w_5\, dx \\
&= 2\,a \int_{\mathbb{R}} \xi \wedge w \wedge w_5\, dx - 2\,a \int_{\mathbb{R}} \xi \wedge w_2 \wedge w_5\, dx \\
&= -\,a \int_{\mathbb{R}} \partial^5 \xi\, (\wedge w)^2\, dx + 5\,a \int_{\mathbb{R}} \partial^3 \xi\, (\wedge w_1)^2\, dx - a \int_{\mathbb{R}} (5\,\partial\xi - \partial^3\xi)\, (\wedge w_2)^2\, dx \\
&\quad - 3\,a \int_{\mathbb{R}} \partial \xi\, (\wedge w_1)^2\, dx.
\end{aligned}
$$

The other terms are calculated the same form

$$
\begin{aligned}
-2 \int_{\mathbb{R}} \xi\, h^{(2)}\, w \wedge w_3\, dx &= \int_{\mathbb{R}} \partial^3 (\xi\, h^{(2)})\, (\wedge w)^2\, dx - 3 \int_{\mathbb{R}} \partial(\xi\, h^{(2)})\, (\wedge w_1)^2\, dx \\
&\quad - \int_{\mathbb{R}} \partial(\xi\, h^{(2)})\, (\wedge w_2)^2\, dx.
\end{aligned}
$$

$$
\begin{aligned}
-2 \int_{\mathbb{R}} \xi\, h^{(1)}\, w \wedge z_3\, dx &= -2 \int_{\mathbb{R}} \partial^2 (\xi\, h^{(1)}) \wedge w \wedge z_1\, dx - 2 \int_{\mathbb{R}} \partial(\xi\, h^{(1)}) \wedge w_1 \wedge z_1\, dx \\
&\quad + 2 \int_{\mathbb{R}} \xi\, h^{(1)} \wedge w_1 \wedge z_2\, dx + 2 \int_{\mathbb{R}} \xi\, h^{(1)} \wedge w_2 \wedge z_3\, dx.
\end{aligned}
$$

Using that $\wedge w_n = (I - (I-\partial^2)) \wedge w_{n-2} = \wedge w_{n-2} - w_{n-2}$ (for n positive integer) and standard estimates follow that

$$
\int_{\mathbb{R}} \mathcal{L}U \cdot U\, dx \geq \partial_t \int_{\mathbb{R}} \xi\, w^2\, dx + \partial_t \int_{\mathbb{R}} \xi\, z^2\, dx - c \int_{\mathbb{R}} \xi\, w^2\, dx - c \int_{\mathbb{R}} \xi\, z^2\, dx. \tag{4.10}
$$

Multiplying (4.10) for $e^{-M\,t}$, and integrate in time t for $t \in [0,\, T]$ and $U = (w,\, z) \in \mathcal{D}$.

$$
\begin{aligned}
\int_0^T \int_{\mathbb{R}} e^{-M\,t} \mathcal{L}U \cdot U\, dx\, dt &\geq \int_0^T e^{-M\,t} \left(\partial_t \int_{\mathbb{R}} \xi\, w^2\, dx \right) dt + \int_0^T e^{-M\,t} \left(\partial_t \int_{\mathbb{R}} \xi\, z^2\, dx \right) dt \\
&\quad - c \int_0^T \int_{\mathbb{R}} \xi\, e^{-M\,t} w^2\, dx\, dt - c \int_0^T \int_{\mathbb{R}} \xi\, e^{-M\,t} z^2\, dx\, dt \\
&= e^{-M\,t} \int_{\mathbb{R}} \xi\, w^2(x,\, t)\, dx \,|_0^T + M \int_0^T \int_{\mathbb{R}} \xi\, e^{-M\,t} w^2\, dx\, dt \\
&\quad + e^{-M\,t} \int_{\mathbb{R}} \xi\, z^2(x,\, t)\, dx \,|_0^T + M \int_0^T \int_{\mathbb{R}} \xi\, e^{-M\,t} z^2\, dx\, dt \\
&\quad - c \int_0^T \int_{\mathbb{R}} \xi\, w^2\, dx\, dt - c \int_0^T \int_{\mathbb{R}} \xi\, z^2\, dx\, dt.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_0^T \int_{\mathbb{R}} e^{-M\,t} \mathcal{L}U \cdot U\, dx\, dt &\geq e^{-M\,t} \int_{\mathbb{R}} \xi(x,\, T)\, w^2(x,\, T)\, dx + e^{-M\,t} \int_{\mathbb{R}} \xi(x,\, T)\, z^2(x,\, T)\, dx \\
&\quad + (M-c) \int_0^T \int_{\mathbb{R}} \xi\, e^{-M\,t} w^2\, dx\, dt + (M-c) \int_0^T \int_{\mathbb{R}} \xi\, e^{-M\,t} z^2\, dx\, dt \\
&\geq \int_0^T \int_{\mathbb{R}} \xi e^{-M\,t} w^2 dx\, dt + \int_0^T \int_{\mathbb{R}} \xi\, e^{-M\,t} z^2\, dx\, dt = \int_0^T \int_{\mathbb{R}} \xi\, e^{-M\,t} (w^2 + z^2)\, dx\, dt
\end{aligned}
$$

provided M is chosen large enough. Then

$$
\langle \mathcal{L}U,\, U \rangle \;\geq\; \langle U,\, U \rangle, \qquad \forall \quad U \in \mathcal{D}.
$$

Let $\mathcal{L}^* = 2\,\xi\,(-\partial_t + A\wedge\partial^5 + B^{1,2}\wedge\partial^3)$ the formal adjoint of $\mathcal{L}$. Let $\mathcal{D}^*$ such that $\mathcal{D}^* = \{W = (w, z) \in C_0^\infty(\mathbb{R}\times[0, T]) \times C_0^\infty(\mathbb{R}\times[0, T]) : w(x, T) = 0 \text{ and } z(x, T) = 0\}$.
The same form for $\mathcal{L}^*$ the formal adjoint of $\mathcal{L}$ we show that

$$\langle \mathcal{L}^* W, W\rangle \geq \langle W, W\rangle \qquad \forall \quad W \in \mathcal{D}^*. \tag{4.11}$$

From (4.11) we have that $\mathcal{L}^*$ is one to one. Therefore $\langle \mathcal{L}^* W, \mathcal{L}^* W\rangle$ is an inner product on $\mathcal{D}^*$. Denote by X the completion of $\mathcal{D}^*$ with respect to this inner product. By the Riesz representation Theorem, there exists a unique solution $V \in X$, such that for any $W \in \mathcal{D}^*$

$$\langle \xi\, C^{(0)}, W\rangle = \langle \mathcal{L}^* V, \mathcal{L}^* W\rangle$$

where we used that $\xi\, C^{(0)} \in X$. Then if $Z = \mathcal{L}^* V$ we have

$$\langle Z, \mathcal{L}^* W\rangle = \langle \xi\, C^{(0)}, W\rangle \quad \text{or} \quad \langle \mathcal{L}^* W, Z\rangle = \langle W, \xi\, C^{(0)}\rangle$$

hence $Z = \mathcal{L}^* V$ is a weak solution of $\mathcal{L} Z = \xi\, C^{(0)}$ with $Z \in L^2(\mathbb{R}\times[0, T]) \times L^2(\mathbb{R}\times[0, T]) \simeq L^2([0, T] : L^2(\mathbb{R})) \times L^2([0, T] : L^2(\mathbb{R}))$.

Remark 4.1 *To obtain higher regularity of the solution, we repeat the proof with higher derivatives included in the inner product. It is a standard approximation procedure to obtain a result for general initial data.*

The following estimate is related to the existence of solutions theorem.
Lemma 4.2. *Let $\theta, \phi, w, z \in C^k([0, +\infty) : H^N(W_{0\,i\,0}))$ for all k, N which satisfy (4.5), (4.6) and $a < 0$. For each α there exist positive, nondecreasing functions G, E and M such that for all $t \geq 0$*

$$\begin{aligned} \partial_t \int_{\mathbb{R}} \xi\, w_\alpha^2\, dx + \partial_t \int_{\mathbb{R}} \xi\, z_\alpha^2\, dx \quad &\leq \quad G(\|\theta\|_\lambda,\ \|\phi\|_\lambda)\,(\|w\|_\alpha^2 + \|z\|_\alpha^2) \\ &\quad + E(\|\theta\|_\lambda,\ \|\phi\|_\lambda)\,(\|\theta\|_\alpha^2 + \|\phi\|_\alpha^2) + M(\|\theta\|_\alpha,\ \|\phi\|_\alpha) \end{aligned} \tag{4.12}$$

where $\|\cdot\|_\alpha$ is the norm in $H^\alpha(W_{0\,i\,0})$ and $\lambda = \max\{1, \alpha\}$.
Proof. We begin by applying ∂ to (4.5), our equation become

$$\begin{aligned} \partial_t w_1 \quad &= \quad -\, a\wedge w_6 + a\wedge w_4 + 6\wedge\theta\wedge w_4 + 2\,b\wedge\phi\wedge z_4 + 24\wedge\theta_1\wedge w_3 + 8\,b\wedge\phi_1\wedge z_3 \\ &\quad + 18\wedge\theta_2\wedge w_2 + 6\,b\wedge\phi_2\wedge z_2 - 6\wedge\theta\wedge w_2 + 2\,b\wedge\phi\wedge z_2 - 6\wedge\theta_1\wedge w_1 + 2\,b\wedge\phi_1\wedge z_1 \end{aligned}$$

follow that

$$\begin{aligned} \partial_t w_1 \quad &= \quad -\, a\wedge w_6 + a\wedge w_4 + 6\wedge\theta\wedge w_4 + 2\,b\wedge\phi\wedge z_4 + 24\wedge\theta_1\wedge w_3 + 8\,b\wedge\phi_1\wedge z_3 \\ &\quad + p_1(\wedge\theta_2,\ \wedge\phi_2,\ \ldots,\ \wedge\theta,\ \wedge\phi) \end{aligned}$$

where

$$p_1 \quad = \quad 18\wedge\theta_2\wedge w_2 + 6\,b\wedge\phi_2\wedge z_2 - 6\wedge\theta\wedge w_2 + 2\,b\wedge\phi\wedge z_2 - 6\wedge\theta_1\wedge w_1 + 2\,b\wedge\phi_1\wedge z_1.$$

The similar form applying ∂^2 to (4.5), follow that

$$\begin{aligned} \partial_t w_2 \quad &= \quad -\, a\wedge w_7 + a\wedge w_5 + 6\wedge\theta\wedge w_5 + 2\,b\wedge\phi\wedge z_5 + 30\wedge\theta_1\wedge w_4 + 10\,b\wedge\phi_1\wedge z_4 \\ &\quad + p_2(\wedge\theta_3,\ \wedge\phi_3,\ \ldots,\ \wedge\theta,\ \wedge\phi) \end{aligned}$$

where

$$\begin{aligned} p_2 \quad &= \quad 42\wedge\theta_2\wedge w_3 + 14\,b\wedge\phi_2\wedge z_3 - 6\wedge\theta\wedge w_3 + 2\,b\wedge\phi\wedge z_3 + 18\wedge\theta_3\wedge w_2 + 2\,b\wedge\phi_3\wedge z_2 \\ &\quad - 6\wedge\theta_2\wedge w_1 + 2\,b\wedge\phi_2\wedge z_1 - 12\wedge\theta_1\wedge w_2 + 4\,b\wedge\phi_1\wedge z_2. \end{aligned}$$

Applying ∂^α to (4.5), our equation become

$$\begin{aligned}\partial_t w_\alpha \quad &= \quad -\,a\wedge w_{\alpha+5}+\sum_{j=3}^{\alpha+3}h_1^{(j)}\wedge w_j+r_1\wedge\theta_{\alpha+1}+r_2(\wedge\theta_\alpha,\,\wedge\theta_{\alpha-1},\,\ldots,\,\wedge\theta)\\ &\qquad +\sum_{j=3}^{\alpha+3}h_2^{(j)}\wedge z_j+s_1\wedge\phi_{\alpha+1}+s_2(\wedge\phi_\alpha,\,\wedge\phi_{\alpha-1},\,\ldots,\,\wedge\phi)\end{aligned}\tag{4.13}$$

where $h_1^{(j)}$ and $h_2^{(j)}$ are smooth functions depending on $\wedge\theta_i,\,\wedge\theta_{i-1},\,\ldots,\,\wedge\theta\,;\,\wedge\phi_i,\,\wedge\phi_{i-1},\,\ldots,\,\wedge\phi$ with $i=3+\alpha-j$.
Multiplying (4.13) by $2\,\xi\,w_\alpha$, and integrate over $x\in\mathbb{R}$, as follows

$$\begin{aligned}&2\int_{\mathbb{R}}\xi\,w_\alpha\,\partial_t w_\alpha\,dx=-\,2\,a\int_{\mathbb{R}}\xi\,w_\alpha\wedge w_{\alpha+5}\,dx+2\sum_{j=3}^{\alpha+3}\int_{\mathbb{R}}\xi\,h_1^{(j)}\,w_\alpha\wedge w_j\,dx\\ &+2\int_{\mathbb{R}}\xi\,r_1\,w_\alpha\wedge\theta_{\alpha+1}\,dx+2\int_{\mathbb{R}}\xi\,w_\alpha\,r_2(\wedge\theta_\alpha,\,\wedge\theta_{\alpha-1},\,\ldots,\,\wedge\theta)\,dx+\sum_{j=3}^{\alpha+3}\int_{\mathbb{R}}\xi\,h_2^{(j)}\,w_\alpha\wedge z_j\,dx\\ &+2\int_{\mathbb{R}}\xi\,s_1 w_\alpha\wedge\phi_{\alpha+1}\,dx+2\int_{\mathbb{R}}\xi\,w_\alpha\,s_2(\wedge\phi_\alpha,\,\wedge\phi_{\alpha-1},\ldots,\,\wedge\phi)\,dx.\end{aligned}\tag{4.14}$$

Each term in (4.14) is treated separately. The first two terms yield

$$2\int_{\mathbb{R}}\xi\,w_\alpha\,\partial_t w_\alpha\,dx=\partial_t\int_{\mathbb{R}}\xi\,w_\alpha^2\,dx-\int_{\mathbb{R}}\xi_t\,w_\alpha^2\,dx.$$

$$\begin{aligned}-\,2\,a\int_{\mathbb{R}}\xi\,w_\alpha\wedge w_{\alpha+5}\,dx \quad &= \quad -\,2\,a\int_{\mathbb{R}}\xi\wedge(I-\partial^2)w_\alpha\wedge w_{\alpha+5}\,dx\\ &= \quad -\,2\,a\int_{\mathbb{R}}\xi\wedge w_\alpha\wedge w_{\alpha+5}\,dx+2\,a\int_{\mathbb{R}}\xi\wedge w_{\alpha+2}\wedge w_{\alpha+5}\,dx\\ &= \quad a\int_{\mathbb{R}}\partial^5\xi\,(\wedge w_\alpha)^2\,dx-5\,a\int_{\mathbb{R}}\partial^3\xi\,(\wedge w_{\alpha+1})^2\,dx+5\,a\int_{\mathbb{R}}\partial\xi\,(\wedge w_{\alpha+2})^2\,dx\\ &\qquad -\,a\int_{\mathbb{R}}\partial^3\xi\,(\wedge w_{\alpha+2})^2\,dx+3\,a\int_{\mathbb{R}}\partial\xi\,(\wedge w_{\alpha+3})^2\,dx\\ &= \quad 3\,a\int_{\mathbb{R}}\partial\xi\,(\wedge w_{\alpha+3})^2\,dx-a\int_{\mathbb{R}}(\partial^3\xi-5\,\partial\xi)\,(\wedge w_{\alpha+2})^2\,dx\\ &\qquad -\,5\,a\int_{\mathbb{R}}\partial^3\xi\,(\wedge w_{\alpha+1})^2\,dx+a\int_{\mathbb{R}}\partial^5\xi\,(\wedge w_\alpha)^2\,dx.\end{aligned}$$

