Electronic Journal of Qualitative Theory of Differential Equations
2011, No. **44**, 1-11; **http://www.math.u-szeged.hu/ejqtde/**

Forced oscillation of second-order superlinear dynamic equations on time scales

Yuangong Sun*

School of Mathematics, University of Jinan, Jinan, Shandong 250022, China

Abstract. In this paper, by constructing a class of Philos type functions on time scales, we investigate the oscillation of the following second-order forced nonlinear dynamic equation

$$x^{\Delta\Delta}(t) - p(t)|x(q(t))|^{\lambda-1}x(q(t)) = e(t), \quad t \in \mathbb{T}$$

where $\mathbb{T}$ is a time scale, $p, e : \mathbb{T} \to \mathbb{R}$ are right dense continuous functions with $p > 0$, $\lambda > 1$ is a constant, and $q(t) = t$ or $q(t) = \sigma(t)$. Our results not only unify the oscillation of second-order forced differential equations and their discrete analogues, but also complement several results in the literature.

Keywords: Time scales, oscillation; dynamic equations; second-order

2000 MSC: 34N05; 34C10; 39A21

1 Introduction

Following Hilger's landmark paper [1], a rapidly expanding body of literature has sought to unify, extend and generalize ideas from discrete calculus, quantum calculus and continuous calculus to arbitrary time-scale calculus, where a time scale is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, e.g., $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ for $q > 1$ (which has important applications in quantum theory), $\mathbb{T} = h\mathbb{N}$ with $h > 0$, $\mathbb{T} = \mathbb{N}^2$ and $\mathbb{T} = \mathbb{T}_n$ the space of the harmonic numbers. For an introduction to time scale calculus and dynamic equations, we refer to the seminal books by Bohner and Peterson [2,3].

Recently, many authors have expounded on various aspects of time scales, among which, the oscillation theory has attracted considerable attention, e.g. see [4-19] and the references cited therein. We are here concerned with the following second-order forced

*Email address: sunyuangong@yahoo.cn

dynamic equation

$$x^{\Delta\Delta}(t) - p(t)|x(q(t))|^{\lambda-1}x(q(t)) = e(t), \quad t \in \mathbb{T} \tag{1}$$

where $\mathbb{T}$ is a time scale unbounded above with $t_0 \in \mathbb{T}$; p and e are real-valued right dense continuous functions on $\mathbb{T}$ with $p > 0$, $\lambda > 1$ is a constant, $q(t) = t$ or $q(t) = \sigma(t)$.

A solution of Eq. (1) is a nontrivial real function $x : \mathbb{T} \to \mathbb{R}$ such that $x \in C^2_{rd}[t_x, \infty)_{\mathbb{T}}$ with $t_x \geq t_0$ and $[t_x, \infty)_{\mathbb{T}} = [t_x, \infty) \cap \mathbb{T}$, and x satisfies Eq. (1) on $\mathbb{T}$. A function x is an oscillatory solution of Eq. (1) if and only if x is a solution of Eq. (1) that is neither eventually positive nor eventually negative. Eq. (1) is oscillatory if and only if every solution of Eq. (1) is oscillatory.

Some equations related to Eq. (1) have been extensively studied by many authors in [20-27]. For the oscillation of the second-order forced dynamic Eq. (1), the oscillation results in [6] can be applied to Eq. (1) with $q(t) = \sigma(t)$ and oscillatory potentials. Following the idea in [27], the authors established several oscillation criteria for Eq. (1) with $p(t) > 0$ and $q(t) = \sigma(t)$ in [18] and [19], while the case of $q(t) = t$ remains unstudied.

The main purpose of this paper is to further study the oscillation of Eq. (1) in the superlinear case when $q(t) = \sigma(t)$ and $q(t) = t$, respectively. We will show that the results in [18] and [19] seem to be invalid when the time scale $\mathbb{T}$ only contains isolated points. We also extend the results to the case of $q(t) = t$. Based on the usual Philos type functions for differential equations, we first construct a class of explicit functions on time scales for Eq. (1). Then, several oscillation criteria for Eq. (1) are established in both the case $q(t) = \sigma(t)$ and the case $q(t) = t$, which complement those results in [18] and [19].

2 Time scale essentials

The definitions below merely serve as a preliminary introduction to the time-scale calculus; they can be found in the context of a much more robust treatment than is allowed here in the text [2] and the references therein.

