Electronic Journal of Qualitative Theory of Differential Equations
2018, No. **69**, 1–10; https://doi.org/10.14232/ejqtde.2018.1.69 www.math.u-szeged.hu/ejqtde/

Existence of solutions for a class of quasilinear degenerate $p(x)$-Laplace equations

Qing-Mei Zhou[✉][1] and **Jian-Fang Wu**[2]

[1]Library, Northeast Forestry University, Harbin, 150040, P.R. China
[2]Department of Applied Mathematics, Harbin Engineering University, Harbin, 150001, P.R. China

Received 7 April 2018, appeared 11 August 2018
Communicated by Gabriele Bonanno

Abstract. We study the existence of weak solutions for a degenerate $p(x)$-Laplace equation. The main tool used is the variational method, more precisely, the Mountain Pass Theorem.

Keywords: Sobolev spaces with variable exponent, variational methods, Mountain Pass Theorem.

2010 Mathematics Subject Classification: 35J20, 35J60, 35D05, 35J70.

1 Introduction

We study the existence of weak solutions for a degenerate $p(x)$-Laplace equation. The main tool used is the variational method, more precisely, the Mountain Pass Theorem. The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions has been a new and interesting topic. Such problems arise from the study of electrorheological fluids (see Růžička [31]), and elastic mechanics (see Zhikov [35]). It also has wide applications in different research fields, such as image processing model (see e.g., [16, 22]), stationary thermorheological viscous flows (see [2]) and the mathematical description of the processes filtration of an idea barotropic gas through a porous medium (see [3]).

In recent years, many problems on $p(x)$-Laplace type have been studied by many authors using various methods, for example, variational method (see, e.g., [1,5–14,17,20,21,23,27,30,34, 36]), topological method (see, e.g., [15, 24]), sub-supersolution method (see, e.g., [18]), Nehari manifold method (see, e.g., [28]), monotone mapping theory (see, e.g., [29]) and fibering map approach [32].

In this paper, we considered the following quasilinear degenerate $p(x)$-Laplace problem:

$$\begin{cases} -\operatorname{div}(a(x)|\nabla u|^{p(x)-2}\nabla u) = \lambda\big(b(x)|u|^{q(x)-2}u - c(x)|u|^{r(x)-2}u\big), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{P}$$

✉ Corresponding author. Email: zhouqingmei2008@163.com

where Ω is a smooth boundary domain in $\mathbb{R}^N$, $\lambda \in \mathbb{R}^N$, and $p, q, r \in C_+(\overline{\Omega})$, where $C_+(\overline{\Omega})$ is defined by $C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}), \inf_{x\in\overline{\Omega}} p(x) > 1\}$, $q^+ := \sup_{x\in\overline{\Omega}} q(x) < p^- := \inf_{x\in\overline{\Omega}} p(x) \leq p^+ < r^- = \inf_{x\in\overline{\Omega}} r(x) \leq r(x) < p^*(x)$, where $p^*(x) = \frac{Np(x)}{N-p(x)}$, $a(x), b(x) > 0$ for $x \in \overline{\Omega}$.

We make the following assumptions:

(h_a) $0 < a \in L^1_{\mathrm{loc}}(\Omega)$, $a^{-\frac{1}{p(x)-1}} \in L^1_{\mathrm{loc}}(\Omega)$ and $a^{-\frac{\xi(x)}{p(x)-\xi(x)}} \in L^1(\Omega)$, where $\xi \in C_+(\overline{\Omega})$ with $\xi(x) < p(x)$.

(h_b) $0 < b \in L^{\alpha(x)}(\Omega)$ and $\alpha \in C_+(\overline{\Omega})$.

(h_c) $0 < c \in L^{\gamma(x)}(\Omega)$ and $\gamma \in C_+(\overline{\Omega})$.

(h_q) $q^+ < \frac{(\alpha(x)-1)\xi^*(x)}{\alpha(x)}$.

(h_r) $r(x) < \frac{(\gamma(x)-1)\xi^*(x)}{\gamma(x)}$.

To study (P) by means of variational methods, we introduce the functional associated

$$\varphi(u) = \int_\Omega \frac{a(x)}{p(x)}|\nabla u|^{p(x)}dx - \lambda \int_\Omega \frac{b(x)}{q(x)}|u|^{q(x)}dx + \lambda \int_\Omega \frac{c(x)}{r(x)}|u|^{r(x)}dx$$

for $u \in W^{1,p(x)}_{a(x)}(\Omega)$, where the Sobolev space $W^{1,p(x)}_{a(x)}(\Omega)$ which is called weighted variable exponent Sobolev space, is introduced in [26].