The other terms in (4.14) are treated the similar form, using integrating by parts.

$$
\begin{aligned}
&\partial_t \int_{\mathbb{R}} \xi\, {w_\alpha}^2\, dx - \int_{\mathbb{R}} \xi_t\, {w_\alpha}^2\, dx + 3\, a \int_{\mathbb{R}} \partial\xi\, (\wedge w_{\alpha+3})^2\, dx - a \int_{\mathbb{R}} (\partial^5\xi - 5\, \partial\xi)\, (\wedge w_{\alpha+2})^2\, dx \\
&- 5\, a \int_{\mathbb{R}} \partial^3\xi\, (\wedge w_{\alpha+1})^2\, dx + a \int_{\mathbb{R}} \partial^5\xi\, (\wedge w_\alpha)^2\, dx \\
&- \int_{\mathbb{R}} \partial^3(\xi\, h_1^{(\alpha+3)})\, (\wedge w_\alpha)^2\, dx + 3 \int_{\mathbb{R}} \partial(\xi\, h_1^{(\alpha+3)})\, (\wedge w_{\alpha+1})^2\, dx \\
&\int_{\mathbb{R}} \partial(\xi\, h_1^{(\alpha+3)})\, (\wedge w_{\alpha+2})^2\, dx + \int_{\mathbb{R}} \partial^2(\xi\, h_1^{(\alpha+2)})(\wedge w_\alpha)^2\, dx \\
&-2 \int_{\mathbb{R}} \xi\, h_1^{(\alpha+2)}\, (\wedge w_{\alpha+1})^2\, dx - 2 \int_{\mathbb{R}} \xi\, h_1^{(\alpha+2)}\, (\wedge w_{\alpha+2})^2\, dx \\
&+ 2 \sum_{j=3}^{\alpha+1} \int_{\mathbb{R}} \xi\, h_1^{(j)}\, w_\alpha \wedge w_j\, dx + 2 \int_{\mathbb{R}} \xi\, r_1\, w_\alpha \wedge \theta_{\alpha+1}\, dx + 2 \int_{\mathbb{R}} \xi\, r_2\, w_\alpha\, dx \\
&-2 \int_{\mathbb{R}} \partial(\xi\, h_2^{(\alpha+3)}\, w_\alpha) \wedge z_{\alpha+2}\, dx - 2 \int_{\mathbb{R}} \partial(\xi\, h_2^{(\alpha+2)}\, w_\alpha) \wedge z_{\alpha+1}\, dx \\
&+ 2 \sum_{j=3}^{\alpha+1} \int_{\mathbb{R}} \xi\, h_2^{(j)}\, w_\alpha \wedge z_j\, dx + 2 \int_{\mathbb{R}} \xi\, s_1\, w_\alpha \wedge \phi_{\alpha+1}\, dx + 2 \int_{\mathbb{R}} \xi\, s_2\, w_\alpha\, dx = 0. \qquad (4.15)
\end{aligned}
$$

Performing similar calculations to (4.6), our equation become

$$
\begin{aligned}
\partial_t z_\alpha \;\; = \;\; & \wedge z_{\alpha+5} + \sum_{j=3}^{\alpha+3} k_1^{(j)} \wedge w_j + m_1 \wedge \theta_{\alpha+1} + m_2(\wedge\theta_\alpha,\, \wedge\theta_{\alpha-1},\, \ldots,\, \wedge\theta) \\
& + \sum_{j=3}^{\alpha+3} k_2^{(j)} \wedge z_j + n_1 \wedge \phi_{\alpha+1} + n_2(\wedge\phi_\alpha,\, \wedge\phi_{\alpha-1},\, \ldots,\, \wedge\phi) \qquad (4.16)
\end{aligned}
$$

where $k_1^{(j)}$ and $k_2^{(j)}$ are smooth functions depending on $\wedge\theta_i,\, \wedge\theta_{i-1},\, \ldots,\, \wedge\theta, \quad \wedge\phi_i,\, \wedge\phi_{i-1},\, \ldots,\, \wedge\phi$ with $i = 3 + \alpha - j$.

We now multiply (4.16) by $2\, \xi\, z_\alpha$, integrate over $x \in \mathbb{R}$ and performing calculations we obtain

$$
\begin{aligned}
&-\partial_t \int_{\mathbb{R}} \xi\, z_\alpha^2\, dx + \int_{\mathbb{R}} \xi_t\, z_\alpha^2\, dx - 3 \int_{\mathbb{R}} \partial\xi\, (\wedge z_{\alpha+3})^2\, dx + \int_{\mathbb{R}} (\partial^5\xi - 5\, \partial\xi)\, (\wedge z_{\alpha+2})^2\, dx \\
&+ 5 \int_{\mathbb{R}} \partial^3\xi\, (\wedge z_{\alpha+1})^2\, dx - \int_{\mathbb{R}} \partial^5\xi\, (\wedge z_\alpha)^2\, dx - 2 \int_{\mathbb{R}} \partial(\xi\, k_1^{(\alpha+3)}\, z_\alpha) \wedge w_{\alpha+2}\, dx \\
&- 2 \int_{\mathbb{R}} \partial(\xi\, k_1^{(\alpha+2)}\, z_\alpha) \wedge w_{\alpha+1}\, dx + 2 \sum_{j=3}^{\alpha+1} \int_{\mathbb{R}} \xi\, k_1^{(j)}\, z_\alpha \wedge w_j\, dx + 2 \int_{\mathbb{R}} \xi\, m_1\, z_\alpha \wedge \theta_{\alpha+1}\, dx + 2 \int_{\mathbb{R}} \xi\, m_2\, z_\alpha\, dx \\
&- \int_{\mathbb{R}} \partial^3(\xi\, k_2^{(\alpha+3)})\, (\wedge z_\alpha)^2\, dx + 3 \int_{\mathbb{R}} \partial(\xi\, k_2^{(\alpha+3)})\, (\wedge z_{\alpha+1})^2\, dx + \int_{\mathbb{R}} \partial(\xi\, k_2^{(\alpha+3)})\, (\wedge z_{\alpha+2})^2\, dx \\
&+ \int_{\mathbb{R}} \partial^2(\xi\, k_2^{(\alpha+2)})\, (\wedge z_\alpha)^2\, dx - 2 \int_{\mathbb{R}} \xi\, k_2^{(\alpha+2)}\, (\wedge z_{\alpha+1})^2\, dx - 2 \int_{\mathbb{R}} \xi\, k_2^{(\alpha+2)}\, (\wedge z_{\alpha+2})^2\, dx \\
&+ 2 \sum_{j=3}^{\alpha+1} \int_{\mathbb{R}} \xi\, k_2^{(j)}\, z_\alpha \wedge z_j\, dx + 2 \int_{\mathbb{R}} \xi\, n_1\, z_\alpha \wedge \phi_{\alpha+1}\, dx + 2 \int_{\mathbb{R}} \xi\, n_2\, z_\alpha\, dx = 0. \qquad (4.17)
\end{aligned}
$$

Adding (4.15) with (4.17) we have the following identity

$$
\begin{aligned}
&- \partial_t \int_{\mathbb{R}} \xi\, w_\alpha^2\, dx - \partial_t \int_{\mathbb{R}} \xi\, z_\alpha^2\, dx + \int_{\mathbb{R}} \xi_t\, w_\alpha^2\, dx + \int_{\mathbb{R}} \xi_t\, z_\alpha^2\, dx + 3\, a \int_{\mathbb{R}} \partial\xi\, (\wedge w_{\alpha+3})^2\, dx \\
&- 3 \int_{\mathbb{R}} \partial\xi\, (\wedge z_{\alpha+3})^2\, dx - a \int_{\mathbb{R}} (\partial^5\xi - 5\, \partial\xi)\, (\wedge w_{\alpha+2})^2\, dx + \int_{\mathbb{R}} (\partial^5\xi - 5\, \partial\xi)\, (\wedge z_{\alpha+2})^2\, dx \\
&- 5\, a \int_{\mathbb{R}} \partial^3\xi\, (\wedge w_{\alpha+1})^2 dx + 5 \int_{\mathbb{R}} \partial^3\xi\, (\wedge z_{\alpha+1})^2\, dx + a \int_{\mathbb{R}} \partial^5\xi\, (\wedge w_\alpha)^2\, dx - \int_{\mathbb{R}} \partial^5\xi\, (\wedge z_\alpha)^2\, dx \\
&- \int_{\mathbb{R}} \partial^3(\xi\, h_1^{(\alpha+3)})\, (\wedge w_\alpha)^2\, dx + 3 \int_{\mathbb{R}} \partial(\xi\, h_1^{(\alpha+3)})\, (\wedge w_{\alpha+1})^2\, dx + \int_{\mathbb{R}} \partial(\xi\, h_1^{(\alpha+3)})\, (\wedge w_{\alpha+2})^2\, dx \\
&- \int_{\mathbb{R}} \partial^3(\xi\, k_2^{(\alpha+3)})\, (\wedge z_\alpha)^2\, dx + 3 \int_{\mathbb{R}} \partial(\xi\, k_2^{(\alpha+3)})\, (\wedge z_{\alpha+1})^2\, dx + \int_{\mathbb{R}} \partial(\xi\, k_2^{(\alpha+3)})\, (\wedge z_{\alpha+2})^2\, dx \\
&+ \int_{\mathbb{R}} \partial^2(\xi\, h_1^{(\alpha+2)})\, (\wedge w_\alpha)^2\, dx - 2 \int_{\mathbb{R}} \xi\, h_1^{(\alpha+2)}\, (\wedge w_{\alpha+1})^2\, dx - 2 \int_{\mathbb{R}} \xi\, h_1^{(\alpha+2)}\, (\wedge w_{\alpha+2})^2\, dx \\
&+ \int_{\mathbb{R}} \partial^2(\xi\, k_2^{(\alpha+2)})\, (\wedge z_\alpha)^2\, dx - 2 \int_{\mathbb{R}} \xi\, k_2^{(\alpha+2)}\, (\wedge z_{\alpha+1})^2\, dx - 2 \int_{\mathbb{R}} \xi\, k_2^{(\alpha+2)}\, (\wedge z_{\alpha+2})^2\, dx \\
&- 2 \int_{\mathbb{R}} \partial(\xi\, h_2^{(\alpha+3)}\, w_\alpha)\, \wedge z_{\alpha+2}\, dx - 2 \int_{\mathbb{R}} \partial(\xi\, h_2^{(\alpha+2)}\, w_\alpha) \wedge z_{\alpha+1}\, dx \\
&- 2 \int_{\mathbb{R}} \partial(\xi\, k_1^{(\alpha+3)}\, z_\alpha)\, \wedge w_{\alpha+2}\, dx - 2 \int_{\mathbb{R}} \partial(\xi\, k_1^{(\alpha+2)}\, z_\alpha)\, \wedge w_{\alpha+1}\, dx \\
&+ 2 \sum_{j=3}^{\alpha+1} \int_{\mathbb{R}} \xi\, h_1^{(j)} w_\alpha \wedge w_j\, dx + 2 \sum_{j=3}^{\alpha+1} \int_{\mathbb{R}} \xi\, k_2^{(j)}\, z_\alpha \wedge z_j\, dx \\
&+ 2 \sum_{j=3}^{\alpha+1} \int_{\mathbb{R}} \xi\, h_2^{(j)}\, w_\alpha \wedge z_j\, dx + 2 \sum_{j=3}^{\alpha+1} \int_{\mathbb{R}} \xi\, k_1^{(j)}\, z_\alpha \wedge w_j\, dx \\
&+ 2 \int_{\mathbb{R}} \xi\, r_1\, w_\alpha \wedge \theta_{\alpha+1}\, dx + 2 \int_{\mathbb{R}} \xi\, m_1\, z_\alpha \wedge \theta_{\alpha+1}\, dx + 2 \int_{\mathbb{R}} \xi\, s_1\, w_\alpha \wedge \phi_{\alpha+1}\, dx + 2 \int_{\mathbb{R}} \xi\, n_1\, z_\alpha \wedge \phi_{\alpha+1}\, dx \\
&+ 2b_2 \int_{\mathbb{R}} \xi\, r_2\, w_\alpha\, dx + 2 \int_{\mathbb{R}} \xi\, m_2\, z_\alpha\, dx + 2 \int_{\mathbb{R}} \xi\, s_2\, w_\alpha\, dx + 2 \int_{\mathbb{R}} \xi\, n_2\, z_\alpha\, dx = 0.
\end{aligned}
$$