Definition 2.1 Define the forward (backward) jump operator $\sigma(t)$ at t for $t < \sup \mathbb{T}$ (respectively $\rho(t)$ at t for $t > \inf \mathbb{T}$) by

$$\sigma(t) = \inf\{s > t : s \in \mathbb{T}\}, \quad (\rho(t) = \sup\{s < t : t \in \mathbb{T}\}), \quad t \in \mathbb{T}.$$

Also define $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$, if $\sup \mathbb{T} < \infty$, and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$, if $\inf \mathbb{T} > -\infty$. The graininess functions are given by $\mu(t) = \sigma(t) - t$ and $v(t) = t - \rho(t)$.

Throughout this paper, the assumption is made that $\mathbb{T}$ is unbounded above and has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. Also assume throughout that $a < b$ are points in $\mathbb{T}$. The jump operators σ and ρ allow the classification of points in a time scale in the following way: If $\sigma(t) > t$ the point t is right-scattered, while if $\rho(t) < t$ then t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. If $t < \sup \mathbb{T}$ and $\sigma(t) = (t)$ the point t is right-dense; if $t > \inf \mathbb{T}$ and $\rho(t) = t$ then t is left-dense. Points that are right-dense and left-dense at the same time are called dense. The composition $f \circ \sigma$ is often denoted f^σ.

Definition 2.2 A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous (denoted $f \in C_{rd}(\mathbb{T}, \mathbb{R})$) if it is continuous at each right-dense point and if there exists a finite left limit in all left-dense points.

Every right-dense continuous function has a delta antiderivative [2, Theorem 1.74]. This implies that the delta definite integral of any right-dense continuous function exists. Likewise every left-dense continuous function f on the time scale, denoted $f \in C_{ld}(\mathbb{T}, \mathbb{R})$, has a nabla antiderivative [2, Theorem 8.45]

Definition 2.3 Fix $t \in \mathbb{T}$ and let $y : \mathbb{T} \to \mathbb{R}$. Define $y^\Delta(t)$ to be the number (if it exists) with the property that given $\epsilon > 0$ there is a neighborhood U of t such that, for all $s \in U$,

$$|[y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|.$$

Call $y^\Delta(t)$ the (delta) derivative of y at t.

Definition 2.4 If $F^\Delta(t) = f(t)$ then define the (Cauchy) delta integral by

$$\int_a^b f(s)\Delta s = F(b) - F(a).$$

The following theorem is due to Hilger [1].

Theorem 2.5 Assume that $f : \mathbb{T} \to \mathbb{T}$ and let $t \in \mathbb{T}^\kappa$.

(1) If f is differentiable at t, then f is continuous at t.

(2) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

(3) If f is differentiable and t is right-dense, then

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(4) If f is differentiable at t, then $f^\sigma(t) = f(t) + \mu(t)f^\Delta(t)$.

(5) If f and g are differentiable at t, then fg is differentiable at t with

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t).$$

3 Main results

We first construct Philos type functions on time scales for Eq. (1). In [18] and [19] the authors only sketchily defined Philos type functions on time scales for Eq. (1), while they did not answer how to construct these functions explicitly.

Let $t_0 \in \mathbb{T}$, $\mathbb{D}_0 = \{(t,s) \in \mathbb{R}^2 : t > s \geq t_0\}$ and $\mathbb{D} = \{(t,s) \in \mathbb{R}^2 : t \geq s \geq t_0\}$. Recall to introduce a usual Philos type function class $\mathcal{X}$ in [26] and [27]. The function $H(t,s) \in C(\mathbb{D}, \mathbb{R})$ is said to belong to the class $\mathcal{X}$ if $H(t,s) \geq 0$ on $\mathbb{D}$ and $H(t,s) > 0$ on $\mathbb{D}_0$.

Just as shown in the sequel, the results in [18] and [19] seem to be invalid when the time scale $\mathbb{T}$ only contains isolated points. Therefore, we are here concerned with the time scale $\mathbb{T}$ which only contains isolated points.