We are now in the position to state our main results.

Theorem 1.1. *Suppose that (h_a), (h_b), (h_c), (h_q) and (h_r) hold.*

(i) If $\lambda > 0$, then problem (P) has a nontrivial solution which is a minimizer of the associated integral functional of φ.

(ii) If $\lambda < 0$, then problem (P) has a sequence of solutions $\{\pm u_n\}$ such that $\varphi(\pm u_n) \to +\infty$, as $n \to +\infty$.

The rest of this paper is organized as follows. In Section 2, we recall some necessary preliminaries, which will be used in our investigation in Section 3. In Section 3, we prove the main results of the paper.

2 Preliminaries

In order to discuss problem (P), we need some theories on $W^{1,p(x)}_{a(x)}(\Omega)$ which we will call weighted variable exponent Sobolev space. For more details on the basic properties of these spaces, we refer the reader to Kufner and B. Opic [26], Kim, Wang and Zhang [25].

Denoted by $\mathcal{U}(\Omega)$ the set of all measurable real functions defined on Ω, elements in $\mathcal{U}(\Omega)$ which are equal to each other almost everywhere are considered as one element.

Write

$$L^{p(x)}(\Omega) = \left\{ u \in \mathcal{U}(\Omega) : \int_\Omega |u(x)|^{p(x)}dx < +\infty \right\},$$

with the norm $|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx \leq 1\}$, and

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

with the norm $\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{p(x)} + |\nabla u|_{p(x)}$.

Lemma 2.1 ([19]).

(1) Poincaré's inequality in $W_0^{1,p(x)}(\Omega)$ holds, that is, there exists a positive constant C such that $|u|_{L^{p(x)}(\Omega)} \leq C|\nabla u|_{L^{p(x)}(\Omega)}, \forall u \in W_0^{1,p(x)}(\Omega)$.

(2) If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous.

Lemma 2.2 ([19]). *Set $\rho(u) = \int_\Omega |u(x)|^{p(x)}dx$. For $u \in L^{p(x)}(\Omega)$, we have*

(1) if $|u|_{p(x)} > 1$, then $|u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{p(x)}^{p^+}$;

(2) if $|u|_{p(x)} < 1$, then $|u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$.

Lemma 2.3 ([15]). *Assume that $h \in L_+^\infty(\Omega)$, $p \in C_+(\overline{\Omega})$. If $|u|^{h(x)} \in L^{p(x)}(\Omega)$, then we have*

$$\min\left\{|u|_{h(x)p(x)}^{h^-}, |u|_{h(x)p(x)}^{h^+}\right\} \leq \left||u|^{h(x)}\right|_{p(x)} \leq \max\left\{|u|_{h(x)p(x)}^{h^-}, |u|_{h(x)p(x)}^{h^+}\right\}.$$

We consider $W_{a(x)}^{1,p(x)}(\Omega)$ as an appropriate Sobolev space for studding problem (P), which is defined as a completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| = |\nabla u|_{L_{a(x)}^{p(x)}(\Omega)},$$

where $L_{a(x)}^{p(x)}(\Omega) = \{u \in \mathcal{U}(\Omega) : \int_\Omega a(x)|u|^{p(x)}dx < +\infty\}$ is equipped with the norm

$$|u|_{L_{a(x)}^{p(x)}(\Omega)} = \inf\left\{\sigma > 0 : \int_\Omega a(x)\left|\frac{u}{\sigma}\right|^{p(x)} dx \leq 1\right\}.$$

The Sobolev space $W_{a(x)}^{1,p(x)}(\Omega)$ which is called weighted variable exponent Sobolev space, is introduced in [26], where $a(x)$ is a measurable, nonnegative real valued function for $x \in \Omega$.