where

$$
\begin{aligned}
&\partial_t \int_{\mathbb{R}} \xi\, w_\alpha^2\, dx + \partial_t \int_{\mathbb{R}} \xi\, z_\alpha^2\, dx = \int_{\mathbb{R}} \xi_t\, w_\alpha^2\, dx + \int_{\mathbb{R}} \xi_t\, z_\alpha^2\, dx + 3\, a \int_{\mathbb{R}} \partial\xi\, (\wedge w_{\alpha+3})^2\, dx \\
&- 3 \int_{\mathbb{R}} \partial\xi\, (\wedge z_{\alpha+3})^2\, dx - a \int_{\mathbb{R}} (\partial^5\xi - 5\, \partial\xi)\, (\wedge w_{\alpha+2})^2\, dx + \int_{\mathbb{R}} (\partial^5\xi - 5\, \partial\xi)\, (\wedge z_{\alpha+2})^2\, dx \\
&- 5\, a \int_{\mathbb{R}} \partial^3\xi\, (\wedge w_{\alpha+1})^2 dx + 5 \int_{\mathbb{R}} \partial^3\xi\, (\wedge z_{\alpha+1})^2\, dx + a \int_{\mathbb{R}} \partial^5\xi\, (\wedge w_\alpha)^2\, dx - \int_{\mathbb{R}} \partial^5\xi\, (\wedge z_\alpha)^2\, dx \\
&- \int_{\mathbb{R}} \partial^3(\xi\, h_1^{(\alpha+3)})\, (\wedge w_\alpha)^2\, dx + 3 \int_{\mathbb{R}} \partial(\xi\, h_1^{(\alpha+3)})\, (\wedge w_{\alpha+1})^2\, dx + \int_{\mathbb{R}} \partial(\xi\, h_1^{(\alpha+3)})\, (\wedge w_{\alpha+2})^2\, dx \\
&- \int_{\mathbb{R}} \partial^3(\xi\, k_2^{(\alpha+3)})\, (\wedge z_\alpha)^2\, dx + 3 \int_{\mathbb{R}} \partial(\xi\, k_2^{(\alpha+3)})\, (\wedge z_{\alpha+1})^2\, dx + \int_{\mathbb{R}} \partial(\xi\, k_2^{(\alpha+3)})\, (\wedge z_{\alpha+2})^2\, dx \\
&+ \int_{\mathbb{R}} \partial^2(\xi\, h_1^{(\alpha+2)})\, (\wedge w_\alpha)^2\, dx - 2 \int_{\mathbb{R}} \xi\, h_1^{(\alpha+2)}\, (\wedge w_{\alpha+1})^2\, dx - 2 \int_{\mathbb{R}} \xi\, h_1^{(\alpha+2)}\, (\wedge w_{\alpha+2})^2\, dx \\
&+ \int_{\mathbb{R}} \partial^2(\xi\, k_2^{(\alpha+2)})\, (\wedge z_\alpha)^2\, dx - 2 \int_{\mathbb{R}} \xi\, k_2^{(\alpha+2)}\, (\wedge z_{\alpha+1})^2\, dx - 2 \int_{\mathbb{R}} \xi\, k_2^{(\alpha+2)}\, (\wedge z_{\alpha+2})^2\, dx \\
&- 2 \int_{\mathbb{R}} \partial(\xi\, h_2^{(\alpha+3)}\, w_\alpha)\, \wedge z_{\alpha+2}\, dx - 2 \int_{\mathbb{R}} \partial(\xi\, h_2^{(\alpha+2)}\, w_\alpha) \wedge z_{\alpha+1}\, dx \\
&- 2 \int_{\mathbb{R}} \partial(\xi\, k_1^{(\alpha+3)}\, z_\alpha)\, \wedge w_{\alpha+2}\, dx - 2 \int_{\mathbb{R}} \partial(\xi\, k_1^{(\alpha+2)}\, z_\alpha)\, \wedge w_{\alpha+1}\, dx \\
&+ 2 \sum_{j=3}^{\alpha+1} \int_{\mathbb{R}} \xi\, h_1^{(j)} w_\alpha \wedge w_j\, dx + 2 \sum_{j=3}^{\alpha+1} \int_{\mathbb{R}} \xi\, k_2^{(j)}\, z_\alpha \wedge z_j\, dx \\
&+ 2 \sum_{j=3}^{\alpha+1} \int_{\mathbb{R}} \xi\, h_2^{(j)}\, w_\alpha \wedge z_j\, dx + 2 \sum_{j=3}^{\alpha+1} \int_{\mathbb{R}} \xi\, k_1^{(j)}\, z_\alpha \wedge w_j\, dx \\
&+ 2 \int_{\mathbb{R}} \xi\, r_1\, w_\alpha \wedge \theta_{\alpha+1}\, dx + 2 \int_{\mathbb{R}} \xi\, m_1\, z_\alpha \wedge \theta_{\alpha+1}\, dx + 2 \int_{\mathbb{R}} \xi\, s_1\, w_\alpha \wedge \phi_{\alpha+1}\, dx + 2 \int_{\mathbb{R}} \xi\, n_1\, z_\alpha \wedge \phi_{\alpha+1}\, dx \\
&+ 2b_2 \int_{\mathbb{R}} \xi\, r_2\, w_\alpha\, dx + 2 \int_{\mathbb{R}} \xi\, m_2\, z_\alpha\, dx + 2 \int_{\mathbb{R}} \xi\, s_2\, w_\alpha\, dx + 2 \int_{\mathbb{R}} \xi\, n_2\, z_\alpha\, dx.
\end{aligned}
$$

Since $a < 0$, then the third, fourth in the equality are non positive, then

$$
\begin{aligned}
&\partial_t \int_{\mathbb{R}} \xi\, w_\alpha^2\, dx + \partial_t \int_{\mathbb{R}} \xi\, z_\alpha^2\, dx \le \int_{\mathbb{R}} \xi_t\, w_\alpha^2\, dx + \int_{\mathbb{R}} \xi_t\, z_\alpha^2\, dx + \\
&- a \int_{\mathbb{R}} (\partial^5\xi - 5\,\partial\xi)\,(\wedge w_{\alpha+2})^2\, dx + \int_{\mathbb{R}} (\partial^5\xi - 5\,\partial\xi)\,(\wedge z_{\alpha+2})^2\, dx \\
&- 5\,a \int_{\mathbb{R}} \partial^3\xi\,(\wedge w_{\alpha+1})^2 dx + 5 \int_{\mathbb{R}} \partial^3\xi\,(\wedge z_{\alpha+1})^2\, dx + a \int_{\mathbb{R}} \partial^5\xi\,(\wedge w_\alpha)^2\, dx - \int_{\mathbb{R}} \partial^5\xi\,(\wedge z_\alpha)^2\, dx \\
&- \int_{\mathbb{R}} \partial^3(\xi\, h_1^{(\alpha+3)})\,(\wedge w_\alpha)^2\, dx + 3 \int_{\mathbb{R}} \partial(\xi\, h_1^{(\alpha+3)})\,(\wedge w_{\alpha+1})^2\, dx + \int_{\mathbb{R}} \partial(\xi\, h_1^{(\alpha+3)})\,(\wedge w_{\alpha+2})^2\, dx \\
&- \int_{\mathbb{R}} \partial^3(\xi\, k_2^{(\alpha+3)})\,(\wedge z_\alpha)^2\, dx + 3 \int_{\mathbb{R}} \partial(\xi\, k_2^{(\alpha+3)})\,(\wedge z_{\alpha+1})^2\, dx + \int_{\mathbb{R}} \partial(\xi\, k_2^{(\alpha+3)})\,(\wedge z_{\alpha+2})^2\, dx \\
&+ \int_{\mathbb{R}} \partial^2(\xi\, h_1^{(\alpha+2)})\,(\wedge w_\alpha)^2\, dx - 2 \int_{\mathbb{R}} \xi\, h_1^{(\alpha+2)}(\wedge w_{\alpha+1})^2\, dx - 2 \int_{\mathbb{R}} \xi\, h_1^{(\alpha+2)}(\wedge w_{\alpha+2})^2\, dx \\
&+ \int_{\mathbb{R}} \partial^2(\xi\, k_2^{(\alpha+2)})\,(\wedge z_\alpha)^2\, dx - 2 \int_{\mathbb{R}} \xi\, k_2^{(\alpha+2)}(\wedge z_{\alpha+1})^2\, dx - 2 \int_{\mathbb{R}} \xi\, k_2^{(\alpha+2)}(\wedge z_{\alpha+2})^2\, dx \\
&- 2 \int_{\mathbb{R}} \partial(\xi\, h_2^{(\alpha+3)}\, w_\alpha)\, \wedge z_{\alpha+2}\, dx - 2 \int_{\mathbb{R}} \partial(\xi\, h_2^{(\alpha+2)}\, w_\alpha) \wedge z_{\alpha+1}\, dx \\
&- 2 \int_{\mathbb{R}} \partial(\xi\, k_1^{(\alpha+3)} z_\alpha)\, \wedge w_{\alpha+2}\, dx - 2 \int_{\mathbb{R}} \partial(\xi\, k_1^{(\alpha+2)} z_\alpha)\, \wedge w_{\alpha+1}\, dx \\
&+ 2 \sum_{j=3}^{\alpha+1} \int_{\mathbb{R}} \xi\, h_1^{(j)} w_\alpha \wedge w_j\, dx + 2 \sum_{j=3}^{\alpha+1} \int_{\mathbb{R}} \xi\, k_2^{(j)}\, z_\alpha \wedge z_j\, dx \\
&+ 2 \sum_{j=3}^{\alpha+1} \int_{\mathbb{R}} \xi\, h_2^{(j)}\, w_\alpha \wedge z_j\, dx + 2 \sum_{j=3}^{\alpha+1} \int_{\mathbb{R}} \xi\, k_1^{(j)}\, z_\alpha \wedge w_j\, dx \\
&+ 2 \int_{\mathbb{R}} \xi\, r_1\, w_\alpha \wedge \theta_{\alpha+1}\, dx + 2 \int_{\mathbb{R}} \xi\, m_1\, z_\alpha \wedge \theta_{\alpha+1}\, dx + 2 \int_{\mathbb{R}} \xi\, s_1\, w_\alpha \wedge \phi_{\alpha+1}\, dx + 2 \int_{\mathbb{R}} \xi\, n_1\, z_\alpha \wedge \phi_{\alpha+1}\, dx \\
&+ 2\, b_2 \int_{\mathbb{R}} \xi\, r_2\, w_\alpha\, dx + 2 \int_{\mathbb{R}} \xi\, m_2\, z_\alpha\, dx + 2 \int_{\mathbb{R}} \xi\, s_2\, w_\alpha\, dx + 2 \int_{\mathbb{R}} \xi\, n_2\, z_\alpha\, dx.
\end{aligned}
$$

Using that

$$\wedge w_n = \wedge w_{n-2} - w_{n-2} \tag{4.18}$$

and standard estimates, the Lemma follows.

5 Uniqueness and Existence of a Local Solution

In this section, we study the uniqueness and the existence of local strong solutions in the Sobolev space $H^N(\mathbb{R})$ for $N \ge 3$ for problem (2.8), (2.9). To establish the existence of strong solutions for (2.8), (2.9) we use the a priori estimate together with an approximation procedure.

Theorem 5.1 *(Uniqueness). Let $a < 0$, $(u_0(x),\, v_0(x)) \in H^3(\mathbb{R}) \times H^3(\mathbb{R})$ with N an integer ≥ 3, $i \in \mathbb{N}$. We suppose that there is at least one local strong solution of* (2.8), (2.9) *in the interval* $[0,\, T]$. *Then there is at most one strong solution $(u,\, v) \in L^\infty([0,\, T] : H^N(\mathbb{R})) \times L^\infty([0,\, T] : H^N(\mathbb{R}))$ of* (2.8), (2.9) *with initial data $u(x,\, 0) = u_0(x)$ and $v(x,\, 0) = v_0(x)$.*

Proof. Assume that $(u,\, v)$ and $(u',\, v')$ are two solutions of (2.8), (2.9) in $L^\infty([0,\, T] : H^N(\mathbb{R})) \times L^\infty([0,\, T] : H^N(\mathbb{R}))$ with the same initial data $(u_0(x),\, v_0(x))$. From (2.8), (2.9), $u_t,\ v_t,\ u'_t,\ v'_t \in L^\infty([0,\, T] : L^2(\mathbb{R}))$, so the integrations below are justified. Therefore, the difference $(u - u')$ satisfies

$$(u - u')_t - a\,(u - u')_3 + u\, u_1 - u'\, u'_1 = 2\, b\, v\, v_1 - 2\, b\, v'\, v'_1. \tag{5.1}$$

Multiplying (5.1) by $2\,\xi\,(u-u')$, and integrating over $x\in\mathbb{R}$ our equation becomes

$$\begin{aligned}
&2\int_{\mathbb{R}}\xi\,(u-u')\,(u-u')_t\,dx-2\,a\int_{\mathbb{R}}\xi\,(u-u')\,(u-u')_3\,dx\\
&+2\int_{\mathbb{R}}\xi\,(u-u')\,u\,u_1\,dx-2\int_{\mathbb{R}}\xi\,(u-u')\,u'\,u'_1\,dx\\
&=4\,b\int_{\mathbb{R}}\xi\,(u-u')\,v\,v_1\,dx-4\,b\int_{\mathbb{R}}\xi\,(u-u')\,v'\,v'_1\,dx.
\end{aligned}\tag{5.2}$$

Each term in (5.2) is treated separately integrating by parts. The first two terms yield

$$2\int_{\mathbb{R}}\xi\,(u-u')\,(u-u')_t\,dx \;=\; \partial_t\int_{\mathbb{R}}\xi\,(u-u')^2\,dx-\int_{\mathbb{R}}\xi_t\,(u-u')^2\,dx.$$

$$-\,2\,a\int_{\mathbb{R}}\xi\,(u-u')\,(u-u')_3\,dx \;=\; a\int_{\mathbb{R}}\partial^3\xi\,(u-u')^2\,dx-3\,a\int_{\mathbb{R}}\partial\xi\,(u-u')_1^2\,dx.$$

The other terms in (5.2) are also treated the similar form, using integrating by parts.

$$\begin{aligned}
&\partial_t\int_{\mathbb{R}}\xi\,(u-u')^2\,dx-\int_{\mathbb{R}}\xi_t\,(u-u')^2\,dx-3\,a\int_{\mathbb{R}}\partial\xi\,(u-u')_1^2\,dx+a\int_{\mathbb{R}}\partial^3\xi\,(u-u')^2\,dx\\
&+\int_{\mathbb{R}}\xi\,(u-u')\,\partial(u^2)\,dx-\int_{\mathbb{R}}\xi\,(u-u')\,\partial({u'}^2)\,dx\\
&=2\,b\int_{\mathbb{R}}\xi\,(u-u')\,\partial(v^2)\,dx-2\,b\int_{\mathbb{R}}\xi\,(u-u')\,\partial({v'}^2)\,dx
\end{aligned}$$

then

$$\begin{aligned}
&\partial_t\int_{\mathbb{R}}\xi\,(u-u')^2\,dx-\int_{\mathbb{R}}\xi_t\,(u-u')^2\,dx-3\,a\int_{\mathbb{R}}\partial\xi\,(u-u')_1^2\,dx+a\int_{\mathbb{R}}\partial^3\xi\,(u-u')^2\,dx\\
&+\int_{\mathbb{R}}\xi\,(u-u')\,\partial(u^2-{u'}^2)\,dx=2\,b\int_{\mathbb{R}}\xi\,(u-u')\,\partial(v^2-{v'}^2)\,dx.
\end{aligned}\tag{5.3}$$

