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On stabilizability of the upper equilibrium of the asymmetrically excited inverted pendulum

Dedicated to Professor László Hatvani on the occasion of his 75th birthday

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Abstract. Using purely elementary methods, necessary and sufficient conditions are given for the existence of T-periodic and $2T$-periodic solutions around the upper equilibrium of the mathematical pendulum when the suspension point is vibrating vertically with asymmetric high frequency. The equation of the motion is of the form

$$\ddot{\theta} - \frac{1}{l}\left(g + a(t)\right)\theta = 0,$$

where

$$a(t) := \begin{cases} A_h, & \text{if } kT \le t < kT + T_h, \\ -A_e, & \text{if } kT + T_h \le t < (kT + T_h) + T_e, \end{cases} \qquad (k = 0, 1, \dots);$$

A_h, A_e, T_h, T_e are positive constants ($T_h + T_e = T$); g and l denote the acceleration of gravity and the length of the pendulum, respectively. An extended Oscillation Theorem is given. The exact stability regions for the upper equilibrium are presented.

Keywords: inverted pendulum, asymmetric excitation, periodic step function coefficient, stabilization, stability regions.

2010 Mathematics Subject Classification: 34A26, 34A30, 34D20, 70J25.

1 Introduction

Since A. Stephenson discovered [21] that the upper (unstable) equilibrium of the mathematical pendulum can be stabilized by vibrating of the point of suspension vertically with sufficiently high frequency many papers (see, e.g., [2, 4, 8, 15–17, 20] and the references therein) have been devoted to the description of this phenomenon (see also [1, 5, 19]). Investigating the small oscillation around the upper equilibrium V. I. Arnold [1] and, later, M. Levi and W. Weckesser [17] estimated the stability zones on the parameter plane. In [6] with László Hatvani, we gave

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a more precise estimate for the stability regions than Levi and Weckesser. It is well known [1] that the boundary curves of these zones correspond to the equations of motions having T-periodic and $2T$-periodic solutions, where T is the period of the vibration of the suspension point. In the joint work [7] with Professor László Hatvani, we gave necessary and sufficient conditions for the parameters in the equation of motions so that the equation have periodic solutions of $2T$ or $4T$, in that case the suspension point of the pendulum moves vertically by a symmetric effect. In the present article we investigate the equation of motion of the pendulum when its suspension point moves under the influence of an asymmetric, T-periodic force and give necessary and sufficient conditions for the parameters so that the equation of motion have periodic solutions of T or $2T$. Applying these conditions we can give an *extended* Oscillation Theorem in the sense that setting special value for each independent parameter, this theorem corresponds an oscillation theorem of the corresponding equation. The conditions define the exact stability regions on the parameter space. The conditions and their proofs are based upon purely elementary methods; we do not use even Floquet's theory [1,5,19].

In Section 2 we set up the model describing the small oscillations of the excited pendulum around the upper equilibrium. The model is a non-autonomous second order linear differential equation with a T-periodic step function coefficient. We reduce this equation to an equivalent dynamical system on the plane. In Section 3 we construct periodic solutions of period T and $2T$ to this equivalent system. In Section 4 we give an oscillation theorem and deduce stability conclusion, and present the stability regions on the parameter space introduced in [6].

2 Technical background

It is well-known [1, 5, 19] that motions of the mathematical pendulum are described by the second order differential equation

$$\ddot{\psi} + \frac{g}{l}\sin\psi = 0 \qquad (-\infty < \psi < \infty), \tag{2.1}$$

where the state variable ψ denotes the angle between the rod of the pendulum and the direction downward measured counter-clockwise; g and l are positive constants. The lower equilibrium position $\psi \equiv 0 \pmod 0$ is stable, and the upper one $\psi \equiv \pi \pmod{2\pi}$ is unstable. We want to stabilize the upper equilibrium position, so introducing the new angle variable $\theta = \psi - \pi$ and linearizing equation (2.1) we obtain the linear second order differential equation

$$\ddot{\theta} - \frac{g}{l}\theta = 0,$$

which describes the small oscillations of the pendulum around the upper equilibrium position $\theta \equiv 0 \pmod{2\pi}$.

Suppose that the suspension point is vibrating vertically with the T-periodic acceleration

$$a(t) := \begin{cases} A_h, & \text{if } kT \le t < kT + T_h, \\ -A_e, & \text{if } kT + T_h \le t < (kT + T_h) + T_e, \end{cases} \qquad (k = 0, 1, \dots); \tag{2.2}$$

A_h, A_e, T_h, T_e are positive constants ($T_h + T_e = T$). If $p = p(t)$ and $\dot{p}$ denote the displacement and the velocity in the vibration of the suspension point respectively, and $p(0) = 0$, $\dot{p}(0) < 0$,

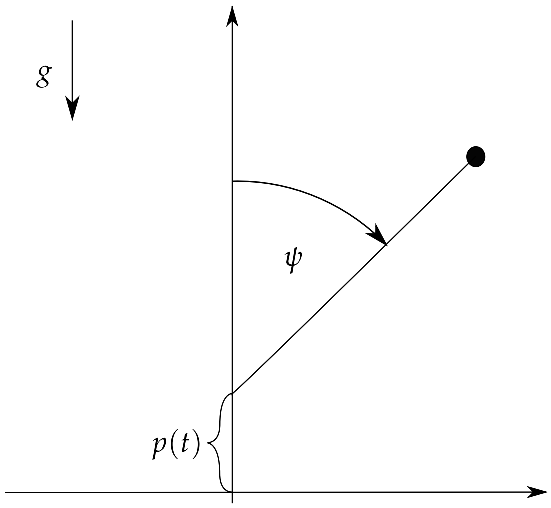

Figure 2.1: Vertically excited inverted pendulum.

then it can be seen that the motion of the point is represented by the function

$$p(t) := \begin{cases} \frac{1}{2}A_h(t-kT)(t-kT-T_h) & \text{if } kT \le t < kT+T_h, \\ -\frac{1}{2}A_e(t-kT-T_h)^2 & \\ \quad +\frac{1}{2}A_eT_e(t-kT-T_h) & \text{if } kT+T_h \le t < (k+1)T, \end{cases} \qquad (k=0,1,\dots), \tag{2.3}$$

(see Figure 2.1).