Lemma 2.4 ([32, Theorem 2.5]). *Assume that (h_p), (h_b) and (h_q) are satisfied. Then we have the following compact embedding $W_{a(x)}^{1,p(x)}(\Omega) \hookrightarrow L_{b(x)}^{q(x)}(\Omega)$.*

A function $u \in W_{a(x)}^{1,p(x)}(\Omega)$ is said to be a weak solution of (P) if

$$\begin{aligned}\int_\Omega a(x)|\nabla u|^{p(x)-2}\nabla u\nabla v dx &= \lambda \int_\Omega b(x)|u|^{q(x)-2}uv dx \\ &\quad - \lambda \int_\Omega c(x)|u|^{r(x)-2}uv dx, \qquad \forall v \in W_{a(x)}^{1,p(x)}(\Omega).\end{aligned}$$

Then

$$\langle \varphi'(u), v\rangle = \int_\Omega a(x)|\nabla u|^{p(x)-2}\nabla u\nabla v dx - \lambda \int_\Omega b(x)|u|^{q(x)-2}uv dx + \lambda \int_\Omega c(x)|u|^{r(x)-2}uv dx$$

for all $u, v \in W_{a(x)}^{1,p(x)}(\Omega)$. It is well known that the weak solution of (P) corresponds to the critical point of the functional φ on $W_{a(x)}^{1,p(x)}(\Omega)$.

In order to prove Theorem 1.1, we need a lemma.

Let X be a reflexive and separable Banach space, then there are $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{\text{span}\{e_j : j = 1, 2, 3, \cdots\}}, \qquad X^* = \overline{\text{span}\{e_j^* : j = 1, 2, 3, \cdots\}}$$

and

$$\langle e_j^*, e_i \rangle = \begin{cases} 1, & j = i, \\ 0, & j \neq i. \end{cases}$$

For convenience, we write

$$X_j = \text{span}\{e_j\}, \qquad Y_k = \oplus_{j=1}^{k} X_j \quad \text{and} \quad Z_k = \oplus_{j=k}^{\infty} X_j. \tag{2.1}$$

Lemma 2.5 ([33]). *X is a Banach space, $\varphi \in C^1(X, \mathbb{R})$ is an even functional, the subspaces Y_k and Z_k are defined in (2.1). If for each $k = 1, 2, 3, \ldots$, there exists $\rho_k > d_k > 0$ such that*

(1) $\max_{u \in Y_k, \|u\| = \rho_k} \varphi(u) \leq 0$;

(2) $\inf_{u \in Z_k, \|u\| = d_k} \varphi(u) \to \infty$ *as* $k \to \infty$.

(3) *The functional φ satisfies the (P.S.) condition.*

Then, φ has an unbounded sequence of critical values.

3 Proof of the main result

Throughout the paper, the letters c, c_i, $i = 1, 2, 3, \ldots$ denote positive constants which may change from line to line.

First, we recall that in view of Lemma 2.3,

$$\begin{aligned} \int_\Omega \frac{b(x)}{q(x)} |u|^{q(x)} dx &\leq \frac{2}{q^-} |b|_{\alpha(x)} \left| |u|^{q(x)} \right|_{\alpha'(x)} \\ &\leq \frac{2}{q^-} |b|_{\alpha(x)} \left[|u|^{q^+}_{q(x)\alpha'(x)} + |u|^{q^-}_{q(x)\alpha'(x)} \right]. \end{aligned}$$

Note that $1 < q(x)\alpha'(x) < \xi^*(x)$ for all $x \in \overline{\Omega}$, then by Lemma 2.4, we have $W^{1,p(x)}_{a(x)}(\Omega) \hookrightarrow L^{q(x)\alpha'(x)}_{b(x)}(\Omega)$ (compact embedding). Furthermore, there exists a positive constant c such that the following inequality holds $|u|_{q(x)\alpha'(x)} \leq c\|u\|$. Thus,

$$\int_\Omega \frac{b(x)}{q(x)} |u|^{q(x)} dx \leq \frac{2}{q^-} |b|_{\alpha(x)} [c^{q^+} \|u\|^{q^+} + c^{q^-} \|u\|^{q^-}].$$

Proof. We start by proving the first assertion (i) of Theorem 1.1, if $\lambda > 0$, so the functional φ is coercive. In fact, let $\|u\| > 1$. From the Lemma 2.2 we have

$$\varphi(u) \geq \frac{1}{p^+} \|u\|^{p^-} - \frac{4\lambda}{q^-} |b|_{\alpha(x)} c^{q^+} \|u\|^{q^+}. \tag{3.1}$$