Moreover

$$\begin{aligned}
\int_{\mathbb{R}}\xi\,(u-u')\,\partial(u^2-{u'}^2)\,dx \;&=\; -\int_{\mathbb{R}}\partial(\xi\,(u-u'))\,(u^2-{u'}^2)\,dx\\
&=\; -\int_{\mathbb{R}}\partial\xi\,(u-u')\,(u^2-{u'}^2)\,dx-\int_{\mathbb{R}}\xi\,(u-u')_1\,(u^2-{u'}^2)\,dx\\
&=\; -\int_{\mathbb{R}}\partial\xi\,(u-u')\,(u-u')\,(u+u')\,dx-\int_{\mathbb{R}}\xi\,(u-u')\,(u-u')_1\,(u+u')\,dx\\
&=\; -\int_{\mathbb{R}}\partial\xi\,(u+u')\,(u-u')^2\,dx-\int_{\mathbb{R}}\xi\,(u+u')\,(u-u')\,(u-u')_1\,dx\\
&=\; -\int_{\mathbb{R}}\partial\xi\,(u+u')\,(u-u')^2\,dx+\frac{1}{2}\int_{\mathbb{R}}\partial(\xi\,(u+u'))\,(u-u')^2\,dx.
\end{aligned}$$

$$\begin{aligned}
2\,b\int_{\mathbb{R}}\xi\,(u-u')\,\partial(v^2-{v'}^2)\,dx \;&=\; -2\,b\int_{\mathbb{R}}\partial(\xi\,(u-u'))\,(v^2-{v'}^2)\,dx\\
&=\; -2\,b\int_{\mathbb{R}}\partial\xi\,(u-u')\,(v^2-{v'}^2)\,dx-2\,b\int_{\mathbb{R}}\xi\,(u-u')_1\,(v^2-{v'}^2)\,dx\\
&=\; -2\,b\int_{\mathbb{R}}\partial\xi\,(u-u')\,(v-v')\,(v+v')\,dx-2\,b\int_{\mathbb{R}}\xi\,(u-u')_1\,(v-v')\,(v+v')\,dx.
\end{aligned}$$

hence in (5.3) we have

$$\partial_t \int_{\mathbb{R}} \xi\,(u-u')^2\,dx - \int_{\mathbb{R}} \xi_t\,(u-u')^2\,dx - 3\,a \int_{\mathbb{R}} \partial\xi\,(u-u')_1^2\,dx + a \int_{\mathbb{R}} \partial^3\xi\,(u-u')^2\,dx$$
$$- \int_{\mathbb{R}} \partial\xi\,(u+u')\,(u-u')^2\,dx + \frac{1}{2}\int_{\mathbb{R}} \partial(\xi\,(u+u'))\,(u-u')^2\,dx$$
$$= -\,2\,b \int_{\mathbb{R}} \partial\xi\,(u-u')\,(v-v')\,(v+v')\,dx - 2\,b \int_{\mathbb{R}} \xi\,(u-u')_1\,(v-v')\,(v+v')\,dx.$$

where using that $a<0$ we obtain

$$\partial_t \int_{\mathbb{R}} \xi\,(u-u')^2\,dx - \int_{\mathbb{R}} \xi_t\,(u-u')^2\,dx + a \int_{\mathbb{R}} \partial^3\xi\,(u-u')^2\,dx$$
$$- \int_{\mathbb{R}} \partial\xi\,(u+u')\,(u-u')^2\,dx + \frac{1}{2}\int_{\mathbb{R}} \partial(\xi\,(u+u'))\,(u-u')^2\,dx$$
$$\le -\,2\,b \int_{\mathbb{R}} \partial\xi\,(u-u')\,(v-v')\,(v+v')\,dx - 2\,b \int_{\mathbb{R}} \xi\,(u-u')_1\,(v-v')\,(v+v')\,dx. \tag{5.4}$$

The difference $(v-v')$ satisfies

$$(v-v')_t + (v-v')_3 + 3\,u\,v_1 - 3\,u'\,v_1' = 0. \tag{5.5}$$

Multiplying (5.5) by $2\,\xi\,(v-v')$, integrating over $x\in\mathbb{R}$ and performing the similar calculations our equation becomes

$$\partial_t \int_{\mathbb{R}} \xi\,(v-v')^2\,dx - \int_{\mathbb{R}} \xi_t\,(v-v')^2\,dx + 3 \int_{\mathbb{R}} \partial\xi\,(v-v')_1^2\,dx - \int_{\mathbb{R}} \partial^3\xi\,(v-v')^2\,dx$$
$$+ 6 \int_{\mathbb{R}} \xi\,(v-v')\,u\,v_1\,dx - 6 \int_{\mathbb{R}} \xi\,(v-v')\,u'\,v_1'\,dx = 0$$

then

$$\partial_t \int_{\mathbb{R}} \xi\,(v-v')^2\,dx - \int_{\mathbb{R}} \xi_t\,(v-v')^2\,dx + 3 \int_{\mathbb{R}} \partial\xi\,(v-v')_1^2\,dx - \int_{\mathbb{R}} \partial^3\xi\,(v-v')^2\,dx$$
$$= -\,6 \int_{\mathbb{R}} \xi\,(v-v')\,(u\,v_1 - u'\,v_1')\,dx. \tag{5.6}$$

Moreover

$$u\,v_1 - u'\,v_1' = u\,(v_1 - v_1') + (u-u')\,v_1' = u\,(v-v')_1 + (u-u')\,v_1'$$

hence in (5.6) we have

$$\partial_t \int_{\mathbb{R}} \xi\,(v-v')^2\,dx - \int_{\mathbb{R}} \xi_t\,(v-v')^2\,dx + 3 \int_{\mathbb{R}} \partial\xi\,(v-v')_1^2\,dx - \int_{\mathbb{R}} \partial^3\xi\,(v-v')^2\,dx$$
$$= -\,3 \int_{\mathbb{R}} \xi\,(v-v')\,[u\,(v-v')_1 + (u-u')\,v_1']\,dx.$$

then

$$\partial_t \int_{\mathbb{R}} \xi\,(v-v')^2\,dx - \int_{\mathbb{R}} \xi_t\,(v-v')^2\,dx + 3 \int_{\mathbb{R}} \partial\xi\,(v-v')_1^2\,dx - \int_{\mathbb{R}} \partial^3\xi\,(v-v')^2\,dx$$
$$= -\,3 \int_{\mathbb{R}} \xi\,(v-v')\,u\,(v-v')_1\,dx - 3 \int_{\mathbb{R}} \xi\,(v-v')\,(u-u')\,v_1'\,dx$$
$$= -\,\frac{3}{2}\int_{\mathbb{R}} \xi\,u\,\partial[(v-v')^2]\,dx - 3 \int_{\mathbb{R}} \xi\,v_1'\,(v-v')\,(u-u')\,dx$$
$$= \frac{3}{2}\int_{\mathbb{R}} \partial(\xi\,u)\,(v-v')^2\,dx - 3 \int_{\mathbb{R}} \xi\,v_1'\,(v-v')\,(u-u')\,dx$$

where

$$\partial_t \int_{\mathbb{R}} \xi\,(v-v')^2\,dx - \int_{\mathbb{R}} \xi_t\,(v-v')^2\,dx - \int_{\mathbb{R}} \partial^3\xi\,(v-v')^2\,dx$$
$$\le \frac{3}{2}\int_{\mathbb{R}} \partial(\xi\,u)\,(v-v')^2\,dx - 3\int_{\mathbb{R}} \xi\,v_1'\,(v-v')\,(u-u')\,dx \tag{5.7}$$

Adding (5.4) with (5.7) and using straightforward calculus it follows that ($\xi \in W_{0\,i\,0}$)

$$\partial_t \int_{\mathbb{R}} \xi\,(u-u')^2\,dx + \partial_t \int_{\mathbb{R}} \xi\,(v-v')^2\,dx \le c\left(\int_{\mathbb{R}} \xi\,(u-u')^2\,dx + \int_{\mathbb{R}} \xi\,(v-v')^2\,dx\right)$$

for some positive constant c. Using that $u(x,\,0)-u'(x,\,0)\equiv 0$ and $v(x,\,0)-v'(x,\,0)\equiv 0$, and Gronwall's inequality it follows that

$$\int_{\mathbb{R}} \xi\,(u-u')^2\,dx + \int_{\mathbb{R}} \xi\,(v-v')^2\,dx \le 0.$$

We conclude that $u \equiv u'$ and $v \equiv v'$. This proves the uniqueness of the solution. We construct the mapping

$$\mathcal{Z} : L^\infty([0,\,T] :\, H^s(\mathbb{R})) \times L^\infty([0,\,T] :\, H^s(\mathbb{R})) \to L^\infty([0,\,T] :\, H^s(\mathbb{R})) \times L^\infty([0,\,T] :\, H^s(\mathbb{R}))$$

where the initial condition is given by $u^{(0)} = u_0(x)$, $v^{(0)} = v_0(x)$ and the first approximation is given by $u^{(n)} = \mathcal{Z}(u^{(n-1)})$, $v^{(n)} = \mathcal{Z}(v^{(n-1)})$ for $n \ge 1$, where $u^{(n-1)}$ is in a place of θ, $v^{(n-1)}$ is in a place of ϕ, $u^{(n)}$ is in a place of u, $v^{(n)}$ is in a place of v in the equations (4.5), (4.6) which are the solution of the equations (4.5), (4.6). By Lemma 4.1, $(u^{(n)},\,v^{(n)})$ exists and is unique in $C((0,\,+\infty) :\, H^{(N)}(\mathbb{R})) \times C((0,\,+\infty) :\, H^{(N)}(\mathbb{R}))$. A choice of C_0 and the use of the a priori estimate in Section 4 show that $\mathcal{Z} : \mathbb{B}_{C_0} \to \mathbb{B}_{C_0}$ where $\mathbb{B}_{C_0}$ is a bounded ball in $L^\infty([0,\,T] :\, H^s(\mathbb{R}))$.

Theorem 5.2 *(Local solution). Let $a < 0$ and N an integer ≥ 3. If $(u_0(x),\,v_0(x)) \in H^N(\mathbb{R}) \times H^N(\mathbb{R})$, then there are $T > 0$ and $(u,\,v)$ such that $(u,\,v)$ is a strong solution of* (2.8), (2.9), *$(u,\,v) \in L^\infty([0,\,T] :\, H^N(\mathbb{R})) \times L^\infty([0,\,T] :\, H^N(\mathbb{R}))$ and $u(x,\,0) = u_0(x)$, $v(x,\,0) = v_0(x)$.*

Proof. We prove that for $u_0,\,v_0 \in H^\infty(\mathbb{R}) = \bigcap_{k\ge 0} H^k(\mathbb{R})$ there exists a solution $(u,\,v) \in L^\infty([0,\,T] :\, H^N(\mathbb{R})) \times L^\infty([0,\,T] :\, H^N(\mathbb{R}))$ with initial data $(u(x,\,0),\,v(x,\,0)) = (u_0(x),\,v_0(x))$. where the time of existence $T > 0$ only depends on the norm of $u_0(x)$ and $v_0(x)$. We define a sequence of approximations to equations (4.5)-(4.6)

$$\begin{aligned} w_t^{(n)} \;&=\; -\,a \wedge w_5^{(n)} + 6 \wedge w^{(n-1)} \wedge w_3^{(n)} + a \wedge w_3^{(n)} - 2\,b \wedge z^{(n)} \wedge z_3^{(n)} + 18 \wedge w_1^{(n-1)} \wedge w_2^{(n)} \\ &\quad - 6\,b \wedge z_1^{(n-1)} \wedge z_2^{(n)} - 6 \wedge w^{(n-1)} \wedge w_1^{(n)} + 2\,b \wedge z^{(n-1)} \wedge z_1^{(n)} \end{aligned} \tag{5.8}$$

$$\begin{aligned} z_t^{(n)} \;&=\; \wedge z_5^{(n)} + 3 \wedge w^{(n-1)} \wedge z_3^{(n)} - \wedge z_3^{(n)} + 6 \wedge w_1^{(n-1)} \wedge z_2^{(n)} \\ &\quad + 3 \wedge w_2^{(n-1)} \wedge z_1^{(n)} - 3 \wedge w^{(n)} \wedge z_1^{(n)} \end{aligned} \tag{5.9}$$

where the initial conditions $u^{(n)}(x,\,0) = u_0(x) - \partial^2 u_0(x)$ and $v^{(n)}(x,\,0) = v_0(x) - \partial^2 v_0(x)$. The first approximation are given by $u^{(0)}(x,\,0) = u_0(x) - \partial^2 u_0(x)$ and $v^{(0)}(x,\,0) = v_0(x) - \partial^2 v_0(x)$. Equations (5.8)-(5.9) are linear equations at each iteration which can be solved in any interval of time in which the coefficients are defined. This is shown in Lemma 4.1. By Lemma 4.2, it follows that

$$\begin{aligned} \partial_t \int_{\mathbb{R}} \xi\,[w_\alpha^{(n)}]^2\,dx + \partial_t \int_{\mathbb{R}} \xi\,[z_\alpha^{(n)}]^2\,dx \;&\le\; G(\|w^{(n-1)}\|_\lambda,\, \|z^{(n-1)}\|_\lambda)\,(\|w^{(n)}\|_\alpha^2 + \|z^{(n)}\|_\alpha^2\,) \\ &\quad + E(\|w^{(n-1)}\|_\lambda,\, \|z^{(n-1)}\|_\lambda)\,(\|w^{(n-1)}\|_\alpha^2 + \|z^{(n-1)}\|_\alpha^2\,) \\ &\quad + M(\|w^{(n-1)}\|_\alpha,\, \|z^{(n-1)}\|_\alpha). \end{aligned} \tag{5.10}$$

Let

$$N = \partial_t \int_{\mathbb{R}} \xi\,[w_\alpha^{(n)}]^2\,dx + \partial_t \int_{\mathbb{R}} \xi\,[z_\alpha^{(n)}]^2\,dx \tag{5.11}$$

where $\xi \in W_{0\,i\,0}$. ($\Rightarrow \xi(x,\,0) = \xi(x,\,t)$). Then we have

$$\begin{aligned} N &\leq G(\|w^{(n-1)}\|_\lambda,\ \|z^{(n-1)}\|_\lambda)\,(\|w^{(n)}\|_\alpha^2 + \|z^{(n)}\|_\alpha^2) \\ &\quad + E(\|w^{(n-1)}\|_\lambda,\ \|z^{(n-1)}\|_\lambda)\,(\|w^{(n-1)}\|_\alpha^2 + \|z^{(n-1)}\|_\alpha^2) \\ &\quad + M(\|w^{(n-1)}\|_\alpha,\ \|z^{(n-1)}\|_\alpha) \end{aligned} \tag{5.12}$$

where $\lambda = \max\{1,\,\alpha\}$.
Choose $\alpha = 1$ and $c \geq \|u_0 - \partial^2 u_0\|_1 \geq \|u_0\|_3$; $c' \geq \|v_0 - \partial^2 v_0\|_1 \geq \|v_0\|_3$. For each iterate n, $w^{(n)}$, $z^{(n)}$: $\|w^{(n)}(\cdot,\,t)\|$, $\|z^{(n)}(\cdot,\,t)\|$ are continuous in $t \in [0,\,T]$ and $\|w^{(n)}(\cdot,\,0)\|_1 = \|\varphi - \partial^2\varphi\|_1 \leq c$, $\|z^{(n)}(\cdot,\,0)\|_1 = \|\psi - \partial^2\psi\|_1 \leq c'$. Define $c_0 = \left(1 + \frac{c^2}{2}\right)$, $c_0' = \left(1 + \frac{c'^2}{2}\right)$. Let $T_0^{(n)}$ be the maximum time such that: $\|w^{(k)}(\cdot,\,t)\|_1 \leq c_0$, $\|z^{(k)}(\cdot,\,t)\|_1 \leq c_0'$ for $0 \leq t \leq T_0^{(n)}$ and $0 \leq k \leq n$, i. e.,

$$T_0^{(n)} = \sup\{\, t \,:\, \|w^{(k)}(\cdot\,,\widetilde{t})\|_1 \leq c_0\,,\ \|z^{(k)}(\cdot\,,\widetilde{t})\|_1 \leq c_0' \quad \text{for} \quad 0 \leq \widetilde{t} \leq t, \quad 0 \leq k \leq n\,\}.$$