The maximum amplitudes of the vibration in the first and second phase within one period $T_h + T_e = T$ are expressed by the formulae

$$D_h = \frac{1}{8}A_h{T_h}^2, \qquad D_e = \frac{1}{8}A_e{T_e}^2,$$

and, presuming the natural condition that the velocity of the point of suspension is continuous, the six parameters of the vibration satisfy the following two assumptions:

$$\frac{A_h}{A_e} = \frac{T_e}{T_h}, \qquad \frac{D_h}{D_e} = \frac{T_h}{T_e}. \tag{2.4}$$

Since the suspending rod is rigid, the acceleration of the vibration is continuously added to the gravity, and the equation of motion of the pendulum is

$$\ddot{\theta} - \frac{1}{l}(g + a(t))\theta = 0. \tag{2.5}$$

Every motion of (2.5) has two phases during every period, a hyperbolic and an elliptic one, that are described by the equations

$$\ddot{\theta} - \omega_h^2\theta = 0 \qquad (kT \le t < kT + T_h) \tag{2.6}$$

and

$$\ddot{\theta} + \omega_e^2\theta = 0 \qquad (kT + T_h \le t < kT + T_h + T_e), \tag{2.7}$$

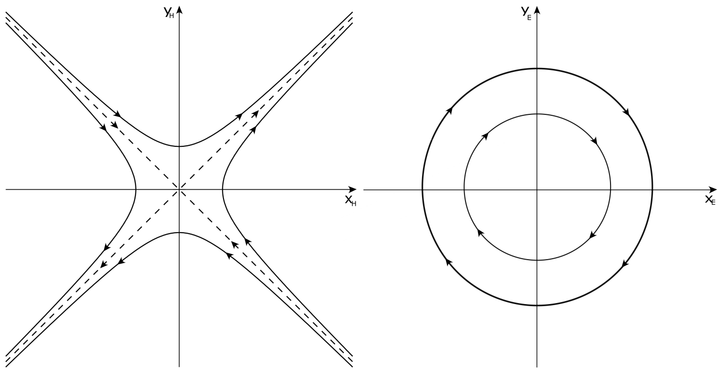

Figure 2.2: Hyperbolic and elliptic rotation.

where

$$\omega_h := \sqrt{\frac{A_h + g}{l}}, \qquad \omega_e := \sqrt{\frac{A_e - g}{l}}, \qquad (A_e > g, \quad k \in \mathbb{N})$$

denote the hyperbolic and the elliptic frequency of the pendulum, respectively.

A fruitful treatment can be found in [11]. Similar to that, we introduce two different phase planes for the two different phases of the motions. Starting with the hyperbolic case, we introduce the new phase variables

$$x_h = \theta, \qquad y_h = \frac{\dot{\theta}}{\omega_h},$$

in which (2.6) has the following symmetric form:

$$\dot{x}_h = \omega_h y_h, \qquad \dot{y}_h = \omega_h x_h. \tag{2.8}$$

Using polar coordinates r_h, φ_h and the transformation rules

$$x_h = r_h \cos\varphi_h, \quad y_h = r_h \sin\varphi_h \qquad (r_h > 0, \ -\infty < \varphi_h < \infty),$$

(2.8) can be rewritten into the system

$$\dot{r}_h = r_h \omega_h \sin 2\varphi_h, \qquad \dot{\varphi}_h = \omega_h \cos 2\varphi_h. \tag{2.9}$$

The derivative of $H_h(x, y) := x_h^2 - y_h^2$ with respect to system (2.8) equals identically zero, i.e., H_h is a first integral of (2.8), so the trajectories of the system are hyperbolae; (2.9) describes "hyperbolic rotations" (see Figure 2.2). We will need the solution of the second equation in (2.9). This equation is separable, so we can write

$$\int_0^t \frac{\dot{\varphi}_h(s)\ \mathrm{d}s}{\cos 2\varphi_h(s)} = \omega_h t, \quad 0 \le t \le T_h,$$

and so

$$\int_{\varphi_0}^{\varphi_h(t)} \frac{\mathrm{d}\varphi}{\cos 2\varphi} = \omega_h t, \quad \varphi_0 := \varphi_h(0) \neq -\frac{\pi}{4}. \tag{2.10}$$

Let $G(\varphi) := \int \mathrm{d}\varphi / \cos 2\varphi$. Then

$$G(\varphi) = -\frac{1}{2} \ln \left| \tan\left(\frac{\pi}{4} - \varphi\right) \right|,$$

whence

$$G(\varphi) := \begin{cases} -\dfrac{1}{2}\ln\tan\left(\dfrac{\pi}{4}-\varphi\right) & \text{if } -\pi/4 < \varphi < \pi/4, \\ -\dfrac{1}{2}\ln\tan\left(\varphi-\dfrac{\pi}{4}\right) & \text{if } -3\pi/4 < \varphi < -\pi/4. \end{cases} \tag{2.11}$$

From (2.10) we obtain

$$\varphi_h(t) = G^{-1}\left(\omega_h t + G(\varphi_0)\right).$$

Especially,

$$\varphi_h(T_h - 0) = G^{-1}\left(\omega_h T_h + G(\varphi_0)\right), \tag{2.12}$$

where $\varphi_h(T_h-0)$ denotes the left-hand side limit of φ at T. Now, we can give the solution of the second equation of (2.9):

$$\varphi_h(t;\varphi_0) := \begin{cases} \dfrac{\pi}{4} - \arctan\left(e^{-2\omega_h t}\tan\left(\dfrac{\pi}{4}-\varphi_0\right)\right) & \text{if } -\pi/4 < \varphi_0 < \pi/4, \\ \dfrac{\pi}{4} + \arctan\left(e^{-2\omega_h t}\tan\left(\varphi_0-\dfrac{\pi}{4}\right)\right) & \text{if } -3\pi/4 < \varphi_0 < -\pi/4. \end{cases} \tag{2.13}$$

Let us repeat the same procedure for the second phase of the period with the new phase variables $x_e = \theta$, $y_e = \dot{\theta}/\omega_e$. Then we get the systems

$$\dot{x}_e = \omega_e y_e, \qquad \dot{y}_e = -\omega_e x_e, \tag{2.14}$$

$$\dot{r}_e = 0, \qquad \dot{\varphi}_e = -\omega_e. \tag{2.15}$$

Now $H_e(x,y) := x_e^2 + y_e^2$ is a first integral, and the trajectories of (2.14) are circles around the origin; (2.15) describes uniform "elliptic (ordinary) rotations".