Now, based on any functions $H_1, H_2 \in \mathcal{X}$, we define the following Philos type function class on time scales for Eq. (1)

$$\mathcal{X}_{\mathbb{T}} = \{H_1(\sigma(t), s)H_2(\sigma^2(t), s) : H_1, H_2 \in \mathcal{X},\ (t,s) \in \mathbb{T}^2\},$$

where $\sigma^2 = \sigma \circ \sigma$. Denote

$$\mathcal{H}(t,s) := H_1(\sigma(t), s)H_2(\sigma^2(t), s).$$

Then, $\mathcal{H}(t, \sigma(s)) \geq 0$ for $t_0 \leq s \leq t$ and $\mathcal{H}(t, \sigma(s)) = 0$ only holds at $s = t$. Straightforward computation yields

$$\begin{aligned} \mathcal{H}^{\Delta_s}(t,s) &= [H_1(\sigma(t), s)H_2(\sigma^2(t), s)]^{\Delta_s} \\ &= H_1(\sigma(t), s)H_2^{\Delta_s}(\sigma^2(t), s) + H_1^{\Delta_s}(\sigma(t), s)H_2(\sigma^2(t), \sigma(s)] \end{aligned} \tag{2}$$

and

$$\begin{aligned} \mathcal{H}^{\Delta^2_{s^2}}(t,s) &= H_1(\sigma(t), \sigma(s))H_2^{\Delta^2_{s^2}}(\sigma^2(t), s) + H_1^{\Delta_s}(\sigma(t), s)H_2^{\Delta_s}(\sigma^2(t), s)] \\ &+ H_1^{\Delta^2_{s^2}}(\sigma(t), s)H_2(\sigma^2(t), \sigma^2(s)) + H_1^{\Delta_s}(\sigma(t), s)H_2^{\Delta_s}(\sigma^2(t), \sigma(s))]. \end{aligned} \tag{3}$$

It is not difficult to verify

$$\mathcal{H}(t, \sigma(t)) = \mathcal{H}^{\Delta_s}(t, \sigma(t)) = 0, \quad t \in \mathbb{T}. \tag{4}$$

Before giving the main results of this paper, we first recall to introduce Theorem 2.2 in [18] as followings:

Theorem A [18]. Assume that $q(t) = \sigma(t)$. If there exists a function $H(t,s) \in C_{rd}(\mathbb{D}_{\mathbb{T}}, \mathbb{R})$ which has a nonpositive continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ and a non-negative continuous second-order Δ-partial derivative $H^{\Delta^2_{s^2}}(t,s)$ with respect to the second variable, such that $H(t,t)=0$, $H(t,s)>0$ on $\mathbb{D}_{\mathbb{T}0}$, $H^{\Delta_s}(\sigma(t),\sigma(t))=0$,

$$\lim_{t\to\infty}\frac{H^{\Delta_s}(\sigma(t),t_0)}{H(\sigma(t),t_0)}=O(1),$$

$$\limsup_{t\to\infty}\frac{1}{H(\sigma(t),t_0)}\int_{t_0}^{\sigma(t)}[H(\sigma(t),\sigma(s))e(s)-G(t,s)]\Delta s=+\infty,$$

$$\liminf_{t\to\infty}\frac{1}{H(\sigma(t),t_0)}\int_{t_0}^{\sigma(t)}[H(\sigma(t),\sigma(s))e(s)+G(t,s)]\Delta s=-\infty,$$

where

$$G(t,s)=(\lambda-1)\lambda^{\frac{\lambda}{1-\lambda}}[H^{\Delta_{s^2}^2}(\sigma(t),s)]^{\frac{\lambda}{\lambda-1}}[H(\sigma(t),\sigma(s))p(s)]^{\frac{1}{1-\lambda}},$$

then Eq. (1) with $q(t)=\sigma(t)$ is oscillatory.

We show that Theorem A seems to be invalid for the case when the time scale $\mathbb{T}$ only contains isolated points. In fact, in the proof of Theorem 2.2 in [18], the authors used a basic inequality

$$F(x)=ax-bx^{\lambda}\le(\lambda-1)\lambda^{\frac{\lambda}{1-\lambda}}a^{\frac{\lambda}{\lambda-1}}b^{\frac{1}{1-\lambda}},\quad a,x\ge0,\ b>0 \tag{5}$$

to estimate

$$\int_{t_0}^{\sigma(t)}[H^{\Delta_{s^2}^2}(\sigma(t),s)x(\sigma(s))-H(\sigma(t),\sigma(s))p(s)x^{\lambda}(\sigma(s))]\Delta s.$$