Note that $q^+ < p^-$, so φ is coercive and has a minimizer which is a solution of (P). This minimizer is nonzero. Indeed, for $t > 0$ small enough and $v_0 \in W^{1,p(x)}_{a(x)}(\Omega)$,

$$\begin{aligned}
\varphi(tv_0) &= \int_\Omega \frac{a(x)}{p(x)} t^{p(x)} |\nabla v_0|^{p(x)} dx - \lambda \int_\Omega \frac{b(x)}{q(x)} t^{q(x)} |v_0|^{q(x)} dx + \lambda \int_\Omega \frac{c(x)}{r(x)} t^{r(x)} |v_0|^{r(x)} dx \\
&\le \frac{t^{p^-}}{p^-} \int_\Omega a(x) |\nabla v_0|^{p(x)} dx - \frac{\lambda t^{q^+}}{q^+} \int_\Omega b(x) |v_0|^{q(x)} dx + \frac{\lambda t^{r^-}}{r^-} \int_\Omega c(x) |v_0|^{r(x)} dx \\
&\le \frac{t^{p^-}}{p^-} \int_\Omega a(x) |\nabla v_0|^{p(x)} dx - \frac{\lambda t^{q^+}}{q^+} \int_\Omega b(x) |v_0|^{q(x)} dx + \frac{\lambda t^{p^-}}{r^-} \int_\Omega c(x) |v_0|^{r(x)} dx \\
&\le c_1 t^{p^-} - c_2 t^{q^+} \\
&< 0,
\end{aligned}$$

because $q^+ < p^-$.

Now, we are to check the second assertion (ii) of Theorem 1.1. Since $W^{1,p(x)}_{a(x)}(\Omega)$ is a reflexive and separable Banach space, it is worth to recall that there $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{\mathrm{span}\{e_j : j = 1, 2, 3, \cdots\}}, \qquad X^* = \overline{\mathrm{span}\{e_j^* : j = 1, 2, 3, \cdots\}}$$

and

$$\langle e_j^*, e_i \rangle = \begin{cases} 1, & j = i, \\ 0, & j \neq i. \end{cases}$$

Set

$$X_j = \mathrm{span}\{e_j\}, \qquad Y_k = \oplus_{j=1}^k X_j \quad \text{and} \quad Z_k = \oplus_{j=k}^\infty X_j.$$

Next, via Lemma 2.5, we are to prove that the Problem (P) has infinitely many solutions whether $\lambda < 0$.

Set $\delta_k = \inf_{v \in Y_k, \|v\|=1} \int_\Omega \frac{c(x)}{r(x)} |v|^{r(x)} dx$. Let $t > 1$ and $v \in Y_k$, with $\|v\| = 1$, we have

$$\begin{aligned}
\varphi(tv) &= \int_\Omega \frac{a(x)}{p(x)} t^{p(x)} |\nabla v|^{p(x)} dx - \lambda \int_\Omega \frac{b(x)}{q(x)} t^{q(x)} |v|^{q(x)} dx + \lambda \int_\Omega \frac{c(x)}{r(x)} t^{r(x)} |v|^{r(x)} dx \\
&\le c_1 t^{p^+} + c_2 t^{q^+} - c_3 t^{r^-} \delta_k.
\end{aligned}$$

Since $r^- > \max\{p^+, q^+\}$, so we may find $t_0 \in [1, \infty)$ satisfies $\varphi(t_0 v) < 0$ and thus there exists large $\rho_k > 0$ such that

$$\max_{u \in Y_k, \|u\| = \rho_k} \varphi(u) < 0.$$

(2) Let $\beta_k = \sup_{v \in Z_k, \|v\| \le 1} \int_\Omega \frac{c(x)}{r(x)} |v|^{r(x)} dx$. By $Z_{k+1} \subset Z_k$ we see that $0 \le \beta_{k+1} \le \beta_k$ and $\beta_k \to 0$ when $k \to \infty$. Indeed, from the definition of β_k we may find u_k such that

$$\left| \beta_k - \int_\Omega \frac{c(x)}{r(x)} |u_k|^{r(x)} dx \right| < \frac{1}{k}, \qquad \forall k \ge 1.$$

Note that $\{u_k\}$ is bounded in $W^{1,p(x)}_{a(x)}(\Omega)$. Thus, we may assume without loss of generality that $u_k \rightharpoonup u_0$ in $W^{1,p(x)}_{a(x)}(\Omega)$ and hence, $e_j^*(u_k) \to 0$ as $k \to \infty$. Thus, we must have $e_j^*(u_0) = 0$ for all $j \ge 1$, so $u_0 = 0$.