Equation (2.5) has a piecewise continuous coefficient, so we have to modify the standard definition of a solution of a continuous second order differential equation. A function $\theta : \mathbb{R}^+ \to \mathbb{R}$ is a solution of (2.5) if it is continuously differentiable on $\mathbb{R}^+$, it is twice differentiable on the set

$$S := \mathbb{R}^+ \setminus \left(\{kT\}_{k\in\mathbb{N}} \cup \{kT - T_e\}_{k\in\mathbb{N}}\right),$$

and it satisfies equation (2.5) on the set S. Any solution θ consists of solutions $x_h\colon [kT, kT + T_h) \to \mathbb{R}$ and $x_e : [kT + T_h, (k+1)T) \to \mathbb{R}$ of (2.8) and (2.14) respectively ($k \in \mathbb{N}$). To guarantee the continuity of $\dot{\theta}$ on $\mathbb{R}$ we have to require the "connecting conditions"

$$\begin{aligned} x_e(kT+T_h) &= \lim_{t\to kT+T_h-0} x_h(t), \\ x_h((k+1)T) &= \lim_{t\to(k+1)T-0} x_e(t); \\ \omega_e y_e(kT+T_h) &= \lim_{t\to kT+T_h-0} \omega_h y_h(t), \\ \omega_h y_h((k+1)T) &= \lim_{t\to(k+1)T-0} \omega_e y_e(t). \end{aligned} \tag{2.16}$$

Geometrically this means that when we illustrate the hyperbolic and elliptic phases in a common coordinate system, then the ends of the continuous parts of dynamics there acts a linear transformation on the phase point (a contraction or a dilation)

$$(x,y) \mapsto (x, dy) =: (x, \hat{y}) \qquad (0 < d = \text{const.},\ d \neq 1)$$

in the direction of y-axis. Namely, $d = \omega_h/\omega_e$ at $t = T_h + kT$, and $d = \omega_e/\omega_h$ at $t = (k+1)T$, $k \in \mathbb{N}$.

The steps of dynamics of the system can be described as follows. The phase point starts from (x_0, y_0) and moves along a hyperbola during the interval $[0, T_h)$. At the moment $t = T_h$ a dilation or a contraction of measure ω_h/ω_e happens parallel with y-axis. Then the phase point turns clockwise around the origin by $\omega_e T_e$. Finally, a contraction/dilation of measure ω_e/ω_h happens. These four steps are repeated ad infinitum, see Figure 2.3.

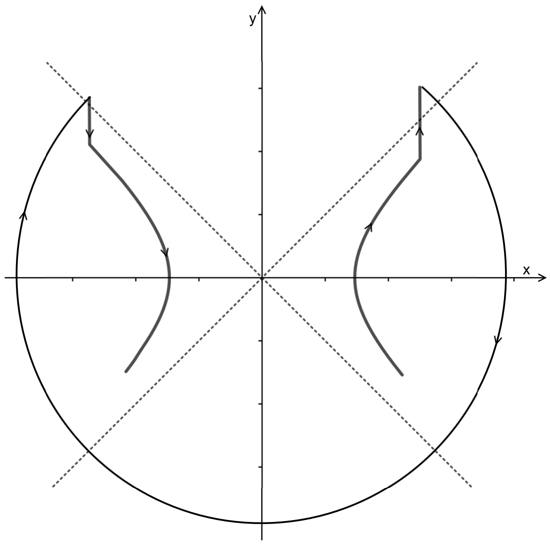

Figure 2.3: The phase space of the inverted pendulum, if $\omega_h > \omega_e$.

Let us consider this system in polar coordinates. Denote by (r_R, φ_R), and $(r_C, \varphi_C) = (\rho(r, \varphi; d), \phi(\varphi; d))$ the image of the point (r, φ) at the rotation of a clockwise angle α and the contraction-dilatation, respectively. Then, obviously, $r_R(r, \varphi) = r,\ \varphi_R(r, \varphi) = \varphi - \alpha$; furthermore,

$$\rho(r, \varphi; d) = \sqrt{x^2 + d^2 y^2} = r\sqrt{1 + (d^2 - 1)\sin^2\varphi} = f(\varphi; d)r,$$
$$f(\varphi, d) := \sqrt{1 + (d^2 - 1)\sin^2\varphi}, \qquad (d > 0, -\infty < \varphi < \infty).$$

It is easy to see that $\tan\phi(\varphi; d) = dy/x = d\tan\varphi$ ($x \neq 0$, i.e., $\varphi \not\equiv \pi/2 \pmod{\pi}$), so

$$\phi(\varphi; d) := \begin{cases} \arctan(d\tan\varphi) + \left[\frac{\varphi + \frac{\pi}{2}}{\pi}\right] \cdot \pi & \text{if } \varphi \neq (2k+1)\dfrac{\pi}{2}, \\ \varphi & \text{if } \varphi = (2k+1)\dfrac{\pi}{2}, \qquad (k \in \mathbb{Z}), \end{cases}$$

where $[x]$ denotes the integer part of $x \in \mathbb{R}$.

The detailed description of properties of functions f and ϕ can be found in [10]. During our calculations we will use from these properties that f is even and ϕ is odd, furthermore $\phi(\cdot + k\pi; d) = \phi(\cdot; d) + k\pi$ $(k \in \mathbb{Z})$; $\phi\left(\phi(\varphi; d); 1/d\right) = \varphi$ $(\varphi \in \mathbb{R})$.

3 The construction of periodic solutions

Let us start a trajectory $t \mapsto (r(t), \varphi(t))$ from r_0, φ_0 at $t_0 = 0$. For the first five notable points of the trajectory we introduce the notations $q := \omega_h/\omega_e$,

$$\begin{aligned}
&r_0 := r(0), && \varphi_0 :\equiv \varphi(0) \pmod{2\pi},\ -2\pi < \varphi_0 \le 0;\\
&r_1 := r(T_h - 0), && \varphi_1 := \varphi(T_h - 0);\\
&r_2 := r(T_h) = f(\varphi_1; q)\, r_1, && \varphi_2 := \varphi(T_h) = \phi(\varphi_1; q);\\
&r_3 := r(T - 0)(= r_2), && \varphi_3 := \varphi(T - 0);\\
&r_4 := r(T) = f(\varphi_3; 1/q) r_3, && \varphi_4 := \varphi(T) = \phi(\varphi_3; 1/q).
\end{aligned} \tag{3.1}$$

If $q > 1$ then the first jump is a dilation and the next one is a contraction, and so on, however, in the case $q < 1$ the first impulsive step is a contraction and the next one is a dilation and so on. If $q = 1$ then $\omega_h = \omega_e$ and so $A_e = A_h + 2g$. In this case the phase point does not make jump: from a hyperbola passes to a circle around the origin, see Figure 3.1.

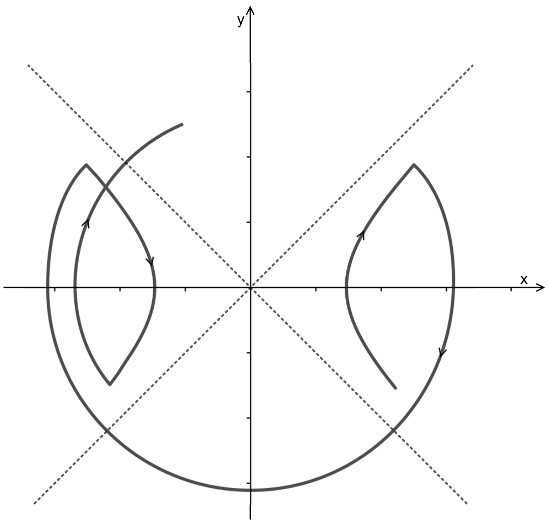

Figure 3.1: The trajectory if $q = 1$.