Note that $H(\sigma(t),\sigma(t))=0$ and t is an isolated point, we do not use the inequality (5) to get that

$$\int_{t_0}^{\sigma(t)}[H^{\Delta_{s^2}^2}(\sigma(t),s)x(\sigma(s))-H(\sigma(t),\sigma(s))p(s)x^{\lambda}(\sigma(s))]\Delta s\le\int_{t_0}^{\sigma(t)}G(t,s)\Delta s.$$

Based on the definition of $H(t,s)$, we can only conclude that

$$\int_{t_0}^{t}[H^{\Delta_{s^2}^2}(\sigma(t),s)x(\sigma(s))-H(\sigma(t),\sigma(s))p(s)x^{\lambda}(\sigma(s))]\Delta s\le\int_{t_0}^{t}G(t,s)\Delta s.$$

Therefore, the term

$$\int_{t}^{\sigma(t)}H^{\Delta_{s^2}^2}(\sigma(t),s)x(\sigma(s))\Delta s$$

remains unestimated.

To complement those results in [18] and [19], we here focus on the oscillation of Eq. (1) on time scales which only contain isolated points.

Theorem 3.1. Assume that $q(t)=\sigma(t)$. If there exist a function $\mathcal{H}(t,s)\in\mathcal{X}_{\mathbb{T}}$ and a right dense continuous function $\phi(t)>0$ on $\mathbb{T}$ such that

$$\limsup_{t\to\infty}\frac{\mathcal{H}^{\Delta_s}(t,t_0)}{\mathcal{H}(t,t_0)}<\infty \tag{6}$$

$$\limsup_{t\to\infty} \frac{\mu(t)|\mathcal{H}^{\Delta^2_{s^2}}(t,t)|\phi(\sigma(t))}{\mathcal{H}(t,t_0)} < \infty \tag{7}$$

$$\limsup_{t\to\infty} \frac{1}{\mathcal{H}(t,t_0)} \sum_{s=t_0}^{\rho(t)} [\mathcal{H}(t,\sigma(s))e(s) - \mathcal{G}(t,s)]\, \Delta s = +\infty \tag{8}$$

$$\liminf_{t\to\infty} \frac{1}{\mathcal{H}(t,t_0)} \sum_{s=t_0}^{\rho(t)} [\mathcal{H}(t,\sigma(s))e(s) + \mathcal{G}(t,s)]\, \Delta s = -\infty \tag{9}$$

where

$$\mathcal{G}(t,s) = (\lambda-1)\lambda^{\frac{\lambda}{1-\lambda}} |\mathcal{H}^{\Delta^2_{s^2}}(t,s)|^{\frac{\lambda}{\lambda-1}} [\mathcal{H}(t,\sigma(s))p(s)]^{\frac{1}{1-\lambda}}$$

then all solutions of Eq. (1) satisfying $|x(t)| = O(\phi(t))$ are oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1). Without loss of generality, assume that $x(t) > 0$ for $t \geq t_0$ and $t \in \mathbb{T}$. Multiplying (1) by $\mathcal{H}(t,\sigma(s))$ and integrating from t_0 to $\sigma(t)$ yield

$$\int_{t_0}^{\sigma(t)} \mathcal{H}(t,\sigma(s))e(s)\Delta s = \int_{t_0}^{\sigma(t)} \mathcal{H}(t,\sigma(s))[x^{\Delta\Delta}(s) - p(s)x^{\lambda}(\sigma(s))]\Delta s.$$

By using the integration by parts formula two times, the definition of $\mathcal{H}$ and (4), we get

$$\begin{aligned} & \int_{t_0}^{\sigma(t)} \mathcal{H}(t,\sigma(s))e(s)\Delta s \\ = \quad & -\mathcal{H}(t,t_0)x^{\Delta}(t_0) + \mathcal{H}^{\Delta_s}(t,t_0)x(t_0) \\ & + \int_{t_0}^{\sigma(t)} [\mathcal{H}^{\Delta^2_{s^2}}(t,s)x(\sigma(s)) - \mathcal{H}(t,\sigma(s))p(s)x^{\lambda}(\sigma(s))]\Delta s. \end{aligned} \tag{10}$$