Moreover, if $u \in W^{1,p(x)}_{a(x)}(\Omega)$ with $\|u\| > 1$, we deduce,

$$\begin{aligned}\varphi(u) &= \int_\Omega \frac{a(x)}{p(x)}|\nabla u|^{p(x)}dx - \lambda \int_\Omega \frac{b(x)}{q(x)}|u|^{q(x)}dx + \lambda \int_\Omega \frac{c(x)}{r(x)}|u|^{r(x)}dx \\ &\geq \frac{1}{p^+}\|u\|^{p^-} - \lambda \int_\Omega \frac{b(x)}{q(x)}|u|^{q(x)}dx - c_4\beta_k\|u\|^{r^+} \\ &\geq \frac{1}{p^+}\|u\|^{p^-} - c_4\beta_k\|u\|^{r^+},\end{aligned} \tag{3.2}$$

because $\lambda < 0$.

Putting $d_k = \left(\frac{1}{2p^+C_4\beta_k}\right)^{\frac{1}{r^+-p^-}}$, in this case $\|u\| \to \infty$ since $\beta_k \to 0$. Hence, taking $\|u\| = d_k$, it follows from (3.2) that

$$\inf_{u \in Z_k, \|u\| = d_k} \varphi(u) \geq \frac{1}{2p^+}d_k^{p^-} \to \infty, \quad \text{as } k \to \infty.$$

(3) The functional φ verifies the Palais–Smale condition *(P.S.)* on $W^{1,p(x)}_{a(x)}(\Omega)$. In fact, let $\{u_n\}$ be a $(P.S.)_c$ sequence, that is, $\varphi(u_n) \to c$ and $\varphi'(u_n) \to 0$ in $(W^{1,p(x)}_{a(x)}(\Omega))^*$. For $\|u_n\|$ large enough, we have

$$\begin{aligned}r^-c + 1 &\geq r^-\varphi(u_n) - \varphi'(u_n)u_n \\ &= \int_\Omega \left[\frac{r^-}{p(x)} - 1\right] a(x)|\nabla u|^{p(x)}dx - \lambda \int_\Omega \left[\frac{r^-}{q(x)} - 1\right] b(x)|u|^{q(x)}dx \\ &\quad + \lambda \int_\Omega \left[\frac{r^-}{r(x)} - 1\right] c(x)|u|^{r(x)}dx \\ &\geq \left[\frac{r^-}{p^+} - 1\right] \|u\|^{p^-}.\end{aligned} \tag{3.3}$$

This implies that $\{u_n\}$ is bounded sequence in $W^{1,p(x)}_{a(x)}(\Omega)$. Up to a subsequence, still denoted by $\{u_n\}$, we may assume that

$$\int_\Omega \frac{a(x)}{p(x)}|\nabla u_n|^{p(x)}dx \to \sigma, \qquad n \to +\infty \tag{3.4}$$

and also there exists $u_0 \in W^{1,p(x)}_{a(x)}(\Omega)$ satisfies

$$\begin{aligned} u_n &\rightharpoonup u_0 \quad \text{in } W^{1,p(x)}_{a(x)}(\Omega), \\ u_n(x) &\to u_0(x) \quad \text{a.e. } x \in \Omega, \\ u_n &\to u_0 \quad \text{in } L^{q(x)}_{b(x)}(\Omega), \qquad u_n \to u_0 \quad \text{in } L^{r(x)}_{c(x)}(\Omega).\end{aligned}$$

Invoking Lemma 2.4, we deduce that

$$\lim_{n\to\infty} \int_\Omega b(x)(|u_n|^{q(x)-2}u_n - |u_0|^{q(x)-2}u_0)(u_n - u_0) = 0$$

and

$$\lim_{n\to\infty} \int_\Omega c(x)(|u_n|^{r(x)-2}u_n - |u_0|^{r(x)-2}u_0)(u_n - u_0) = 0.$$

Hence, we must have

$$\lim_{n\to\infty}\int_\Omega a(x)(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u_0|^{p(x)-2}\nabla u_0)(\nabla u_n - \nabla u_0) = 0.$$

Below we shall prove that $\{u_n\}$ has a strongly convergent subsequence in the following two cases, respectively.
Case (I): $\sigma = 0$.