Since systems of (2.6) and (2.7) are linear, it is obvious that if $t \mapsto (x(t), y(t))$ is a solution of a system then $t \mapsto (-x(t), -y(t))$ is also a solution. So, it is sufficient to consider the half plane of the right-hand side, namely, when $-\pi/2 \leqq \varphi_0 < \pi/2$.

Definition 3.1. A solution of the equation (2.5) is called T-periodic if the corresponding trajectory $t \mapsto (r(t), \varphi(t))$ satisfies that

$$r_4 = r_0, \quad \varphi_4 \equiv \varphi_0 \pmod{2\pi}.$$

Definition 3.2. A solution of the equation (2.5) is called $2T$-periodic but not T-periodic if the corresponding trajectory $t \mapsto (r(t), \varphi(t))$ satisfies that

$$r_4 = r_0, \qquad \varphi_4 \equiv \varphi_0 - \pi \pmod{2\pi}.$$

Using (3.1) and the Definition 3.1, it can be seen that if a solution is T-periodic, then $r_3 = f(\varphi_4; q) r_4 = f(\varphi_0; q) r_0$.

From equations (2.9) we obtain that every hyperbola satisfies some differential equation

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = r \tan 2\varphi \qquad \left(-\frac{\pi}{4} + m\frac{\pi}{2} < \varphi < \frac{\pi}{4} + m\frac{\pi}{2},\ m \in \{-1, 0, 1\}\right). \tag{3.2}$$

(3.2) is separable, so integrating it we have

$$\frac{r}{r_0} = \sqrt{\frac{|\cos 2\varphi_0|}{|\cos 2\varphi|}} \quad \Big(-\frac{\pi}{4} + m\frac{\pi}{2} < \varphi_0,\ \varphi < \frac{\pi}{4} + m\frac{\pi}{2},\ m \in \{-1,0,1\}\Big). \tag{3.3}$$

If the solution is T-periodic and $r_3 = r_2$, from (3.1), we have

$$\frac{r_1}{r_0} = \sqrt{\frac{|\cos 2\varphi_0|}{|\cos 2\varphi_1|}} = \frac{f(\varphi_0;q)}{f(\varphi_1;q)} = \sqrt{\frac{1+(q^2-1)\sin^2\varphi_0}{1+(q^2-1)\sin^2\varphi_1}}. \tag{3.4}$$

By the use of the function (see Figure 3.2)

$$h(\varphi) := \frac{|\cos 2\varphi|}{1+(q^2-1)\sin^2\varphi} \tag{3.5}$$

(3.4) can be expressed by $h(\varphi_0) = h(\varphi_1)$.

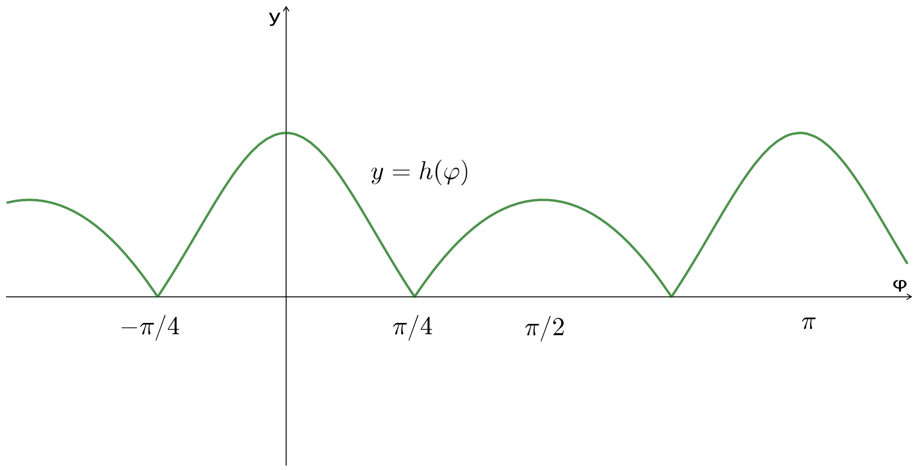

Figure 3.2: The graph of function h; $q > 1$.

An elementary calculation shows that for every q function h is strictly increasing on the closed interval $[\pi/4 + m\pi/2, \pi/2 + m\pi/2]$, and strictly decreasing on $[m\pi/2, \pi/4 + m\pi/2]$ $(m \in \mathbb{Z})$.

If $\varphi_0 \in [0, \pi/4]$ or $\varphi_0 \in [\pi/4, \pi/2]$, then φ_1 must be found in the same interval. Since h is strictly monotonous in these intervals, $h(\varphi_0) = h(\varphi_1)$ cannot be satisfied. So, a T-periodic solution cannot start from such a φ_0.

Function h is even and periodic of period π, so if $\varphi_0 \in (-\pi/4, 0)$ or $\varphi_0 \in (-\pi/2, -\pi/4)$ then there exists exactly one $\varphi_1 \in (0, \pi/4)$ or $\varphi_1 \in (-3\pi/4, -\pi/2)$ for which $h(\varphi_0) = h(\varphi_1)$.

Since equation (2.5) is linear, so a solution $t \mapsto (r(t), \varphi(t))$ is $2T$-periodic but not T-periodic if and only if $r(T) = r(0)$, $\varphi(T) \equiv \varphi(0) - \pi \pmod{2\pi}$. Therefore, the phase point in a $2T$-periodic solutions case can start from same state as in the T-periodic cases.

After these comments we give two lemmas without the proofs about the behaviour of the trajectories in the cases of T- and $2T$-periodic solutions. The exact proofs can be found in [7].

Lemma 3.3. *Let $\varphi_0 \in [-\pi/2, \pi/2)$. Then $t \mapsto (r(t), \varphi(t))$ is a trajectory of a T-periodic solution of* (2.5) *if and only if either* (*a*) $-\pi/4 < \varphi_0 < 0$ *and there is a non-negative integer k such that*

$$\begin{cases} \varphi_1 = -\varphi_0 \\ \varphi_3 = -\varphi_2 - 2k\pi, \end{cases} \tag{3.6}$$

or

(*b*) $-\pi/2 < \varphi_0 < -\pi/4$ *and there is a non-negative integer k such that*

$$\begin{cases} \varphi_1 = -\varphi_0 - \pi \\ \varphi_3 = -\varphi_2 - \pi - 2(k+1)\pi. \end{cases} \tag{3.7}$$

Lemma 3.4. *Let* $\varphi_0 \in [-\pi/2, \pi/2)$*. Then* $t \mapsto (r(t), \varphi(t))$ *is the trajectory of such a* $2T$*-periodic solution of* (2.5) *which is not* T*-periodic if and only if either*

(*a*) $-\pi/4 < \varphi_0 < 0$ *and there is a non-negative integer k such that*

$$\begin{cases} \varphi_1 = -\varphi_0 \\ \varphi_3 = -\varphi_2 - \pi - 2k\pi, \end{cases} \tag{3.8}$$

or

(*b*) $-\pi/2 < \varphi_0 < -\pi/4$ *and there is a non-negative integer k such that*

$$\begin{cases} \varphi_1 = -\varphi_0 - \pi \\ \varphi_3 = -\varphi_2 - 2\pi - 2k\pi. \end{cases} \tag{3.9}$$

The Figure 3.3 and 3.4 shows an example for the trajectories on the phase plane which trajectories correspond to the T- and $2T$-periodic solutions, respectively.