Notice that $\mathcal{H}(t,\sigma(s)) = 0$ only holds at $s = t$ and $x(t) = O(\phi(t))$. Then, there exists an appropriate constant $M > 0$ such that

$$\begin{aligned} & \int_{t_0}^{\sigma(t)} [\mathcal{H}^{\Delta^2_{s^2}}(t,s)x(\sigma(s)) - \mathcal{H}(t,\sigma(s))p(s)x^{\lambda}(\sigma(s))]\Delta s \\ & \qquad \leq M \int_{t}^{\sigma(t)} |\mathcal{H}^{\Delta^2_{s^2}}(t,s)|\phi(\sigma(s))\Delta s \\ & + \int_{t_0}^{t} [|\mathcal{H}^{\Delta^2_{s^2}}(t,s)|x(\sigma(s)) - \mathcal{H}(t,\sigma(s))p(s)x^{\lambda}(\sigma(s))]\Delta s. \end{aligned} \tag{11}$$

By the inequality (5), we have

$$|\mathcal{H}^{\Delta^2_{s^2}}(t,s)|x(\sigma(s)) - \mathcal{H}(t,\sigma(s))p(s)x^{\lambda}(\sigma(s)) \leq \mathcal{G}(t,s), \quad s \in [t_0,t)_{\mathbb{T}}. \tag{12}$$

Thus, from (10)-(12) and noting that $\mathcal{H}(t,\sigma(t)) = 0$ and $\mathcal{H}(t,\sigma(s)) > 0$ on $[t_0,t)_{\mathbb{T}}$ for $t \in \mathbb{T}$, we get

$$\begin{aligned}
&\frac{1}{\mathcal{H}(t,t_0)}\left[\int_{t_0}^{\sigma(t)} \mathcal{H}(t,\sigma(s))e(s)\Delta s - \int_{t_0}^{t} \mathcal{G}(t,s)\Delta s\right] \\
= &\frac{1}{\mathcal{H}(t,t_0)}\left[\sum_{s=t_0}^{t} \mathcal{H}(t,\sigma(s))e(s) - \sum_{s=t_0}^{\rho(t)} \mathcal{G}(t,s)\right] \\
= &\frac{1}{\mathcal{H}(t,t_0)}\sum_{s=t_0}^{\rho(t)}[\mathcal{H}(t,\sigma(s))e(s) - \mathcal{G}(t,s)] \\
\leq &-x^{\Delta}(t_0) + \frac{\mathcal{H}^{\Delta_s}(t,t_0)}{\mathcal{H}(t,t_0)}x(t_0) + \frac{M\mu(t)|\mathcal{H}^{\Delta^2_{s^2}}(t,t)|\phi(\sigma(t))}{\mathcal{H}(t,t_0)}.
\end{aligned}$$

Taking lim sup on both sides of the above inequality as $t \to \infty$ and using conditions (6)-(8), we get a desired contradiction. This completes the proof of Theorem 3.1. □

Theorem 3.2. Assume that $q(t) = t$. If there exist a function $\mathcal{H}(t,s) \in \mathcal{X}_{\mathbb{T}}$ and a right dense continuous function $\phi(t) > 0$ on $\mathbb{T}$ such that

$$\limsup_{t\to\infty} \frac{\mathcal{H}^{\Delta_s}(t,t_0)}{\mathcal{H}(t,t_0)} < \infty \tag{13}$$

$$\limsup_{t\to\infty} \frac{\sum_{s=t}^{\sigma(t)} |\mathcal{H}^{\Delta^2_{s^2}}(t,\rho(s))\phi(s)|}{\mathcal{H}(t,t_0)} < \infty \tag{14}$$

$$\limsup_{t\to\infty} \frac{1}{\mathcal{H}(t,t_0)} \sum_{s=\sigma(t_0)}^{\rho(t)} \left[\mu(s)\mathcal{H}(t,\sigma(s))e(s) - \tilde{\mathcal{G}}(t,s)\right] = +\infty \tag{15}$$

$$\liminf_{t\to\infty} \frac{1}{\mathcal{H}(t,t_0)} \sum_{s=\sigma(t_0)}^{\rho(t)} \left[\mu(s)\mathcal{H}(t,\sigma(s))e(s) + \tilde{\mathcal{G}}(t,s)\right] = -\infty \tag{16}$$

where

$$\tilde{\mathcal{G}}(t,s) = (\lambda-1)\lambda^{\frac{\lambda}{1-\lambda}}[\mu(\rho(s))|\mathcal{H}^{\Delta^2_{s^2}}(t,\rho(s))|]^{\frac{\lambda}{\lambda-1}}[\mu(s)\mathcal{H}(t,\sigma(s))p(s)]^{\frac{1}{1-\lambda}}$$