Integrating (5.12) over $[0,\,t]$ we have that for $0 \leq t \leq T_0^{(n)}$ and $j = 0,\,1$.

$$\begin{aligned} \|w^{(n)}\|_1^2 + \|z^{(n)}\|_1^2 &\leq c^2 + {c'}^2 + G(c_0,\,c_0')\,c_0^2\,t + G(c_0,\,c_0')\,{c_0'}^2\,t \\ &\quad + E(c_0,\,c_0')\,c_0^2\,t + E(c_0,c_0')\,{c_0'}^2\,t + M(c_0,\,c_0')\,t. \end{aligned}$$

Claim $T_0^{(n)}$ does not approach 0.
On the contrary, assume that $T_0^{(n)} \to 0$. Since $\|w^{(n)}(\cdot,\,t)\|$ and $\|z^{(n)}(\cdot,\,t)\|$ are continuous for $t \geq 0$, there exists $\tau \in [0,\,T]$ such that $\|w^{(k)}(\cdot,\,\tau)\|_1 = c_0$ and $\|z^{(k)}(\cdot,\,\tau)\|_1 = c_0'$ for $0 \leq \tau \leq T_0^{(n)}$, $0 \leq k \leq n$. Then

$$\begin{aligned} c_0^2 + {c_0'}^2 &\leq c^2 + {c'}^2 + G(c_0,\,c_0')\,c_0^2\,T_0^{(n)} + G(c_0,\,c_0')\,{c_0'}^2\,T_0^{(n)} \\ &\quad + E(c_0,\,c_0')\,c_0^2\,T_0^{(n)} + E(c_0,\,c_0')\,{c_0'}^2\,T_0^{(n)} + M(c_0,\,c_0')\,T_0^{(n)} \end{aligned}$$

as $n \to +\infty$, we have

$$\left(1 + c^2 + \frac{c^4}{4}\right) + \left(1 + {c'}^2 + \frac{{c'}^4}{4}\right) \leq c^2 + {c'}^2$$

where

$$\left(1 + \frac{c^4}{4}\right) + \left(1 + \frac{c^{,4}}{4}\right) \leq 0$$

which is a contradiction. Consequently $T_0^{(n)} \not\to 0$. Choosing $T = T(c,\,c')$ sufficiently small, and T not depending on n, one concludes that

$$\sup_{0 \leq t \leq T} \|w^{(n)}\|_1^2 + \sup_{0 \leq t \leq T} \|z^{(n)}\|_1^2 \leq C \tag{5.13}$$

for $0 \leq t \leq T$. This show that $T_0^{(n)} \geq T$. Hence, from (5.13) we imply that there exist subsequences $w^{(n_j)} \stackrel{\text{def}}{=} w^{(n)}$, $\quad z^{(n_j)} \stackrel{\text{def}}{=} z^{(n)}$ such that

$$w^{(n)} \stackrel{*}{\rightharpoonup} w \quad \text{weakly on} \quad L^\infty([0,\,T] : H^1(\mathbb{R})) \tag{5.14}$$

$$z^{(n)} \stackrel{*}{\rightharpoonup} z \quad \text{weakly on} \quad L^\infty([0,\,T] : H^1(\mathbb{R})). \tag{5.15}$$

Claim. $u = \wedge w$ and $v = \wedge z$ are solutions.
In the linearized equation (5.8) we have

$$\wedge w_5^{(n)} = \wedge(I - (I - \partial^2))w_3^{(n)} = \wedge w_3^{(n)} - w_3^{(n)} = \partial^2(\underbrace{\wedge w_1^{(n)}}_{\in L^2(\mathbb{R})}) - \underbrace{\partial^2(w_1^{(n)})}_{\in H^{-2}(\mathbb{R})} \in H^{-2}(\mathbb{R}).$$

Since $\wedge = (I-\partial^2)^{-1}$ is bounded in $H^1(\mathbb{R})$, $\wedge w_5^{(n)} \in H^{-2}(\mathbb{R})$. $w^{(n)}$ is still bounded in $L^\infty([0,T] : H^1(\mathbb{R})) \hookrightarrow L^2([0,T] : H^1(\mathbb{R}))$ and since $\wedge : L^2(\mathbb{R}) \to H^2(\mathbb{R})$ is a bounded operator $\|\wedge w_1^{(n)}\|_{H^2(\mathbb{R})} \leq c\,\|w_1^{(n)}\|_{L^2(\mathbb{R})} \leq c'\,\|w_1^{(n)}\|_{H^1(\mathbb{R})}$. Consequently $\wedge w_1^{(n)}$ is bounded in $L^2([0,T] : H^2(\mathbb{R})) \hookrightarrow L^2([0,T] : L^2(\mathbb{R}))$. It follows that $\partial^2(\wedge w_1^{(n)})$ is bounded in $L^2([0,T] : H^{-2}(\mathbb{R}))$ and

$$\wedge w_5^{(n)} \quad \text{is bounded in} \quad L^2([0,T] : H^{-2}(\mathbb{R})). \tag{5.16}$$

Similarly, the other terms are bounded. By (5.8)-(5.9) we have that $w_t^{(n)}$ and $z_t^{(n)}$ is a sum of terms each of which is the product of a coefficient, uniformly bounded on n and a function in $L^2([0,T] : H^{-2}(\mathbb{R}))$ uniformly bounded on n such that $w_t^{(n)}$ is bounded in $L^2([0,T] : H^{-2}(\mathbb{R}))$ and $z_t^{(n)}$ is bounded in $L^2([0,T] : H^{-2}(\mathbb{R}))$. On the other hand, $H^1_{loc}(\mathbb{R}) \overset{c}{\hookrightarrow} H^{1/2}_{loc}(\mathbb{R}) \hookrightarrow H^{-2}(\mathbb{R})$. By the Lions-Aubin compactness Theorem, there are subsequences such that $w^{(n_j)} \overset{\text{def}}{=} w^{(n)}$ and $z^{(n_j)} \overset{\text{def}}{=} z^{(n)}$ such that $w^{(n)} \to w$ strongly on $L^2([0,T] : H^{1/2}_{loc}(\mathbb{R}))$ and $z^{(n)} \to z$ strongly on $L^2([0,T] : H^{1/2}_{loc}(\mathbb{R}))$. Hence, for a subsequence $w^{(n_j)} \overset{\text{def}}{=} w^{(n)}$ and $z^{(n_j)} \overset{\text{def}}{=} z^{(n)}$ we have $w^{(n)} \to w$ a.e. in $L^2([0,T] : H^{1/2}_{loc}(\mathbb{R}))$ and $z^{(n)} \to z$ a.e. in $L^2([0,T] : H^{1/2}_{loc}(\mathbb{R}))$. Moreover, from (5.16), $\wedge w_5^{(n)} \rightharpoonup \wedge w_5$ weakly in $L^2([0,T] : H^{-2}(\mathbb{R}))$. Similarly, for the other nonlinear terms, $\wedge w_3^{(n)} \rightharpoonup \wedge w_3$ weakly in $L^2([0,T] : H^{-2}(\mathbb{R}))$. Since $\|\wedge w^{(n)}\|_{H^3(\mathbb{R})} \leq c\,\|w^{(n)}\|_{H^1(\mathbb{R})} \leq c'\,\|w^{(n)}\|_{H^{1/2}(\mathbb{R})}$ and $w^{(n)} \to w$ strongly on $L^2([0,T] : H^{1/2}_{loc}(\mathbb{R}))$ then $\wedge w^{(n)} \to \wedge w$ strongly on $L^2([0,T] : H^3_{loc}(\mathbb{R}))$. It follow that $\partial(\wedge w^{(n)}) \to \partial(\wedge w)$ strongly on $L^2([0,T] : H^2_{loc}(\mathbb{R}))$ and we have $\wedge w_1^{(n)} \to \wedge w_1$ strongly on $L^2([0,T] : H^2_{loc}(\mathbb{R}))$. Thus, the second term on the right-hand side of (5.8), $\wedge w^{(n-1)}\,\wedge w_3^{(n)} \rightharpoonup \wedge w\,\wedge w_3$ weakly in $L^2([0,T] : L^1_{loc}(\mathbb{R}))$ as $\wedge w_3^{(n)} \rightharpoonup \wedge w_3$ weakly in $L^2([0,T] : H^{-2}(\mathbb{R}))$ and $|\wedge w^{(n)}|^2 \to |\wedge w|^2$ strongly on $L^2([0,T] : H^2_{loc}(\mathbb{R}))$. Similarly, the other terms in (5.8) and (5.9) converge to their corrects limits, implying $w_t^{(n)} \rightharpoonup w_t$ weakly in $L^2([0,T] : L^1_{loc}(\mathbb{R}))$ and $z_t^{(n)} \rightharpoonup z_t$ weakly in $L^2([0,T] : L^1_{loc}(\mathbb{R}))$. Passing to the limit

$$\begin{aligned} w_t &= -\,a\wedge w_5 + 6\wedge w\wedge w_3 + a\wedge w_3 - 2\,b\wedge z\wedge z_3 + 18\wedge w_1\wedge w_2 \\ &\quad -6\,b\wedge z_1\wedge z_2 - 6\wedge w\wedge w_1 + 2\,b\wedge z\wedge z_1 \\ z_t &= \wedge z_5 + 3\wedge w\wedge z_3 - \wedge z_3 + 6\wedge w_1\wedge z_2 + 3\wedge w_2\wedge z_1 - 3\wedge w\wedge z_1. \end{aligned}$$

Thus

$$\begin{aligned} w_t &= \partial^2(-\,a\wedge w_3 + 6\wedge w\wedge w_1 + 2\,b\wedge z\wedge z_1) - (-\,a\wedge w_3 + 6\wedge w\wedge w_1 + 2\,b\wedge z\wedge z_1) \\ z_t &= \partial^2(\wedge z + 3\wedge w\wedge z_1) - (\wedge z + 3\wedge w\wedge z_1) \end{aligned}$$

where

$$\begin{aligned} w_t &= -(I-\partial^2)\,(-\,a\wedge w_3 + 6\wedge w\wedge w_1 + 2\,b\wedge z\wedge z_1) \\ z_t &= -(I-\partial^2)\,(\wedge z + 3\wedge w\wedge z_1). \end{aligned}$$

Hence

$$\begin{aligned} &w_t + (I-\partial^2)\,(-\,a\wedge w_3 + 6\wedge w\wedge w_1 + 2\,b\wedge z\wedge z_1) = 0 \\ &z_t + (I-\partial^2)\,(\wedge z + 3\wedge w\wedge z_1) = 0. \end{aligned}$$

This way, we have (2.8) and (2.9) for $u = \wedge w$ and $v = \wedge z$. Now, we prove that there exists a solution of (2.8), (2.9) with $(u, v) \in L^\infty([0,T] : H^N(\mathbb{R})) \times L^\infty([0,T] : H^N(\mathbb{R}))$ and $N \geq 4$, where T depends only on the norm $\|u_0\|_{H^3(\mathbb{R})}$ and $\|v_0\|_{H^3(\mathbb{R})}$. We already know that there is a solution $(u, v) \in L^\infty([0,T] : H^3(\mathbb{R})) \times L^\infty([0,T] : H^3(\mathbb{R}))$. It is suffices to show that the approximating sequence $(w^{(n)}, z^{(n)})$ is bounded in $L^\infty([0,T] : H^{N-2}(\mathbb{R})) \times L^\infty([0,T] : H^{N-2}(\mathbb{R}))$. Taking $\alpha = N-2$ and consider (5.12) for $\alpha \geq 2$. We define $c_{N-3} = \left(\frac{1}{2}\|u_0\|_N^2 + 1\right)$ and $c'_{N-3} = \left(\frac{1}{2}\|v_0\|_N^2 + 1\right)$. Let $T^{(n)}_{N-3}$ be the largest time such that $\|w^{(k)}(\cdot, t)\|_\alpha \leq c_{N-3}$ and $\|z^{(k)}(\cdot, t)\|_\alpha \leq c'_{N-3}$ for $0 \leq t \leq T^{(n)}_{N-3}$, $0 \leq k \leq n$. i.e.,

$$T^{(n)}_{N-3} = \sup\{t : \|w^{(k)}(\cdot, \tilde{t})\|_\alpha \leq c_{N-3},\ \|z^{(k)}(\cdot, \tilde{t})\|_\alpha \leq c'_{N-3} \quad \text{for} \quad 0 \leq \tilde{t} \leq t, \quad 0 \leq k \leq n\}.$$

Integrating (5.12) over $[0, t]$ we have for $0 \leq t \leq T^{(n)}_{N-3}$ ($\alpha = N-2$ for $N \geq 4$).

$$\begin{aligned}
\|w^{(n)}(\cdot, t)\|^2_\alpha + \|z^{(n)}(\cdot, t)\|^2_\alpha \quad \leq \quad & \|w^{(n)}(\cdot, 0)\|^2_\alpha + \|z^{(n)}(\cdot, 0)\|^2_\alpha \\
& + \int_0^t \{G(\|w^{(n-1)}\|_\alpha,\ z^{(n-1)}\|_\alpha)\,(\|w^{(n)}\|^2_\alpha + \|z^{(n)}\|^2_\alpha) \\
& + E(\|w^{(n-1)}\|_\alpha,\ \ z^{(n-1)}\|_\alpha)\,(\|w^{(n-1)}\|^2_\alpha + \|z^{(n-1)}\|^2_\alpha) \\
& + M(\|w^{(n-1)}\|_\alpha,\ z^{(n-1)}\|_\alpha)\}\,dt \\
\leq \quad & \|u_0\|^2_N + \|v_0\|^2_N \\
& + G(c_{N-3},\, c'_{N-3})\,(c_{N-3})^2\,t + G(c_{N-3},\, c'_{N-3})\,(c'_{N-3})^2\,t \\
& + E(c_{N-3},\, c'_{N-3})\,(c_{N-3})^2\,t + E(c_{N-3},\, c'_{N-3})\,(c'_{N-3})^2\,t \\
& + M(c_{N-3},\, c'_{N-3})\,t.
\end{aligned}$$

Claim. $T^{(n)}_{N-3}$ does not approach 0.