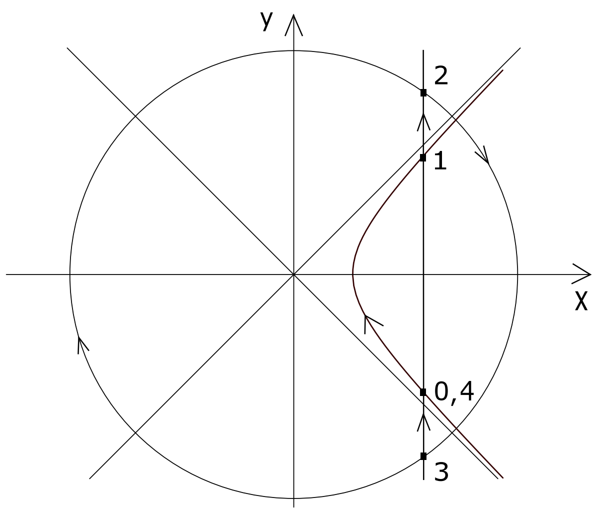

Figure 3.3: Trajectories corresponding to a T-periodic solution.

Now, we can formulate two theorems which yield necessary and sufficient conditions for the existence of T-periodic and $2T$-periodic solutions of (2.5).

Theorem 3.5. *Suppose that $q \neq 1$. Then there is a solution of* (2.5) *of period T if and only if there are positive constants A_h, A_e and T_h, T_e in* (2.2) *and a non-negative integer k such that either*

$$2\arctan\left(q\,\frac{e^{\omega_h T_h}-1}{e^{\omega_h T_h}+1}\right)+2k\pi=\omega_e T_e, \tag{3.10}$$

or

$$2\arctan\left(q\,\frac{e^{\omega_h T_h}+1}{e^{\omega_h T_h}-1}\right)+(2k+1)\pi=\omega_e T_e. \tag{3.11}$$

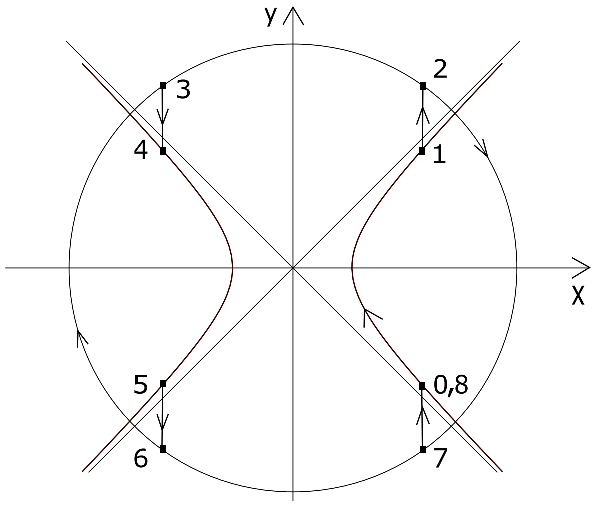

Figure 3.4: Trajectories corresponding to a $2T$-periodic solution.

Theorem 3.6. *Suppose that $q \neq 1$. Then there is a $2T$-periodic solution of* (2.5) *which is not T-periodic if and only if there are positive constants A_h, A_e and T_h, T_e in* (2.2) *and a non-negative integer k such that either*

$$2\arctan\left(q\,\frac{e^{\omega_h T_h}-1}{e^{\omega_h T_h}+1}\right)+(2k+1)\pi=\omega_e T_e, \tag{3.12}$$

or

$$2\arctan\left(q\,\frac{e^{\omega_h T_h}+1}{e^{\omega_h T_h}-1}\right)+2k\pi=\omega_e T_e. \tag{3.13}$$

Remark 3.7.

1. If $q = 1$, the corresponding formulae can be obtained from (3.10)–(3.13) by an obvious modification: $\omega_h = \omega_e = \omega$.

2. We prove only Theorem 3.5. The proof of Theorem 3.6 can be given by similar calculations.

Proof. Necessity. We suppose that θ is a T-periodic solution of equation (2.5), furthermore, case (a) of Lemma 3.3 is satisfied. Using notations (3.1) and the second equation of (2.15) we obtain

$$\varphi_3-\varphi_2=-\omega_e T_e. \tag{3.14}$$

We eliminate φ_2 and φ_3 in (3.14) in terms of φ_0. Since $\varphi_2 = \phi(\varphi_1; q) = \phi(-\varphi_0; q)$, $\varphi_3 = \phi(\varphi_4; q)$; furthermore, $\varphi_3 = -\varphi_2 - 2k\pi$ and so by the periodicity $\varphi_3 = \phi(\varphi_0; q) - 2k\pi$ we can write

$\varphi_3 - \varphi_2 = \phi(\varphi_0; q) - 2k\pi - \phi(-\varphi_0; q) = -\omega_e T_e$. Using the parity of ϕ, (3.14) can be rewritten as

$$2\phi(\varphi_0; q) - 2k\pi = -\omega_e T_e. \tag{3.15}$$

Using (2.12) and (2.13) we obtain

$$\varphi_0 = \arctan \frac{e^{-\omega_h T_h} - 1}{e^{-\omega_h T_h} + 1} = \arctan \frac{1 - e^{\omega_h T_h}}{1 + e^{\omega_h T_h}}. \tag{3.16}$$

Substituting (3.16) into (3.15) we get

$$2 \arctan \left(q \frac{1 - e^{\omega_h T_h}}{1 + e^{\omega_h T_h}} \right) - 2k\pi = -\omega_e T_e. \tag{3.17}$$

Multiplying (3.17) by (-1) we obtain (3.10).

Now, let us suppose that case (b) of Lemma 3.3 is satisfied. Similar calculations lead to the equations:

$$2\phi(\varphi_0; q) + \pi - 2(k+1)\pi = -\omega_e T_e, \tag{3.18}$$

and

$$\varphi_0 = \arctan \frac{e^{-\omega_h T_h} + 1}{e^{-\omega_h T_h} - 1} = \arctan \frac{1 + e^{\omega_h T_h}}{1 - e^{\omega_h T_h}} \tag{3.19}$$

which yield (3.11).