then all solutions of Eq. (1) satisfying $|x(t)| = O(\phi(t))$ are oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1). Say $x(t) > 0$ for $t \geq t_0$ and $t \in \mathbb{T}$. Multiplying Eq. (1) by $\mathcal{H}(t,\sigma(s))$ and integrating from t_0 to $\sigma(t)$ by the integration by parts formula, we get

$$\begin{aligned}
&\int_{t_0}^{\sigma(t)} \mathcal{H}(t,\sigma(s))e(s)\Delta s \\
\leq &-\mathcal{H}(t,t_0))x^{\Delta}(t_0) + \mathcal{H}^{\Delta_s}(t,t_0)x(t_0) \\
&+\int_{t_0}^{\sigma(t)} [|\mathcal{H}^{\Delta^2_{s^2}}(t,s)|x(\sigma(s)) - \mathcal{H}(t,\sigma(s))p(s)x^{\lambda}(s)]\Delta s.
\end{aligned}$$

Noting that $H(t,\sigma(t))=0$ and $\mathbb{T}$ only contains isolated points, we have

$$
\begin{aligned}
& \int_{t_0}^{\sigma(t)} [|\mathcal{H}^{\Delta^2_{s^2}}(t,s)|x(\sigma(s)) - \mathcal{H}(t,\sigma(s))p(s)x^{\lambda}(s)]\Delta s \\
= & \sum_{s=t_0}^{t} \mu(s)[|\mathcal{H}^{\Delta^2_{s^2}}(t,s)|x(\sigma(s)) - \mathcal{H}(t,\sigma(s))p(s)x^{\lambda}(s)] \\
= & \sum_{s=\sigma(t_0)}^{\sigma(t)} \mu(\rho(s))|\mathcal{H}^{\Delta^2_{s^2}}(t,\rho(s))|x(s) - \sum_{s=t_0}^{\rho(t)} \mu(s)\mathcal{H}(t,\sigma(s))p(s)x^{\lambda}(s) \\
\le & \sum_{s=t}^{\sigma(t)} \mu(\rho(s))|\mathcal{H}^{\Delta^2_{s^2}}(t,\rho(s))|x(s) \\
+ & \sum_{s=\sigma(t_0)}^{\rho(t)} [\mu(\rho(s))|\mathcal{H}^{\Delta^2_{s^2}}(t,\rho(s))|x(s) - \mu(s)\mathcal{H}(t,\sigma(s))p(s)x^{\lambda}(s)].
\end{aligned}
$$

Similar to the same argument in Theorem 3.1, we have

$$
\begin{aligned}
& \frac{1}{\mathcal{H}(t,t_0)}\left[\sum_{s=t_0}^{t} \mu(s)\mathcal{H}(t,\sigma(s))e(s) - \sum_{s=t_0}^{\rho(t)} \tilde{\mathcal{G}}(t,s)\right] \\
= & \frac{1}{\mathcal{H}(t,t_0)}\left[\sum_{s=t_0}^{\rho(t)} \mu(s)\mathcal{H}(t,\sigma(s))e(s) - \sum_{s=t_0}^{\rho(t)} \tilde{\mathcal{G}}(t,s)\right] \\
\le & -x^{\Delta}(t_0) + \frac{\mathcal{H}^{\Delta_s}(t,t_0)}{\mathcal{H}(t,t_0)}x(t_0) + \frac{M\sum_{s=t}^{\sigma(t)} \mu(\rho(s))|\mathcal{H}^{\Delta^2_{s^2}}(t,\rho(s))|\phi(s)}{\mathcal{H}(t,t_0)}.
\end{aligned}
$$

This together with (13)-(15) yield a contradiction. The proof of Theorem 3.2 is complete.□

For the special case $\mathbb{T}=\mathbb{Z}$, we have the following oscillation results:

Corollary 3.1. Assume that $q(t)=\sigma(t)$ and $\mathbb{T}=\mathbb{Z}$. If there exist a function $\mathcal{H}(t,s)\in\mathcal{X}_{\mathbb{T}}$ and a right dense continuous function $\phi(t)>0$ on $\mathbb{T}$ such that

$$\limsup_{t\to\infty} \frac{\mathcal{H}^{\Delta_s}(t,t_0)}{\mathcal{H}(t,t_0)} < \infty$$

$$\limsup_{t\to\infty} \frac{\mathcal{H}^{\Delta^2_{s^2}}(t,t)}{\mathcal{H}(t,t_0)}\phi(t+1) < \infty$$

$$\limsup_{t\to\infty} \frac{1}{\mathcal{H}(t,t_0)} \sum_{s=t_0}^{t-1} [\mathcal{H}(t,s+1)e(s) - \mathcal{G}(t,s)] = +\infty$$

$$\liminf_{t\to\infty} \frac{1}{\mathcal{H}(t,t_0)} \sum_{s=t_0}^{t-1} [\mathcal{H}(t,s+1)e(s) + \mathcal{G}(t,s)] = -\infty$$

where

$$\mathcal{G}(t,s)=(\lambda-1)\lambda^{\frac{\lambda}{1-\lambda}}[\mathcal{H}^{\Delta^2_{s^2}}(t,s)]^{\frac{\lambda}{\lambda-1}}[\mathcal{H}(t,s+1)p(s)]^{\frac{1}{1-\lambda}}$$

then all solutions of Eq. (1) satisfying $|x(t)|=O(\phi(t))$ are oscillatory.

Corollary 3.2. Assume that $q(t)=t$ and $\mathbb{T}=\mathbb{Z}$. If there exist a function $\mathcal{H}(t,s)\in\mathcal{X}_{\mathbb{T}}$ and a right dense continuous function $\phi(t)>0$ on $\mathbb{T}$ such that

$$\limsup_{t\to\infty}\frac{\mathcal{H}^{\Delta_s}(t,t_0)}{\mathcal{H}(t,t_0)}<\infty$$

$$\limsup_{t\to\infty}\frac{\mathcal{H}^{\Delta^2_{s^2}}(t,t-1)\phi(t)+\mathcal{H}^{\Delta^2_{s^2}}(t,t)\phi(t+1)}{\mathcal{H}(t,t_0)}<\infty$$

$$\limsup_{t\to\infty}\frac{1}{\mathcal{H}(t,t_0)}\sum_{s=t_0+1}^{t-1}\Big[\mathcal{H}(t,s+1)e(s)-\tilde{\mathcal{G}}(t,s)\Big]=+\infty$$

$$\limsup_{t\to\infty}\frac{1}{\mathcal{H}(t,t_0)}\sum_{s=t_0+1}^{t-1}\Big[\mathcal{H}(t,s+1)e(s)+\tilde{\mathcal{G}}(t,s)\Big]=-\infty$$

where

$$\tilde{\mathcal{G}}(t,s)=(\lambda-1)\lambda^{\frac{\lambda}{1-\lambda}}[\mathcal{H}^{\Delta^2_{s^2}}(t,s-1)]^{\frac{\lambda}{\lambda-1}}[\mathcal{H}(t,s+1)p(s)]^{\frac{1}{1-\lambda}}$$

then all solutions of Eq. (1) satisfying $|x(t)|=O(\phi(t))$ are oscillatory.

To illustrate the usefulness of the results, we state the corresponding theorems in the above for the special case $\mathbb{T}=\mathbb{Z}$. It is not difficult to provide similar results for other specific time scales of interest. On the other hand, all the results obtained in this paper are restricted to those solutions satisfying $|x(t)|=O(\phi(t))$. At present, it seems difficult to obtain sufficient conditions for the oscillation of all solutions of Eq. (1) with $p(t)>0$ and $\lambda>1$ when the time scale $\mathbb{T}$ only contains isolated points. This problem is left for future study.

Acknowledgment

The author thanks the reviewers for their valuable comments on this paper. This work was supported by the National Natural Science Foundation of China (60704039) and the Natural Science Foundation of Shandong Province (ZR2010AL002)

References

[1] S. Hilger, Analysis on measure chains-A unified approach to continuous and discrete calculus, Results Math. 18 (1990) 18-56.

[2] M. Bohner, A. Peterson, Dynamic Equations on Time Scales, An Introduction with Applications, Birkhäser, Boston, 2001.

[3] M. Bohner, A. Peterson (Eds.), Advances in Dynamic Equations on Time Scales, Birkhäser, Boston, 2003.

[4] R.P. Agarwal, M. Bohner, D. O'Regan, A. Peterson, Dynamic equations on time scales: a survey. J. Comput. Appl. Math., 141 (2002), 1-26.