Proof. On the contrary, assume that $T^{(n)}_{N-3} \to 0$. Since $\|w^{(n)}(\cdot, t)\|$ and $\|z^{(n)}(\cdot, t)\|$ are continuous in $t \geq 0$, there exist $\tau \in [0, T^{(n)}_{N-3}]$, such that $\|w^{(k)}(\cdot, \tau)\|_\alpha = c_{N-3}$ and $\|z^{(k)}(\cdot, \tau)\|_\alpha = c'_{N-3}$ for $0 \leq \tau \leq T^{(n)}_{N-3},\ 0 \leq k \leq n$. Then

$$\begin{aligned}
[c_{N-3}]^2 + [c'_{N-3}]^2 \quad \leq \quad & \|u_0\|^2_N + \|v_0\|^2_N \\
& + G(c_{N-3},\, c'_{N-3})\,(c_{N-3})^2\,T^{(n)}_{N-3} + G(c_{N-3},\, c'_{N-3})\,(c'_{N-3})^2\,T^{(n)}_{N-3} \\
& + E(c_{N-3},\, c'_{N-3})\,(c_{N-3})^2\,T^{(n)}_{N-3} + E(c_{N-3},\, c'_{N-3})\,(c'_{N-3})^2\,T^{(n)}_{N-3} \\
& + M(c_{N-3},\, c'_{N-3})\,T^{(n)}_{N-3}.
\end{aligned}$$

As $n \to +\infty$ follow that

$$\left(1 + \|u_0\|^2_N + \frac{\|u_0\|^4_N}{4}\right) + \left(1 + \|v_0\|^2_N + \frac{\|v_0\|^4_N}{4}\right) \leq \|u_0\|^2_N + \|v_0\|^2_N$$

where

$$\left(1 + \frac{\|u_0\|^4_N}{4}\right) + \left(1 + \frac{\|v_0\|^4_N}{4}\right) \leq 0$$

hence $T^{(n)}_{N-3} \not\to 0$. Choosing $T_{N-3} = T_{N-3}(\|u_0\|^2_N,\, \|v_0\|^2_N)$ sufficiently small with T_{N-3} not depending on n, one concludes that

$$\sup_{0 \leq t \leq T_{N-3}} \|w^{(n)}(\cdot, t)\|^2_\alpha + \sup_{0 \leq t \leq T_{N-3}} \|z^{(n)}(\cdot, t)\|^2_\alpha \leq c_{N-3} \tag{5.17}$$

for all $0 \leq t \leq T_{N-3}$. This shows that $T^{(n)}_{N-3} \geq T_{N-3}$. Therefore $w^{(n)}$ is a bounded sequence in $L^\infty([0, T_{N-3}] : H^{N-2}(\mathbb{R}))$ weakly convergent on $w \in L^\infty([0, T_{N-3}] : H^{N-2}(\mathbb{R}))$ and $z^{(n)}$ is a bounded sequence $L^\infty([0, T_{N-3}] : H^{N-2}(\mathbb{R}))$ weakly convergent on $z \in L^\infty([0, T_{N-3}] : H^{N-2}(\mathbb{R}))$. Thus

$$(u,\, v) = (\wedge w,\, \wedge z) \in L^\infty([0, T_N] : H^N(\mathbb{R})) \times L^\infty([0, T_N] : H^N(\mathbb{R})).$$

We denote by T^*_{N-3} the maximal number such that $u = \wedge w \in L^\infty([0, T] : H^N(\mathbb{R}))$ and $v = \wedge z \in L^\infty([0, T] : H^N(\mathbb{R}))$ for all $0 < t < T^*_{N-3}$. In particular $T \equiv T_0 \leq T^*_{N-3}$, and, thus, a time of existence T can be chosen depending only on norm $\|u_0\|_3$ and $\|v_0\|_3$. We now approximate u_0 and v_0 by $\{u^j_0\},\ \{v^j_0\} \in C^\infty_0(\mathbb{R})$ in such a way that

$$\|u_0 - u^j_0\|_{H^N(\mathbb{R})} \overset{j \to +\infty}{\longrightarrow} 0, \qquad \|v_0 - v^j_0\|_{H^N(\mathbb{R})} \overset{j \to +\infty}{\longrightarrow} 0.$$

Let $(u_j,\, v_j)$ be the solution to (2.8)-(2.9) with $u^j(x,\,0)=u_0^j(x)$ and $v^j(x,\,0)=v_0^j(x)$. According to the above argument, there exist T which is independent of n but depending on $\sup_j \|u_0^j\|$ and $\sup_j \|\, v_0^j \,\|$ such that u^j, v^j exists on $[0,\,T]$ and a subsequence

$$\begin{aligned} u^j &\overset{j\to+\infty}{\longrightarrow} u \quad \text{in} \quad L^\infty([0,\,T] :\, H^N(\mathbb{R}))\\ v^j &\overset{j\to+\infty}{\longrightarrow} v \quad \text{in} \quad L^\infty([0,\,T] :\, H^N(\mathbb{R})). \end{aligned}$$

As a consequence of Theorem 5.1 and 5.2 and its proof, one obtains the following result.

Corollary 5.3 *Let $a<0$ and let $(u_0,\, v_0)\in H^3(\mathbb{R})\times H^3(\mathbb{R})$ with $N\geq 3$ such that*

$$u_0^{(\gamma)} \longrightarrow u_0 \quad in \quad H^N(\mathbb{R}), \qquad v_0^{(\gamma)} \longrightarrow v_0 \quad in \quad H^N(\mathbb{R}).$$

Let $(u,\, v)$ and $(u^{(\gamma)},\, v^{(\gamma)})$ be the corresponding unique solutions given by Theorem 5.1. and 5.2. in $L^\infty([0,\,T] :\, H^N(\mathbb{R}))\times L^\infty([0,\,T] :\, H^N(\mathbb{R}))$ with T depending only on $\sup_\gamma \|u_0^{(\gamma)}\|_{H^3(\mathbb{R})}$ and $\sup_\gamma \|v_0^{(\gamma)}\|_{H^3(\mathbb{R})}$. Then

$$\begin{aligned} u^{(\gamma)} &\overset{*}{\rightharpoonup} u \quad \textit{weakly on} \quad L^\infty([0,\,T] :\, H^N(\mathbb{R}))\\ v^{(\gamma)} &\overset{*}{\rightharpoonup} v \quad \textit{weakly on} \quad L^\infty([0,\,T] :\, H^N(\mathbb{R})) \end{aligned}$$

and

$$\begin{aligned} u^{(\gamma)} &\longrightarrow u \quad \textit{strongly on} \quad L^2([0,\,T] :\, H^{N+1}(\mathbb{R}))\\ v^{(\gamma)} &\longrightarrow v \quad \textit{strongly on} \quad L^2([0,\,T] :\, H^{N+1}(\mathbb{R})). \end{aligned}$$

6 Persistence Theorem

As a starting point for the a priori gain of regularity results that will be discussed in the next section, we need to develop some estimates for solutions of (2.8), (2.9) in weighted Sobolev norms. The existence of these weighted estimates is often called the persistence of a property of the initial data $(u_0,\, v_0)$. We show that if $(u_0,\, v_0)\in H^3(\mathbb{R})\cap H^L(W_{0\,i\,0})\times H^3(\mathbb{R})\cap H^L(W_{0\,i\,0})$ for $L\geq 0$, $i\geq 1$ then the solution $(u(\cdot,\,t),\, v(\cdot,\,t))\in H^L(W_{0\,i\,0})\times H^L(W_{0\,i\,0})$ for $t\in[0,\,T]$. The time interval of that persistence is at least as long as the interval guaranteed by the existence Theorem 5.2.

Theorem 6.1 *(Persistence). Let $i\geq 1$, $L\geq 3$ be non negative integers and $0<T<+\infty$. Assume that $(u_0(x),\, v_0(x))=(u(x,\,0),\, v(x,\,0))\in H^3(\mathbb{R})\times H^3(\mathbb{R})$ and $a<0$. If $(u_0(x),\, v_0(x))=(u(x,\,0), v(x,\,0))\in H^L(W_{0\,i\,0})\times H^L(W_{0\,i\,0})$ then*

$$(u,\, v)\in L^\infty([0,\,T] :\, H^3(\mathbb{R})\cap H^L(W_{0\,i\,0}))\times L^\infty([0,\,T] :\, H^3(\mathbb{R})\cap H^L(W_{0\,i\,0})) \tag{6.1}$$

$$\int_0^T \int_{\mathbb{R}} \left|\partial^{L+1}u(x,\,t)\right|^2 \mu_1\, dx\, dt<+\infty \tag{6.2}$$

$$\int_0^T \int_{\mathbb{R}} \left|\partial^{L+1}v(x,\,t)\right|^2 \mu_2\, dx\, dt<+\infty \tag{6.3}$$

where σ is arbitrary, $\mu_1,\, \mu_2\in W_{\sigma,\, i-1,\, 0}$ for $i\geq 1$.

Proof. (Induction on α)

$$u,\, v\in L^\infty([0,\,T] :\, H^3(\mathbb{R})\cap H^\alpha(W_{0\,i\,0})) \quad \text{for} \quad 0\leq\alpha\leq L.$$

We derive formally some a priori estimate for the solution where the bound, involves only the norms of $(u,\, v)$ in $L^\infty([0,\,T] :\, H^3(\mathbb{R}))\times L^\infty([0,\,T] :\, H^3(\mathbb{R}))$ and the norm of $u_0(x)$, $v_0(x)$ in $H^3(W_{0\,i\,0})$. We do this by approximating $(u(x,\,t),\, v(x,\,t))$ by smooth solutions, and weight functions by smooth bounded functions. By Theorem 5.2, we have

$$(u(x,\,t),\, v(x,\,t))\in L^\infty([0,\,T] :\, H^N(\mathbb{R}))\times L^\infty([0,\,T] :\, H^N(\mathbb{R})) \qquad \text{with} \qquad N=\max\{L,\, 3\}.$$

In particular

$$(u_j(x,\,t),\,v_j(x,\,t)) \in L^\infty([0,\,T]\times\mathbb{R})\times L^\infty([0,\,T]\times\mathbb{R}) \qquad \text{for} \qquad 0\le j\le N-1.$$

To obtain (6.1)-(6.2) and (6.3) there are two ways of approximations. We approximate general solutions by smooth solutions, and we approximate general weight functions by bounded weight functions. The first of these procedure has already been discussed, so we shall concentrate on the second. Given a smooth weight function $\mu_1(x)\in W_{\sigma,\,i-1,\,0}$ with $\sigma>0$, we take a sequence $\mu_{\delta_1}(x)$ of smooth bounded weight functions approximating $\mu_1(x)$ from below, uniformly on any half line $(-\infty,\,c)$. Similarly, given a smooth weight function $\mu_2(x)\in W_{\sigma,\,i-1,\,0}$ with $\sigma>0$, we take a sequence $\mu_{\delta_2}(x)$ of smooth bounded weight functions approximating $\mu_2(x)$ from below, uniformly on any half line $(-\infty,\,c)$. Define the weight functions for the αth induction step as

$$\xi_{\delta_1} = -\frac{4\,b}{3\,a}\int_{-\infty}^{x}\mu_{\delta_1}(y,\,t)\,dy \qquad \text{and} \qquad \xi_{\delta_2} = \frac{2}{3}\int_{-\infty}^{x}\mu_{\delta_2}(y,\,t)\,dy \tag{6.4}$$

then the ξ_{δ_1} and ξ_{δ_2} are bounded weight functions which approximate a desired weight functions $\xi_1,\ \xi_2\in W_{0\;i\;0}$ respectively from below, uniformly on a compact set.
For $\alpha=0$ (simple case), multiplying (2.8) by $2\,\xi_\delta\,u$ and integrating over $x\in\mathbb{R}$ we have

$$2\int_{\mathbb{R}}\xi_\delta\,u\,\partial_t u\,dx - 2\,a\int_{\mathbb{R}}\xi_\delta\,u\,u_3\,dx + 12\int_{\mathbb{R}}\xi_\delta\,u^2\,u_1\,dx = 4\,b\int_{\mathbb{R}}\xi_\delta\,u\,v\,v_1\,dx. \tag{6.5}$$

Each term is treated separately. In the first term we have

$$2\int_{\mathbb{R}}\xi_\delta\,u\,\partial_t u\,dx = \partial_t\int_{\mathbb{R}}\xi_\delta\,u^2\,dx - \int_{\mathbb{R}}\partial_t\xi_\delta\,u^2\,dx$$

$$-\,2\,a\int_{\mathbb{R}}\xi_\delta\,u\,u_3\,dx = a\int_{\mathbb{R}}\partial^3\xi_\delta\,u^2\,dx - 3\,a\int_{\mathbb{R}}\partial\xi_\delta\,u_1^2\,dx.$$

For the others terms, using integration by parts, we have

$$12\int_{\mathbb{R}}\xi_\delta\,u^2\,u_1\,dx = -4\int_{\mathbb{R}}\partial\xi_\delta\,u^3\,dx$$

$$4\,b\int_{\mathbb{R}}\xi_\delta\,u\,v\,v_1\,dx = -\,2\,b\int_{\mathbb{R}}\partial(\xi_\delta\,u)\,v^2\,dx.$$

Replacing in (6.5), we obtain

$$\begin{aligned}&\partial_t\int_{\mathbb{R}}\xi_\delta\,u^2\,dx - 3\,a\int_{\mathbb{R}}\partial\xi_\delta\,u_1^2\,dx - \int_{\mathbb{R}}\partial_t\xi_\delta\,u^2\,dx + a\int_{\mathbb{R}}\partial^3\xi_\delta\,u^2\,dx\\ &-\,4\int_{\mathbb{R}}\partial\xi_\delta\,u^3\,dx = -\,2\,b\int_{\mathbb{R}}\partial(\xi_\delta\,u)\,v^2\,dx\end{aligned} \tag{6.6}$$

We multiply (2.9) by $2\,\xi_\delta\,v$ and integrating over $x\in\mathbb{R}$

$$2\int_{\mathbb{R}}\xi_\delta\,v\,\partial_t v\,dx + 2\int_{\mathbb{R}}\xi_\delta\,v\,v_3\,dx + 6\int_{\mathbb{R}}\xi_\delta\,u\,v\,v_1\,dx = 0. \tag{6.7}$$

Each term is treated separately. Performing straightforward calculations as above we obtain

$$\partial_t\int_{\mathbb{R}}\xi_\delta\,v^2\,dx + 3\int_{\mathbb{R}}\partial\xi_\delta\,v_1^2\,dx - \int_{\mathbb{R}}\partial_t\xi_\delta\,v^2\,dx - \int_{\mathbb{R}}\partial^3\xi_\delta\,v^2\,dx = 3\int_{\mathbb{R}}\partial(\xi_\delta\,u)\,v^2\,dx \tag{6.8}$$

Adding (6.6) and (6.8) we have

$$\begin{aligned}&\partial_t\int_{\mathbb{R}}\xi_\delta\,u^2\,dx + \partial_t\int_{\mathbb{R}}\xi_\delta\,v^2\,dx - 3\,a\int_{\mathbb{R}}\partial\xi_\delta\,u_1^2\,dx + 3\int_{\mathbb{R}}\partial\xi_\delta\,u_1^2\,dx\\ =\;&\int_{\mathbb{R}}[\partial_t\xi_\delta - a\,\partial^3\xi_\delta + 4\,\partial\xi_\delta\,u]\,u^2\,dx + \int_{\mathbb{R}}[\partial_t\xi_\delta + \partial^3\xi_\delta + 3\,\partial(\xi_\delta\,u) - 2\,b\,\partial(\xi_\delta\,u)]\,v^2\,dx.\end{aligned} \tag{6.9}$$