Sufficiency. Suppose that (3.10) is satisfied. If

$$\varphi_0 := \arctan \frac{1 - e^{\omega_h T_h}}{1 + e^{\omega_h T_h}} \tag{3.20}$$

then the solution of (2.5) is T-periodic. Indeed. From (3.20) we get $e^{\omega_h T_h} = \frac{1-\tan\varphi_0}{1+\tan\varphi_0}$; furthermore, using also (2.13) we can write

$$\tan\left(\frac{\pi}{4} - \varphi_1\right) = \frac{1 + \tan\varphi_0}{1 - \tan\varphi_0} = \tan\left(\frac{\pi}{4} + \varphi_0\right).$$

Since $\varphi_0 \in (-\pi/4, 0)$ we obtain $\varphi_1 = -\varphi_0$. We show that the second equality in (3.6) is also satisfied. In fact, from (2.9) and (3.10) we obtain

$$2 \arctan \left(q \frac{e^{\omega_h T_h} - 1}{e^{\omega_h T_h} + 1} \right) + 2k\pi = -(\varphi_3 - \varphi_2). \tag{3.21}$$

From (3.1) and (3.20) we can write

$$\begin{aligned} \varphi_2 = \phi(\varphi_1; q) = \phi(-\varphi_0; q) &= -\arctan(q \tan\varphi_0) \\ &= -\arctan\left(q \frac{1 - e^{\omega_h T_h}}{1 + e^{\omega_h T_h}}\right) = \arctan\left(q \frac{e^{\omega_h T_h} - 1}{e^{\omega_h T_h} + 1}\right). \end{aligned}$$

Therefore, (3.21) can be rewritten into the form:

$$2\varphi_2 + 2k\pi = -\varphi_3 + \varphi_2,$$

i.e.,

$$\varphi_3 = -\varphi_2 - 2k\pi.$$

So we have proved that (3.6) is satisfied. Lemma 3.3 guaranties that the solution is T-periodic.

If (3.11) is satisfied, then we define

$$\varphi_0 := -\arctan \frac{e^{\omega_h T_h}+1}{e^{\omega_h T_h}-1} \in \Big(-\frac{\pi}{2}, -\frac{\pi}{4}\Big).$$

Repeating step by step the previous reasoning we get that (3.7) is satisfied, and the solution with this φ_0 is T-periodic. □

Parameters $\omega_h, \omega_e, T_h, T_e$, namely A_h, A_e, T_h, T_e in (3.10)–(3.13) are not independent, see (2.4). Let introduce the new, independent parameters:

$$d := \sqrt{\frac{A_e}{A_h}}, \qquad \varepsilon := \sqrt{\frac{D_e}{l}}, \qquad \mu := \sqrt{\frac{g}{A_e}}. \tag{3.22}$$

Note that $d = 1$ characterizes the symmetrically excited pendulum case [7]. Using (3.22) the equations of Theorem 3.5 and 3.6 can be rewritten into the next form.

Corollary 3.8. *There is a solution of* (2.5) *of period T if and only if there are positive constants d, ε, μ and a non-negative integer k such that either*

$$2\arctan\left(\frac{1}{d}\sqrt{\frac{1+d^2\mu^2}{1-\mu^2}}\,\frac{e^{2\sqrt{2}\varepsilon d\sqrt{1+d^2\mu^2}}-1}{e^{2\sqrt{2}\varepsilon d\sqrt{1+d^2\mu^2}}+1}\right)+2k\pi = 2\sqrt{2}\varepsilon\sqrt{1-\mu^2}, \tag{3.23}$$

or

$$2\arctan\left(\frac{1}{d}\sqrt{\frac{1+d^2\mu^2}{1-\mu^2}}\,\frac{e^{2\sqrt{2}\varepsilon d\sqrt{1+d^2\mu^2}}+1}{e^{2\sqrt{2}\varepsilon d\sqrt{1+d^2\mu^2}}-1}\right)+(2k+1)\pi = 2\sqrt{2}\varepsilon\sqrt{1-\mu^2}. \tag{3.24}$$

Corollary 3.9. *There is a $2T$-periodic solution of* (2.5) *which is not T-periodic if and only if there are positive constants d, ε, μ and a non-negative integer k such that either*

$$2\arctan\left(\frac{1}{d}\sqrt{\frac{1+d^2\mu^2}{1-\mu^2}}\,\frac{e^{2\sqrt{2}\varepsilon d\sqrt{1+d^2\mu^2}}-1}{e^{2\sqrt{2}\varepsilon d\sqrt{1+d^2\mu^2}}+1}\right)+(2k+1)\pi = 2\sqrt{2}\varepsilon\sqrt{1-\mu^2}, \tag{3.25}$$

or

$$2\arctan\left(\frac{1}{d}\sqrt{\frac{1+d^2\mu^2}{1-\mu^2}}\,\frac{e^{2\sqrt{2}\varepsilon d\sqrt{1+d^2\mu^2}}+1}{e^{2\sqrt{2}\varepsilon d\sqrt{1+d^2\mu^2}}-1}\right)+2k\pi = 2\sqrt{2}\varepsilon\sqrt{1-\mu^2}. \tag{3.26}$$

4 An oscillation theorem and its consequences

Equation (2.5) is a special type of Hill's equation. One of main results about Hill's equation is the Oscillation Theorem [18]. The previous corollaries lead us to an oscillation theorem for (2.5) which, using the parameters ε, μ and d we can formulate as follows.

Theorem 4.1. *For every $d > 0$ and for every $0 < \mu < 1$ there exist sequences of functions $\{\varepsilon_n(\mu, d)\}_{n=1}^{\infty}$, $\{\tilde{\varepsilon}_n(\mu, d)\}_{n=1}^{\infty}$ such that* (2.5) *with $\varepsilon = \varepsilon_n$ (respectively, $\varepsilon = \tilde{\varepsilon}_n$) has T-periodic (respectively, $2T$-periodic) solutions. In addition,*

$$0 < \varepsilon_1 < \tilde{\varepsilon}_1 < \tilde{\varepsilon}_2 < \varepsilon_2 < \cdots < \tilde{\varepsilon}_n < \tilde{\varepsilon}_{n+1} < \varepsilon_{n+1} < \varepsilon_{n+2} < \cdots$$

$$\lim_{n\to\infty} \varepsilon_n = \infty, \qquad \lim_{n\to\infty} \tilde{\varepsilon}_n = \infty.$$