Indeed, by using (3.4), we derive that

$$\int_\Omega \frac{a(x)}{p(x)}|\nabla u_n|^{p(x)}dx \to 0, \qquad n \to +\infty.$$

Then, u_n is strongly convergent to 0 in $W^{1,p(x)}_{a(x)}(\Omega)$, the proof is complete.
Case (II): $\sigma > 0$.

It is observed now that (see [4]) for $x, y \in \mathbb{R}^N$, we have the following estimates

$$|x-y|^\theta \le 2^\theta(|x|^{\theta-2}x - |y|^{\theta-2}y)\cdot(x-y), \quad \text{if } \theta \ge 2,$$
$$|x-y|^2 \le \frac{1}{\theta-1}(|x|+|y|)^{2-\theta}(|x|^{\theta-2}x - |y|^{\theta-2}y)\cdot(x-y), \quad \text{if } 1 < \theta < 2,$$

where $x \cdot y$ is the inner product in $\mathbb{R}^N$.

Using the above inequalities, there is $c_5 > 0$ such that

$$\lim_{n\to\infty}\int_\Omega a(x)(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u_0|^{p(x)-2}\nabla u_0)(\nabla u_n - \nabla u_0)dx \ge c_5\int_\Omega a(x)|\nabla u_n - \nabla u_0|^{p(x)}dx. \quad (3.5)$$

Thereby,

$$\int_\Omega a(x)|\nabla u_n - \nabla u_0|^{p(x)}dx \to 0, \qquad n \to +\infty,$$

which implies that

$$u_n \to u_0 \quad \text{in } W^{1,p(x)}_{a(x)}(\Omega),$$

finishing the proof. Therefore, by virtue of Lemma 2.5, the second conclusion of Theorem 1.1 is true.

Acknowledgements

Q.-M. Zhou was supported by the Fundamental Research Funds for the Central Universities (No. DL12BC10), the New Century Higher Education Teaching Reform Project of Heilongjiang Province in 2012 (No. JG2012010012), the Humanities and Social Sciences Foundation of the Educational Commission of Heilongjiang Province of China (No. 12544026); J.-F. Wu was supported by the NNSF of China (Nos. U1706227, 11201095), the Youth Scholar Backbone Supporting Plan Project of Harbin Engineering University, the Postdoctoral Research Startup Foundation of Heilongjiang (LBH-Q14044), the Science Research Funds for Overseas Returned Chinese Scholars of Heilongjiang Province (LC201502), the Fundamental Research Funds for the Central Universities(No. HEUCFM181102).

References

[1] C. O. Alves, M. Ferreira, Existence of solutions for a class of $p(x)$-Laplacian equations involving a concave-convex nonlinearity with critical growth in $\mathbb{R}^N$, *Topol. Methods Nonlinear Anal.* **45**(2015), No. 2, 399–422. https://doi.org/10.12775/TMNA.2015.020; MR3408829; Zbl 1371.35131

[2] S. N. Antontsev, J. F. Rodrigues, On stationary thermo-rheological viscous flows, *Ann. Univ. Ferrara Sez. VII Sci. Mat.* **52**(2006), No. 1, 19–36. https://doi.org/10.1007/s11565-006-0002-9; MR2246902; Zbl 1117.76004

[3] S. N. Antontsev, S. I. Shmarev, A model porous medium equation with variable exponent of nonlinearity: existence, uniqueness and localization properties of solutions, *Nonlinear Anal.* **60**(2005), No. 3, 515–545. https://doi.org/10.1066/j.na.2004.09.026; MR2103951; Zbl 1066.35045

[4] S. Antontsev, S. Shmarev, Elliptic equations with anisotropic nonlinearity and nonstandard growth conditions, in: *Handbook of differential equations: Stationary partial differential equations, Vol. 3*, Amsterdam, Elsevier/North Holland, 2006, pp. 1–100. https://doi.org/10.1016/S1874-5733(06)80005-7; Zbl 1192.35047

[5] G. Barletta, A. Chinnì, Existence of solutions for a Neumann problem involving the $p(x)$-Laplacian, *Electron. J. Differential Equations.* **2013**, No. 158, 1–12. MR3084638; Zbl 1288.35211

[6] G. Barletta, A. Chinnì, D. O'Regan, Existence results for a Neumann problem involving the $p(x)$-Laplacian with discontinuous nonlinearities, *Nonlinear Anal. Real World Appl.* **27**(2016), 312–325. https://doi.org/10.1016/j.nonrwa.2015.08.002; MR3400531; Zbl 1332.35128