Using (2.7) and Gagliardo-Nirenberg's inequality, we obtain

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}} \xi_\delta\, u^2\, dx + \partial_t \int_{\mathbb{R}} \xi_\delta\, v^2\, dx - 3\, a \int_{\mathbb{R}} \partial\xi_\delta\, u_1^2\, dx + 3 \int_{\mathbb{R}} \partial\xi_\delta\, u_1^2\, dx \\
&\leq C \left(\int_{\mathbb{R}} \xi_\delta\, u^2\, dx + \int_{\mathbb{R}} \xi_\delta\, v^2\, dx \right).
\end{aligned}$$

thus

$$\partial_t \int_{\mathbb{R}} \xi_\delta\, u^2\, dx + \partial_t \int_{\mathbb{R}} \xi_\delta\, v^2\, dx \leq C \left(\int_{\mathbb{R}} \xi_\delta\, u^2\, dx + \int_{\mathbb{R}} \xi_\delta\, v^2\, dx \right).$$

We apply Gronwall's lemma to conclude

$$\partial_t \int_{\mathbb{R}} \xi_\delta\, u^2\, dx + \partial_t \int_{\mathbb{R}} \xi_\delta\, v^2\, dx \leq C = C(T,\, \|u_0\|,\, \|v_0\|) \tag{6.10}$$

for $0 \leq t \leq T$ and C not depending on $\delta > 0$, the weighted estimate remains true for $\delta \to 0$.
Now, we assume that the result is true for $(\alpha - 1)$ and we prove that it is true for α. To prove this, we start from the main inequality (3.1)

$$\begin{aligned}
&\frac{1}{4\, b}\, \partial_t \int_{\mathbb{R}} \xi_\delta\, u_\alpha^2\, dx + \frac{1}{6}\, \partial_t \int_{\mathbb{R}} \xi_\delta\, v_\alpha^2\, dx + \int_{\mathbb{R}} \mu_{\delta_1}\, u_{\alpha+1}^2\, dx + \int_{\mathbb{R}} \mu_{\delta_2}\, v_{\alpha+1}^2\, dx \\
&\int_{\mathbb{R}} \theta_{\delta_1}\, u_\alpha^2\, dx + \int_{\mathbb{R}} \theta_{\delta_2}\, u_\alpha^2\, dx + \int_{\mathbb{R}} R_\alpha\, dx = 0
\end{aligned}$$

where

$$\begin{aligned}
\mu_{\delta_1} &= -\frac{3\, a}{4\, b}\, \partial\xi_\delta \qquad for \quad a < 0 \\
\mu_{\delta_2} &= \frac{3}{2}\, \partial\xi_\delta \\
\theta_{\delta_1} &= -\frac{1}{4\, b} \left[\partial_t \xi_\delta - a\, \partial^3 \xi_\delta + 6\, \partial(\xi_\delta\, u) \right] \\
\theta_{\delta_2} &= -\frac{1}{6} \left[\partial_t \xi_\delta + \partial^3 \xi_\delta \right] \\
R_\alpha &= \frac{1}{3\, b} \sum_{\beta=1}^{\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!}\, \xi_\delta\, u_\alpha\, u_\beta\, u_{\alpha+1-\beta} - \sum_{\beta=0}^{\alpha-1} \frac{\alpha!}{\beta!(\alpha-\beta)!}\, \xi_\delta\, u_\alpha\, v_\beta\, v_{\alpha+1-\beta} \\
&\quad + \sum_{\beta=0}^{\alpha-1} \frac{\alpha!}{\beta!(\alpha-\beta)!}\, \xi_\delta\, v_\alpha\, u_\beta\, v_{\alpha+1-\beta}
\end{aligned}$$

then

$$\begin{aligned}
&\frac{1}{4\, b}\, \partial_t \int_{\mathbb{R}} \xi_\delta\, u_\alpha^2\, dx + \frac{1}{6}\, \partial_t \int_{\mathbb{R}} \xi_\delta\, v_\alpha^2\, dx + \int_{\mathbb{R}} \mu_{\delta_1}\, u_{\alpha+1}^2\, dx + \int_{\mathbb{R}} \mu_{\delta_2}\, v_{\alpha+1}^2\, dx \\
&\leq -\int_{\mathbb{R}} \theta_{\delta_1}\, u_\alpha^2\, dx - \int_{\mathbb{R}} \theta_{\delta_2}\, u_\alpha^2\, dx - \int_{\mathbb{R}} R_\alpha\, dx \\
&\leq \left| -\int_{\mathbb{R}} \theta_{\delta_1}\, u_\alpha^2\, dx - \int_{\mathbb{R}} \theta_{\delta_2}\, u_\alpha^2\, dx - \int_{\mathbb{R}} R_\alpha\, dx \right| \\
&\leq \int_{\mathbb{R}} |\theta_{\delta_1}|\, u_\alpha^2\, dx + \int_{\mathbb{R}} |\theta_{\delta_2}|\, u_\alpha^2\, dx + \int_{\mathbb{R}} |R_\alpha|\, dx.
\end{aligned}$$

Using (2.7) and the Gagliardo-Nirenberg in the first term of the right side we obtain

$$\int_{\mathbb{R}} |\theta_{\delta_1}|\, u_\alpha^2\, dx \leq C \int_{\mathbb{R}} \xi_\delta\, u_\alpha^2\, dx, \qquad \int_{\mathbb{R}} |\theta_{\delta_2}|\, u_\alpha^2\, dx \leq C \int_{\mathbb{R}} \xi_\delta\, v_\alpha^2\, dx$$

thus

$$
\begin{aligned}
&\frac{1}{4\,b}\,\partial_t \int_{\mathbb{R}} \xi_\delta\, u_\alpha^2\, dx + \frac{1}{6}\,\partial_t \int_{\mathbb{R}} \xi_\delta\, v_\alpha^2\, dx + \int_{\mathbb{R}} \mu_{\delta_1}\, u_{\alpha+1}^2\, dx + \int_{\mathbb{R}} \mu_{\delta_2}\, v_{\alpha+1}^2\, dx \\
&\leq C\left(\int_{\mathbb{R}} \xi_\delta\, u_\alpha^2\, dx + C\int_{\mathbb{R}} \xi_\delta\, v_\alpha^2\, dx\right) + \int_{\mathbb{R}} |R_\alpha|\, dx.
\end{aligned}
$$

According to (3.17), $\int_{\mathbb{R}} R_\alpha\, dx$ contains a term of

$$
\int_{\mathbb{R}} \xi_\delta\, u_{\nu_1}\, u_{\nu_1}\, u_\alpha\, dx \tag{6.11}
$$

the other terms are estimate the same form. If $\nu_2 \leq \alpha-2$, using integrating by parts and the Hölder inequality

$$
C\left[\left(\int_{\mathbb{R}} \xi_\delta\, u_{\nu_2+1}^2\, dx\right)^{1/2} + \left(\int_{\mathbb{R}} \xi_\delta\, u_{\nu_2}^2\, dx\right)^{1/2}\right]\left(\int_{\mathbb{R}} \xi_\delta\, u_{\alpha-1}^2\, dx\right)^{1/2}
$$

by the induction hypothesis we have is bounded. If $\nu_1 = \nu_2 = \alpha-1$, then by (3.18) we have $\alpha = 3$ and

$$
\begin{aligned}
\left|\int_{\mathbb{R}} \xi_\delta\, u_{\alpha-1}^2\, u_\alpha\, dx\right| &\leq \|u_{\alpha-1}\|_{L^\infty(\mathbb{R})}\left(\int_{\mathbb{R}} \xi_\delta\, u_{\alpha-1}\, u_\alpha\, dx\right) \\
&\leq \|u_{\alpha-1}\|_{L^\infty(\mathbb{R})}\left(\int_{\mathbb{R}} \xi_\delta\, u_{\alpha-1}^2\, dx\right)^{1/2}\left(\int_{\mathbb{R}} \xi_\delta\, u_\alpha^2 dx\right)^{1/2} \\
&\leq \frac{1}{2}\,\|u_{\alpha-1}\|_{L^\infty(\mathbb{R})}\left(\int_{\mathbb{R}} \xi_\delta\, u_{\alpha-1}^2\, dx + \int_{\mathbb{R}} \xi_\delta\, u_\alpha^2\, dx\right) \\
&\leq C\left(\int_{\mathbb{R}} \xi_\delta\, u_\alpha^2\, dx + 1\right).
\end{aligned}
$$

If $\nu_1 = \alpha-2$ and $\nu_2 = \alpha-1$, then by (3.18) we have $\alpha = 4$ and

$$
\begin{aligned}
\int_{\mathbb{R}} \xi_\delta\, u_2\, u_3\, u_4\, dx &\leq \|u_2\,\sqrt{\xi}_\delta\|_{L^\infty(\mathbb{R})}\|u_3\|_{L^2(\mathbb{R})}\int_{\mathbb{R}} u_3\,\sqrt{\xi}_\delta\, u_4\, dx \\
&\leq \|u_2\,\sqrt{\xi_\delta}\|_{L^\infty(\mathbb{R})}\left(\int_{\mathbb{R}} u_3^2\, dx\right)^{1/2}\left(\int_{\mathbb{R}} \xi_\delta\, u_4^2\, dx\right)^{1/2} \\
&\leq C\,\|u_3\|_{L^2(\mathbb{R})}\left(\int_{\mathbb{R}} \xi_\delta\, u_4^2\, dx\right)^{1/2}.
\end{aligned}
$$

The other terms. Using those estimate and straightforward calculus

$$
\begin{aligned}
&\frac{1}{4\,b}\,\partial_t \int_{\mathbb{R}} \xi_\delta\, u_\alpha^2\, dx + \frac{1}{6}\,\partial_t \int_{\mathbb{R}} \xi_\delta\, v_\alpha^2\, dx + \int_{\mathbb{R}} \mu_{\delta_1}\, u_{\alpha+1}^2\, dx + \int_{\mathbb{R}} \mu_{\delta_2}\, v_{\alpha+1}^2\, dx \\
&\leq C + C'\left(\frac{1}{4\,b}\int_{\mathbb{R}} \xi_\delta\, u_\alpha^2\, dx + \frac{1}{6}\int_{\mathbb{R}} \xi_\delta\, v_\alpha^2\, dx\right).
\end{aligned}
$$

Applying the Gronwall's argument, we obtain for $0 \leq t \leq T$,

$$
\begin{aligned}
&\frac{1}{4\,b}\int_{\mathbb{R}} \xi_\delta\, u_\alpha^2\, dx + \frac{1}{6}\int_{\mathbb{R}} \xi_\delta\, v_\alpha^2\, dx + \int_{\mathbb{R}} \mu_{\delta_1}\, u_{\alpha+1}^2\, dx + \int_{\mathbb{R}} \mu_{\delta_2}\, v_{\alpha+1}^2\, dx \\
&\leq C_0\, e^{C_1\, t}\left(\frac{1}{4\,b}\int_{\mathbb{R}} \xi_\delta\, u_{0\ \alpha}^2\, dx + \frac{1}{6}\int_{\mathbb{R}} \xi_\delta\, v_{0\ \alpha}^2\, dx\right).
\end{aligned}
$$

where C_0 and C_1 are independent δ such that letting the parameter $\delta \to 0$ the desired estimates (6.2)-(6.3) are obtained.

7 Main Theorem

In this section we state and prove our main theorem, which states that if the initial data $(u_0,\, v_0)$ decays faster than polynomially on $\mathbb{R} = \{x \in \mathbb{R} :\ x > 0\}$ and possesses certain initial Sobolev regularity, then the solution $(u(x,\, t),\, v(x,\, t)) \in C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R})$ for all $t > 0$. For the main theorem, we take $4 \le \alpha \le L+2$. For $\alpha \le L+2$, we take

$$\mu_1 \in W_{\sigma,\, L-\alpha+2,\, \alpha-3} \implies \xi \in W_{\sigma,\, L-\alpha+3,\, \alpha-3} \tag{7.1}$$

$$\mu_2 \in W_{\sigma,\, L-\alpha+2,\, \alpha-3} \implies \xi \in W_{\sigma,\, L-\alpha+3,\, \alpha-3} \tag{7.2}$$

Lemma 7.1 *(Estimate of Error Terms). If $4 \le \alpha \le L+2$ and the weight functions are chosen as in (7.1), then*

$$\left| \int_0^T \int_{\mathbb{R}} \left(\theta_1\, u_\alpha^2 + \theta_2\, v_\alpha^2 + R \right) dx\, dt \right| \le C \tag{7.3}$$

where C depends only on the norms of u, v in

$$L^\infty([0,\, T] :\ H^\beta(W_{\sigma,\, L-\beta+3,\, \beta-3})) \cap L^2([0,\, T] :\ H^{\beta+1}(W_{\sigma,\, L-\beta+2,\, \beta-3}))$$

for $3 \le \beta \le \alpha - 2$, and the norm of u, v in $L^\infty([0,\, T] :\ H^3(W_{0\, L\, 0}))$.