Proof. Let us introduce the functions

$$
\begin{aligned}
F_k(\varepsilon,\mu,d) &:= 2\arctan\left(\frac{1}{d}\sqrt{\frac{1+d^2\mu^2}{1-\mu^2}}\,\frac{e^{2\sqrt{2}\varepsilon d\sqrt{1+d^2\mu^2}}-1}{e^{2\sqrt{2}\varepsilon d\sqrt{1+d^2\mu^2}}+1}\right)+2k\pi,\\
\tilde{G}_k(\varepsilon,\mu,d) &:= 2\arctan\left(\frac{1}{d}\sqrt{\frac{1+d^2\mu^2}{1-\mu^2}}\,\frac{e^{2\sqrt{2}\varepsilon d\sqrt{1+d^2\mu^2}}+1}{e^{2\sqrt{2}\varepsilon d\sqrt{1+d^2\mu^2}}-1}\right)+2k\pi,\\
\tilde{F}_k(\varepsilon,\mu,d) &:= 2\arctan\left(\frac{1}{d}\sqrt{\frac{1+d^2\mu^2}{1-\mu^2}}\,\frac{e^{2\sqrt{2}\varepsilon d\sqrt{1+d^2\mu^2}}-1}{e^{2\sqrt{2}\varepsilon d\sqrt{1+d^2\mu^2}}+1}\right)+\pi+2k\pi,\\
G_k(\varepsilon,\mu,d) &:= 2\arctan\left(\frac{1}{d}\sqrt{\frac{1+d^2\mu^2}{1-\mu^2}}\,\frac{e^{2\sqrt{2}\varepsilon d\sqrt{1+d^2\mu^2}}+1}{e^{2\sqrt{2}\varepsilon d\sqrt{1+d^2\mu^2}}-1}\right)+\pi+2k\pi,\\
H(\varepsilon,\mu) &:= 2\sqrt{2}\varepsilon\sqrt{1-\mu^2}, \qquad k\in\mathbb{N}.
\end{aligned}
\tag{4.1}
$$

Let us consider $d > 0$ as parameter in formulae (4.1), then, we can visualize the graphs of these functions, see Figure 4.2. Since

$$\partial F_k/\partial\varepsilon > 0, \quad \partial\tilde{F}_k/\partial\varepsilon > 0, \quad \partial G_k/\partial\varepsilon < 0, \quad \partial\tilde{G}_k/\partial\varepsilon < 0 \qquad (k\in\mathbb{N})$$

and

$$\partial^2 F_k/\partial\varepsilon^2 < 0, \quad \partial^2\tilde{F}_k/\partial\varepsilon^2 < 0, \quad \partial^2 G_k/\partial\varepsilon^2 > 0, \quad \partial^2\tilde{G}_k/\partial\varepsilon^2 > 0 \qquad (k\in\mathbb{N}),$$

so the intersection curve of surface $z = F_k(\varepsilon,\mu)$ ($z = \tilde{F}_k(\varepsilon,\mu)$) and the plane $\mu = $ const. is increasing and concave, and intersection curve of surface $z = G_k(\varepsilon,\mu)$ ($z = \tilde{G}_k(\varepsilon,\mu)$) is decreasing and convex; furthermore, intersection of $z = H(\varepsilon,\mu)$ and $\mu = $ const. is a straight line, see Figure 4.1. From these it is easy to see that for every fixed k and every fixed μ the equations $F_k = H, \tilde{G}_k = H, \tilde{F}_k = H, G_k = H$ each have exactly one solution:

$$\varepsilon_{k+1} < \tilde{\varepsilon}_{k+1} < \tilde{\varepsilon}_{k+2} < \varepsilon_{k+2} \qquad (k\in\mathbb{N}),$$

provided that positive parameter d is fixed. According to the Implicit Function Theorem we

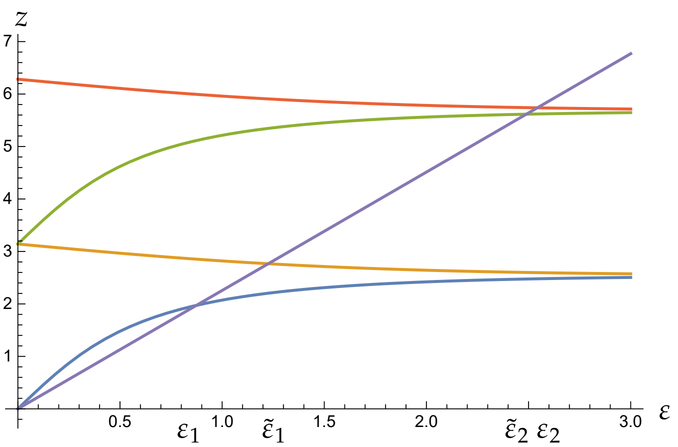

Figure 4.1: Intersection of surfaces; $k = 0$.

can write: $\varepsilon_{k+1} = \varepsilon_{k+1}(\mu;d)$, $\tilde{\varepsilon}_{k+1} = \tilde{\varepsilon}_{k+1}(\mu;d)$, $\tilde{\varepsilon}_{k+2} = \tilde{\varepsilon}_{k+2}(\mu;d), \varepsilon_{k+2} = \varepsilon_{k+2}(\mu;d)$. Moreover, the Implicit Function Theorem shows also that for every $k \in \mathbb{N}$ and every $d > 0$, $\partial\varepsilon/\partial\mu > 0$, namely functions $\varepsilon_k(\mu; d)$ and $\tilde{\varepsilon}_k(\mu; d)$ are increasing. $\square$

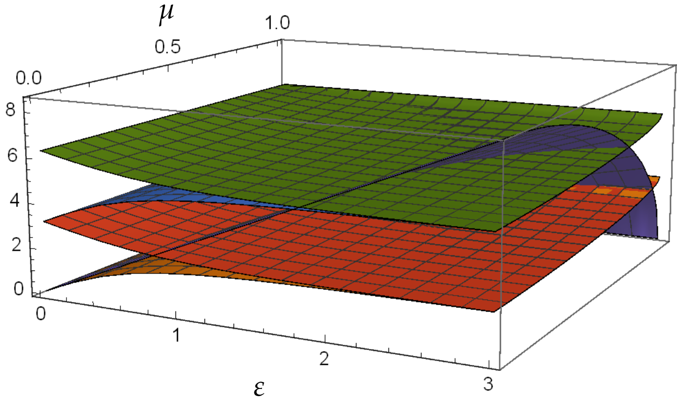

Figure 4.2: Conditions (3.23)–(3.26), when $k = 0, d = 0,7$.

Note that Theorem 4.1 for each d and for each μ corresponds to an oscillation theorem of a corresponding Hill's equation.

For the linear equation (2.5) we use the stability notations accepted in [1]. Equation (2.5) is strongly stable if it is stable in the sense of Lyapunov together with all of its sufficiently small perturbation, i.e., there exists an $\delta > 0$ such that $\ddot{\theta} - ((g + \hat{a}(t))/l)\,\theta = 0$ is stable if $(\hat{A}_h - A_h)^2 + (\hat{A}_e - A_e)^2 + (\hat{T}_h - T_h)^2 < \delta^2$, where the step function $\hat{a}$ belongs to $\hat{A}_h, \hat{A}_e, \hat{T}_h$ in the sense of the definition (2.2), provided that $\hat{T}_e = T - \hat{T}_h$, and the first equality in (2.4) is satisfied for the parameters with $\hat{}$. The set in the $\varepsilon - \mu - d$-space consisting of all the points corresponding to the strongly stable equations is called the stability region of (2.5).