[5] R.P. Agarwal, D. O'Regan, S.H. Saker, Oscillation criteria for second-order nonlinear neutral delay dynamic equations, J. Math. Anal. Appl. 300 (2004) 203-217.

[6] R.P. Agarwal, A. Zafer, Oscillation criteria for second order forced dynamic equations with mixed nonlinearities, Adv. Difference Equations (2009) Article ID 938706, 20 pages.

[7] R.P. Agarwal, D.R. Anderson, A. Zafer, Interval oscillation criteria for second-order forced delay dynamic equations with mixed nonlinearities, Comput. Math. Appl. 59 (2010) 977-993.

[8] D.R. Anderson, Oscillation of second-order forced functional dynamic equations with oscillatory potentials, J. Difference Equ. Appl. 13 (5) (2007) 407-421.

[9] D.R. Anderson, Interval criteria for oscillation of nonlinear second-order dynamic equations on time scales, Nonlinear Anal. 69 (2008) 4614-4623.

[10] D.R. Anderson, A. Zafer, Interval criteria for second-order super-half-linear functional dynamic equations with delay and advanced arguments, J. Diff. Equations and Appl. 16 (2010) 917-930.

[11] A. Zafer, Interval oscillation criteria for second order super-half-linear functional differential equations with delay and advanced arguments, Math. Nachr. 282 (2009) 1334-1341.

[12] M. Bohner, C. Tisdell, Oscillation and nonoscillation of forced second order dynamic equations, Pacific J. Math. 230 (1) (2007) 59-71.

[13] L. Erbe, A. Peterson, S.H. Saker, Kamenev-type oscillation criteria for second-order linear delay dynamic equations, Dynamic Syst. & Appl. 15 (2006) 65-78.

[14] L. Erbe, A. Peterson, S.H. Saker, Hille-Kneser-type criteria for second-order linear dynamic equations, Adv. Differential Equations 2006 (2006) 1-18.

[15] L. Erbe, A. Peterson, S.H. Saker, Oscillation criteria for second-order nonlinear delay dynamic equations, J. Math. Anal. Appl. 333 (2007) 505-522.

[16] T.S. Hassan, Oscillation criteria for half-linear dynamic equations on time scales, J. Math. Anal. Appl. 345 (2008) 176-185.

[17] T.S. Hassan, L. Erbe, A. Peterson, Oscillation of second order superlinear dynamic equations with damping on time scales, Comput. Appl. Math. 59 (2010) 550-558.

[18] T.X. Li, Z.L. Han, S.H. Sun, C.H. Zhang, Forced oscillation of second-order nonlinear dynamic equations on time scales, Electronic Journal of Qualitative Theory of Differential Equations 60 (2009) 1-8.

[19] P.G. Wang, N. Tang, C.X. Gao, Oscillation of forced second order dynamic equations on time scales, Advances in Dynamical Systems and Applications 4 (2009) 123-131.

[20] Y.G. Sun, A note on Nasr's and Wong's papers, J. Math. Anal. Appl. 286 (2003) 363-367.

[21] Y.G. Sun, C.H. Ou, J.S.W. Wong, Interval oscillation theorems for a linear second order differential equation, Comput. Math. Appl. 48 (2004) 1693-1699.

[22] Y.G. Sun, J.S.W. Wong, Oscillation criteria for second order forced ordinary differential equations with mixed nonlinearities, J. Math. Anal. Appl. 334 (2007) 549-560.

[23] Y.G. Sun, F.W. Meng, Interval criteria for oscillation of second-order differential equations with mixed nonlinearities, Appl. Math. Comput. 198 (2008) 375-381.

[24] J.S.W. Wong, Oscillation criteria for a forced second order linear differential equation, J. Math. Anal. Appl. 231 (1999) 235-240.

[25] Q. Kong, Interval criteria for oscillation of second order linear ordinary differential equations, J. Math. Anal. Appl. 229 (1999) 258-270.

[26] Ch.G. Philos, Oscillation theorems for linear differential equation of second order, Archive for Mathematical Logic, 53 (1989), 483-492.

[27] R.P. Agarwal, S.R. Grace, Forced oscillation of nth order nonlinear differential equations, Appl. Math. Lett. 13 (2000), 53-57.

(Received July 3, 2010)