[7] G. Bonanno, A critical point theorem via the Ekeland variational principle, *Nonlinear Anal.* **75**(2012), 2992–3007. https://doi.org/10.1016/j.na.2011.12.003; MR2878492; Zbl 1239.58011

[8] G. Bonanno, A. Chinnì, Discontinuous elliptic problems involving the $p(x)$-Laplacian, *Math. Nachr.* **284**(2011), No. 5–6, 639–652. https://doi.org/10.1002/mana.200810232; MR2663758; Zbl 1214.35076

[9] G. Bonanno, A.Chinnì, Multiple solutions for elliptic problems involving the $p(x)$-Laplacian, *Matematiche (Catania)* **66**(2011), No. 1, 105–113. MR2827189; Zbl 1234.35119

[10] G. Bonanno, A. Chinnì, Existence and multiplicity of weak solutions for elliptic Dirichlet problems with variable exponent, *J. Math. Anal. Appl.* **418**(2014), No. 2, 812–827. https://doi.org/10.1016/j.jmaa.2014.04.016; MR3206681; Zbl 1312.35111

[11] G. Bonanno, A. Chinnì, Existence results of infinitely many solutions for $p(x)$-Laplacian elliptic Dirichlet problems, *Complex Var. Elliptic Equ.* **57**(2012), No. 11, 1233–1246. https://doi.org/10.1080/17476933.2012.662225; MR2989739; Zbl 1258.35096

[12] G. Bonanno, A. Chinnì, S. A. Tersian, Existence results for a two point boundary value problem involving a fourth-order equation, *Electron. J. Qual. Theory Differ. Equ.* **2015**, No. 33, 1–9. https://doi.org/10.14232/ejqtde.2015.1.33; MR3353204; Zbl 1349.34070

[13] F. Cammaroto, A. Chinnì, B. Di Bella, Multiple solutions for a Neumann problem involving the $p(x)$-Laplacian, *Nonlinear Anal.* **71**(2009), No. 10, 4486–4492. https://doi.org/10.1016/j.na.2009.03.009; MR2548679; Zbl 1167.35381

[14] F. Cammaroto, L. Vilasi, Multiplicity results for a Neumann boundary value problem involving the $P(X)$-Laplacian, *Taiwanese J. Math.* **16**(2012), No. 2, 621–634. https://doi.org/10.11650/twjm/1500406606; MR2892903; Zbl 1255.35120

[15] B. Cekic, R. A. Mashiyev, Existence and localization results for $p(x)$-Laplacian via topological Methods, *Fixed Point Theory Appl.* **2010**, Art. ID 120646 7 pp. https://doi.org/10.1155/2010/120646; MR2678552; Zbl 1198.35109

[16] Y. Chen, S. Levine, M. Rao, Variable exponent linear growth functionals in image restoration, *SIAM J. Appl. Math.* **66**(2006), No. 4, 1383–1406. https://doi.org/10.1137/050624522; MR2246061; Zbl 1102.49010

[17] A. Chinnì, R. Livrea, Multiple solutions for a Neumann-type differential inclusion problem involving the $p(\cdot)$-Laplacian, *Discrete Contin. Dyn. Syst. Ser. S.* **5**(2012), No. 4, 753–764. https://doi.org/10.3934/dcdss.2012.5.753; MR2851197; Zbl 1255.35089

[18] X. Fan, On the sub-supersolution method for $p(x)$-Laplacian equations, *J. Math. Anal. Appl.* **330**(2007), No. 1, 665–682. https://doi.org/10.1016/j.jmaa.2006.07.093; MR2302951; Zbl 1206.35103

[19] X. Fan, D. Zhao, On the space $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, *J. Math. Anal. Appl.* **263**(2001), No. 2, 424–446. https://doi.org/10.1006/jmaa.2000.7617; MR1866056; Zbl 1028.46041

[20] B. Ge, L.-L. Liu, Infinitely many solutions for differential inclusion problems in $\mathbb{R}^N$ involving the $p(x)$-Laplacian, *Z. Angew. Math. Phys.* **67**(2016), Art. 8, 16 pp. https://doi.org/10.1007/s00033-015-0612-0; MR3481151; Zbl 1338.35134