Proof. We must estimate R, θ_1 and θ_2. We begin with a term of the form

$$\xi\, u_{\nu_1}\, u_{\nu_2}\, u_\alpha \tag{7.4}$$

(the other terms are calculated the similar form) assuming that $\nu_1 \le \alpha - 2$. By the induction hypothesis, u is bounded on $L^\infty([0,\, T] :\ H^\beta(W_{\sigma,\, L-(\beta-3)^+,\, (\beta-3)^+}))$ for all $\sigma > 0$ and $0 \le \beta \le \alpha - 1$. By Lemma 2.7

$$\sup_t \sup_x \widetilde{\zeta}\, u_\beta^2 < +\infty \tag{7.5}$$

for $0 \le \beta \le \alpha - 2$ and $\widetilde{\zeta} \in W_{\sigma,\, L-(\beta-2)^+,\, (\beta-2)^+}$. Then in the term of the form $\xi u_{\nu_1} u_{\nu_2} u_\alpha$ we estimate u_{ν_1} using (7.5). We estimate u_{ν_2} and u_α using the weight L^2 bounds

$$\int_0^T \int_{\mathbb{R}} \zeta\, u_{\nu_2}^2\, dx\, dt < +\infty \qquad \text{for} \quad \zeta \in W_{\sigma,\, L-(\nu_2-3)^+,\, (\nu_2-4)^+} \tag{7.6}$$

and the same with ν_p replaced by α. It is sufficient to check the powers of t, and the powers x as $x \to +\infty$ and the exponentials x as $x \to -\infty$.
For $x > 1$. In the term (7.4), the factor ξ contributes according to (7.1)-(7.2)

$$\xi(x,\, t) \;=\; t^{(\alpha-3)}\, x^{(L-\alpha+3)}\, t^{-(\alpha-3)}\, x^{-(L-\alpha+3)} \xi(x,\, t) \le c_2 t^{(\alpha-3)}\, x^{(L-\alpha+3)}$$

by (2.6). Then

$$\xi(x,\, t)\, u_{\nu_1}\, u_{\nu_2}\, u_\alpha \le c_2\, t^{(\alpha-3)}\, x^{(L-\alpha+3)}\, u_{\nu_1}\, u_{\nu_2}\, u_\alpha.$$

Moreover

$$\begin{aligned} u_{\nu_1}\, u_{\nu_2}\, u_\alpha \;=\;& t^{\frac{(\nu_1-2)^+}{2}}\, x^{\frac{L-(\nu_1-2)^+}{2}}\, t^{\frac{-(\nu_1-2)^+}{2}}\, x^{\frac{-(L-(\nu_1-2)^+)}{2}}\, u_{\nu_1} \\ & t^{\frac{(\nu_2-4)^+}{2}}\, x^{\frac{L-(\nu_2-3)^+}{2}}\, t^{\frac{-(\nu_2-4)^+}{2}}\, x^{\frac{-(L-(\nu_2-3)^+)}{2}}\, u_{\nu_2} \\ & t^{\frac{(\alpha-4)^+}{2}}\, x^{\frac{L-(\alpha-3)^+}{2}}\, t^{\frac{-(\alpha-4)^+}{2}}\, x^{\frac{-(L-(\alpha-3)^+)}{2}}\, u_\alpha. \end{aligned}$$

It follows that

$$\begin{aligned} & \xi\, u_{\nu_1}\, u_{\nu_2}\, u_\alpha \\ & \quad \le C_2\, t^M\, x^T\, t^{\frac{(\nu_1-2)^+}{2}}\, x^{\frac{L-(\nu_1-2)^+}{2}}\, u_{\nu_1}\, t^{\frac{(\nu_2-4)^+}{2}}\, x^{\frac{L-(\nu_2-3)^+}{2}}\, u_{\nu_2}\, t^{\frac{(\alpha-4)^+}{2}}\, x^{\frac{L-(\alpha-3)^+}{2}}\, u_\alpha \end{aligned} \tag{7.7}$$

where

$$\begin{aligned} M &= (\alpha-3) - \frac{1}{2}(\nu_1-2)^+ - \frac{1}{2}(\nu_2-4)^+ - \frac{1}{2}(\alpha-4)^+ \\ T &= (L-\alpha+3) - \frac{1}{2}(L-(\nu_1-2)^+) - \frac{1}{2}(L-(\nu_2-4)^+) - \frac{1}{2}(L-(\alpha-3)^+). \end{aligned}$$

Claim $M \geq 0$ is large enough, that the extra power of t can be bounded by a constant.
Proof.

$$\begin{aligned} 2M &= 2\alpha - 6 - \nu_1 + 2 - \nu_2 + 4 - \alpha + 4 = \alpha + 4 - (\nu_1+\nu_2) \\ &\overset{(3.18)}{=} \alpha + 4 - (\alpha+1) = 3. \end{aligned}$$

Claim $T \leq 0$ so that the extra power x^T can be bounded as $x \to +\infty$.
Proof.

$$\begin{aligned} 2T &= 2L - 2\alpha + 6 - L + (\nu_1-2)^+ - L + (\nu_2-4)^+ - L + (\alpha-3)^+ \\ &= -L - \alpha + 6 + \nu_1 - 2 + \nu_2 - 4 - 3 = -L - \alpha - 3 + (\nu_1+\nu_2) \\ &\overset{(3.18)}{=} -L - \alpha - 3 + \alpha + 1 = -L - 2 = -(L+2) \leq 0. \end{aligned}$$

Thus

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} \xi\, u_{\nu_1} u_{\nu_2} u_\alpha \, dx\, dt \\ \leq\;& C_3 \int_0^T t^{\frac{(\nu_1-2)^+}{2}} x^{\frac{L-(\nu_1-2)^+}{2}} u_{\nu_1} t^{\frac{(\nu_2-4)^+}{2}} x^{\frac{L-(\nu_2-3)^+}{2}} u_{\nu_2} t^{\frac{(\alpha-4)^+}{2}} x^{\frac{L-(\alpha-3)^+}{2}} u_\alpha \, dx\, dt \\ \leq\;& C \left[\int_0^T t^{(\nu_1-2)^+} x^{L-(\nu_1-2)^+} u_{\nu_1}^2 t^{(\nu_2-4)^+} x^{L-(\nu_2-3)^+} u_{\nu_2}^2 dx\, dt\right]^{1/2} \left[\int_0^T t^{(\alpha-4)^+} x^{L-(\alpha-3)^+} u_\alpha^2 dx\, dt\right]^{1/2} \\ \leq\;& C \left[\int_0^T \int_{\mathbb{R}} \widetilde{\zeta}\, u_{\nu_1}^2\, \zeta\, u_{\nu_2}^2 \, dx\, dt\right]^{1/2} \left[\int_0^T \int_{\mathbb{R}} \zeta\, u_\alpha^2 \, dx\, dt\right]^{1/2} \\ \leq\;& C \left(\sup_t \sup_x \widetilde{\zeta}\, u_{\nu_1}^2\right)^{1/2} \left[\int_0^T \int_{\mathbb{R}} \zeta u_{\nu_2}^2 \, dx\, dt\right]^{1/2} \left[\int_0^T \int_{\mathbb{R}} \zeta\, u_\alpha^2 \, dx\, dt\right]^{1/2} \end{aligned}$$

which is bounded. If $\nu_1 = \nu_2 = \alpha - 1$ then by (3.18)

$$\nu_1 + \nu_2 = \alpha + 1 \iff 2\alpha - 2 \iff \alpha = 3$$

but $\alpha \geq 4$. Thus is not the case. If $\nu_1 = \alpha - 2$, $\nu_2 = \alpha - 1$, then again by (3.18), $\nu_1 + \nu_2 = \alpha + 1 \iff 2\alpha - 3 = \alpha + 1 \iff \alpha = 4$. Thus, if $\alpha = 4$ then we have the term $\int_{\mathbb{R}} \xi\, u_2\, u_3\, u_4\, dx$ and integrating by parts $-\frac{1}{2}\int_{\mathbb{R}} \partial\xi\, u_2\, u_3^2\, dx - \frac{1}{2}\int_{\mathbb{R}} \xi\, u_3^3\, dx$ where $\xi \in W_{\sigma,\, L-1,\, 1}$. For this term we use the interpolation inequality $|u'|_3 \leq |u|_6^{1/2}\, |u''|_2^{1/2}$. Then

$$\int_0^T \int_1^{+\infty} t\, x^{L-1}\, |u_3|^3\, dx\, dt \leq T \sup_t \sup_x |u_2| \left(\int_0^T \int_1^{+\infty} x^{L-1}\, u_2^2\, dx\, dt\right)^{1/2} \left(\int_0^T \int_1^{+\infty} x^{L-1}\, u_4^2\, dx\, dt\right)^{1/2}$$

which is bounded. For $x < 1$ the estimate is similar except for the exponential weight. This completes the estimate of R.
Now we estimate the terms $\theta_1\, u_\alpha^2$ and $\theta_2\, v_\alpha^2$ where θ_1 and θ_2 are given in the fundamental inequality, follow that θ_1 and θ_2 involves derivatives of u and v only up to order 1 and hence $\theta_1\, u_\alpha^2$ and $\theta_2\, v_\alpha^2$ is a sum of terms of the same type we have already encountered in R, so that its integral can be bounded in the same manner. This completes the proof of Lemma 7.1.

Theorem 7.2 *Let $T > 0$, $a < 0$ and $(u,\, v)$ be a solution of* (2.8), (2.9) *in the region $\mathbb{R} \times [0,\, T]$ such that*

$$(u,\, v) \in L^\infty([0,\, T] :\, H^3(W_{0\, L\, 0})) \times L^\infty([0,\, T] :\, H^3(W_{0\, L\, 0})) \tag{7.8}$$

for some $L \geq 2$ and all $\sigma > 0$. Then

$$\begin{aligned} u &\in L^\infty([0,\, T] :\, H^{3+l}(W_{\sigma,\, L-l,\, l})) \cap L^2([0,\, T] :\, H^{4+l}(W_{\sigma,\, L-l-1,\, l})) \\ v &\in L^\infty([0,\, T] :\, H^{3+l}(W_{\sigma,\, L-l,\, l})) \cap L^2([0,\, T] :\, H^{4+l}(W_{\sigma,\, L-l-1,\, l})) \end{aligned}$$

for all $0 \leq l \leq L - 1$.

Remark 7.3 *If the assumption* (7.8) *holds for all $L \geq 2$, the solution is infinitely differentiable in the x-variable. From the equations* (2.8), (2.9) *itself the solution is C^∞ in both of its variables.*

Proof. We use induction on α. For $\alpha = 4$. Let $(u,\, v)$ be a solution of (2.8), (2.9) satisfying (7.8), then the equations itself imply that $u_t\,,\, v_t \in L^\infty([0,\, T] :\, L^2(W_{0\, L\, 0}))$ then

$$u\,,\, v \in C([0,\, T] :\, L^2(W_{0\, L\, 0})) \cap C_w([0,\, T] :\, H^3(W_{0\, L\, 0})).$$

Hence $u,\, v\, :\, [0,\, T] \mapsto H^3(W_{0\, L\, 0})$ are weakly continuous functions. In particular $u(\cdot\,,\, t),\; v(\cdot\,,\, t) \in H^3(W_{0\, L\, 0})$ for all t. Let $t_0 \in (0,\, T)$ and $u(\cdot\,,\, t_0),\, v(\,\cdot\,,\, t_0) \in H^3(W_{0\, L\, 0})$, then there are $\{u_0^{(n)}(\cdot)\}$, $\{v_0^{(n)}(\cdot)\} \subseteq C_0^\infty(\mathbb{R})$ such that $u_0^{(n)}(\cdot) \to u(\cdot\,,\, t_0)$ strong in $H^3(W_{0\, L\, 0})$ and $v_0^{(n)}(\cdot) \to v(\cdot\,,\, t_0)$ strong in $H^3(W_{0\, L\, 0})$. Let $(u^{(n)}(x,\, t),\, v^{(n)}(x,\, t))$ be the unique solution of (2.8), (2.9) with $u^{(n)}(x,\, t_0) \equiv u_0^{(n)}(x)$ and $v^{(n)}(x,\, t_0) \equiv v_0^{(n)}(x)$ then by Theorem 5.1 and 5.2 is guaranteed in a time interval $[t_0,\, t_0 + \delta]$, $\delta > 0$ and the unique solution of (2.8), (2.9) $u^{(n)},\, v^{(n)} \in L^\infty([t_0,\, t_0 + \delta] :\, H^3(W_{0\, L\, 0}))$ with $u^{(n)}(x,\, t_0) \equiv u_0^{(n)}(x) \to u(x,\, t_0) \equiv u_0(x)$ strong in $H^3(W_{0\, L\, 0})$ and $v^{(n)}(x,\, t_0) \equiv v_0^{(n)}(x) \to v(x,\, t_0) \equiv v_0(x)$ strong in $H^3(W_{0\, L\, 0})$. Hence by Theorem 6.1 we do $i = L$ then $u^{(n)},\; v^{(n)} \in L^\infty([t_0,\, t_0 + \delta] :\, H^3(W_{0\, L\, 0}))$ and

$$\int_{t_0}^{t_0+\delta} \int_{\mathbb{R}} |\partial^4 u^{(n)}(x,\, t)|^2\, \eta\, dx\, dt < +\infty, \qquad \int_{t_0}^{t_0+\delta} \int_{\mathbb{R}} |\partial^4 v^{(n)}(x,\, t)|^2\, \eta\, dx\, dt < +\infty$$

where σ is arbitrary and $\eta \in W_{\sigma,\, L-1,\, 0}$ for $L \geq 1$. Thus

$$\begin{aligned} u^{(n)} &\in L^\infty([t_0,\, t_0 + \delta] :\, H^3(W_{0\, L\, 0})) \cap L^2([t_0,\, t_0 + \delta] :\, H^4(W_{\sigma,\, L-1,\, 0})) \\ v^{(n)} &\in L^\infty([t_0,\, t_0 + \delta] :\, H^3(W_{0\, L\, 0})) \cap L^2([t_0,\, t_0 + \delta] :\, H^4(W_{\sigma,\, L-1,\, 0})) \end{aligned}$$

with a bound that depends only on the norm of u_0 and $v_0 \in H^3(W_{0\, i\, 0})$. Furthermore, Theorem 6.1 guarantees the non-uniform bounds

$$\begin{aligned} &\sup_{[t_0,\, t_0+\delta]} \sup_{x \in \mathbb{R}} (1 + |x_+|)^k\, |\partial^\alpha u^{(n)}(x,\, t)| < +\infty \\ &\sup_{[t_0,\, t_0+\delta]} \sup_{x \in \mathbb{R}} (1 + |x_+|)^k\, |\partial^\alpha v^{(n)}(x,\, t)| < +\infty \end{aligned}$$

for each k, n, and α. Therefore, the main inequality (3.1) and (7.3) are justified for each $u^{(n)}$, $v^{(n)}$ in the interval $[t_0,\, t_0 + \delta]$. The multipliers μ_1 and μ_2 may be chosen arbitrarily in its weight class (7.1)-(7.2) and then ξ is defined by (3.15) and the constants c_1, c_2, c_3, c_4 are independent on n. From (3.1) and (7.1)-(7.2) we have

$$\begin{aligned} &\sup_{[t_0,\, t_0+\delta]} \int_{\mathbb{R}} \xi [u_\alpha^{(n)}]^2\, dx + \sup_{[t_0,\, t_0+\delta]} \int_{\mathbb{R}} \xi\, [v_\alpha^{(n)}]^2\, dx \\ &\quad + \int_{t_0}^{t_0+\delta} \int_{\mathbb{R}} \mu_1 [u_{\alpha+1}^{(n)}]^2\, dx\, dt + \int_{t_0}^{t_0+\delta} \int_{\mathbb{R}} \mu_2\, [v_{\alpha+1}^{(n)}]^2\, dx\, dt \leq C \end{aligned} \tag{7.9}$$

where by (7.3), C is independent on n. Estimate (7.9) is proved by induction for $\alpha = 4,\, 5,\, 6,\, \ldots$. Thus $u^{(n)}$, $v^{(n)}$ are also bounded on

$$L^\infty([t_0,\, t_0 + \delta] :\, H^\alpha(W_{\sigma,\, L-\alpha+3,\, \alpha-3})) \cap L^2([t_0,\, t_0 + \delta] :\, H^{\alpha+1}(W_{\sigma,\, L-\alpha+2,\, \alpha-3})) \tag{7.10}$$

for $\alpha \geq 4$. Since $u^{(n)} \to u$ strong in $L^{\infty}([t_0,\, t_0+\delta] : H^3(W_{0\,L\,0}))$ and $v^{(n)} \to v$ strong in $L^{\infty}([t_0,\, t_0+\delta] : H^3(W_{0\,L\,0}))$. By Corollary 5.3, it is follow that u and v belong to the space (7.10). Since $\delta > 0$ is fixed, this result is valid over the whole interval $[0,\, T]$.

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