[21] B. Ge, Q.-M. Zhou, Y.-H. Wu, Eigenvalues of the $p(x)$-biharmonic operator with indefinite weight, *Z. Angew. Math. Phys.* **66**(2014), No. 3, 1007–1021. https://doi.org/10.1007/s00033-014-0465-y; MR3347422; Zbl 1319.35147

[22] P. Harjulehto, P. Hästö, V. Latvala, Minimizers of the variable exponent, non-uniformly convex Dirichlet energy, *J. Math. Pures Appl. (9)* **89**(2008), No. 2, 174–197. https://doi.org/10.1016/j.matpur.2007.10.006; MR2391646; Zbl 1142.49007

[23] S. Heidarkhani, G. A. Afrouzi, S. Moradi, G. Caristi, B. Ge, Existence of one weak solutions for $p(x)$-biharmonic equations with Navier boundary conditions, *Z. Angew. Math. Phys.* **67**(2016), No. 3, Art. 73, 13 pp. https://doi.org/10.1007/s00033-016-0668-5; MR3508996; Zbl 1353.35153

[24] P. Ilias, Dirichlet problem with $p(x)$-Laplacian, *Math. Rep. (Bucur.)* **10(60)**(2008), No. 1, 43–56. MR2428896; Zbl 1199.35086

[25] Y.-H. Kim, L. Wang, C. Zhang, Global bifurcation of a class of degenerate elliptic equations with variable exponents, *J. Math. Anal. Appl.* **371**(2010), No. 2, 624–637. https://doi.org/10.1016/j.jmaa.2010.05.058; MR2670139; Zbl 1198.35089

[26] A. Kufner, B. Opic, How to define reasonably weighted Sobolev spaces, *Comment. Math. Univ. Carolin.* **25**(1984), No. 3, 537–554. MR0775568; Zbl 0557.46025

[27] S. Liang, J. Zhang, Infinitely many small solutions for the $p(x)$-Laplacian operator with nonlinear boundary conditions, *Ann. Mat. Pura Appl. (4)* **192**(2013), No. 1, 1–16. https://doi.org/10.1007/s10231-011-0209-y; MR3011321; Zbl 1266.35093

[28] D. Liu, Existence of multiple solutions for a $p(x)$-Laplace equation, *Electron. J. Differential Equations* **2010**, No. 30, 1–12. MR2602866; Zbl 1192.35073

[29] P. Marcellini, Regularity and existence of solutions of elliptic equations with (p,q)-growth conditions, *J. Differential Equations* **90**(1991), No. 1, 1–30. https://doi.org/10.1016/0022-0396(91)90158-6; MR1094446; Zbl 0724.35043

[30] V. Rădulescu, Nonlinear elliptic equations with variable exponent: old and new, *Nonlinear Anal.* **121**(2015), 336–369. https://doi.org/10.1016/j.na.2014.11.007; MR3348928; Zbl 1321.35030

[31] M. Růžička, *Electrorheological fluids: modeling and mathematical theory*, Springer-Verlag, Berlin, 2000. https://doi.org/10.1007/BFb0104029; MR1810360; Zbl 0962.76001

[32] S. Saiedinezhad, M. B. Ghaemi, The fibering map approach to a quasilinear degenerate $p(x)$-Laplacian equation, *Bull. Iranian Math. Soc.* **41**(2015), No. 6, 1477–1492. MR3437159; Zbl 1373.35144

[33] M. Willem, *Minimax theorems*, Birkhäuser, Boston, 1996. https://doi.org/10.1007/978-1-4612-4146-1; MR1400007; Zbl 0856.49001

[34] Z. Yücedağ, Solutions of nonlinear problems involving $p(x)$-Laplacian operator, *Adv. Nonlinear Anal.* **4**(2015), No. 4, 285–293. https://doi.org/10.1515/anona-2015-0044; MR3420320; Zbl 1328.35058

[35] V. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Izv. Akad. Nauk SSSR Ser. Mat.* **29**(1987), No. 4, 33–66. https://doi.org/10.1070/IM1987v029n01ABEH000958; MR0864171; Zbl 0599.49031

[36] Q.-M. Zhou, On the superlinear problems involving $p(x)$-Laplacian-like operators without AR-condition, *Nonlinear Anal. Real World Appl.* **21**(2015), 161–169. https://doi.org/10.1016/j.nonrwa.2014.07.003; MR3261587; Zbl 1304.35471