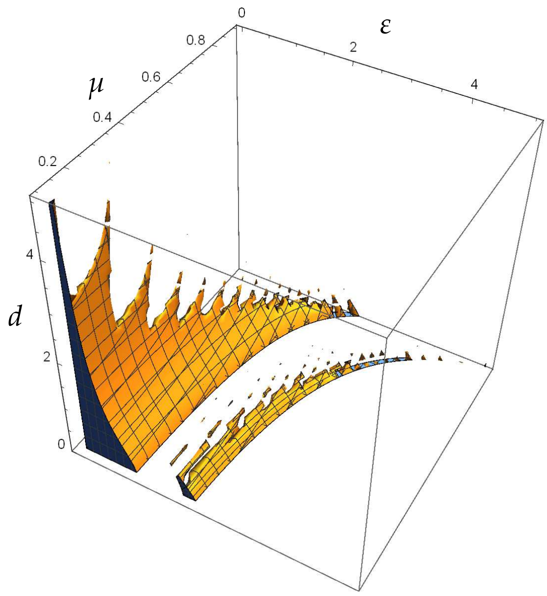

Figure 4.3: A part of stability region, $k = 0$.

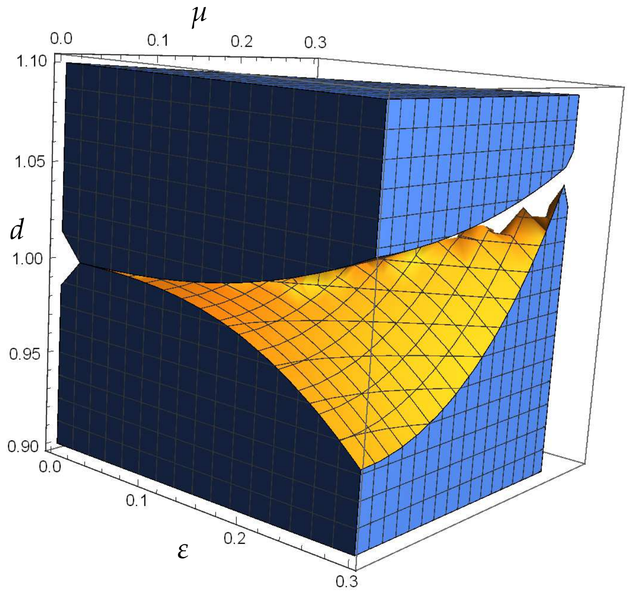

Figure 4.4: The approximating (cave-like part inside the solid) and the exact stability region (the whole solid).

Floquet Theory [1] says the stability region and the instability region are separated by surfaces whose points correspond to the equations of form (2.5) having T- or $2T$-periodic solutions. So, drawing the solution sets of equations (3.23)–(3.26) in the $\varepsilon - \mu - d$-space, we get the boundary surfaces of the stability region, see Figure 4.3. In [6], using a different method, with László Hatvani we gave an approximation for the stability region. Figure 4.4 shows the earlier approximating and the exact stability region.

As we can see, now we have much more chance to stabilize the upper equilibrium of the pendulum than in the earlier approximated case, however, as $\varepsilon \to \infty$ our chance is less and less, because the stability region becomes thin.

4.1 Numerical simulations

Using the stability map we can prepare some computer simulations which demonstrate our previous results. The computer solved the system $\dot{x} = y$, $\dot{y} = \frac{g+a(t)}{l}x$, where $g = 9.81$, $l = 2$; so now we use the "physical" phase plane. Due to Figure 4.3, we can choose the following parameter values: $\varepsilon = 0.2$, $\mu = 0.2$, $d = 1.05$ and thus we obtain $A_e = 245.25$, $A_h = 222.448$, $T_h = 0.056$, $T = 0.1$. From (3.16) we get for the initial values: $x_0 = 1.918$, $y_0 = -0.562$. The calculations were carried out in different long time intervals, and so we can present the next figures.

Figure 4.5 illustrates the first 10 periods, we can not deduce any conclusion from behaviour of this trajectory.

When the simulation runs on a longer interval than $[0, 1]$, see Figure 4.6, we can see that the phase point goes to the half-plane $x < 0$, namely, the origin may be stable.

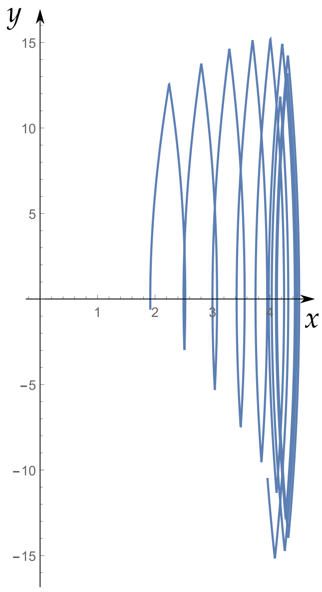

Figure 4.5: Phase curve, when $t \in [0, 1]$.

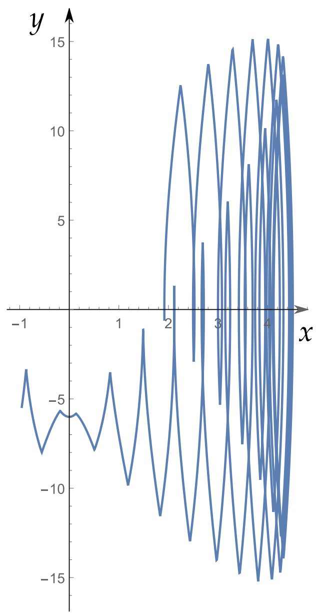

Figure 4.6: Phase curve, when $t \in [0, 2]$.

Following the movement of the phase point during a relatively long time: $t \in [0, 10]$ (respectively, $t \in [0, 20]$) we can see that the solution of the equation of motion is bounded, see Figure 4.7 (respectively, Figure 4.8).

As we can see, the simulations suggest that the upper equilibrium is stable.

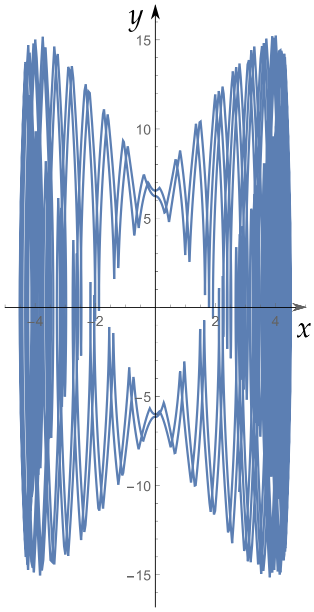

Figure 4.7: Phase curve, when $t \in [0, 10]$.

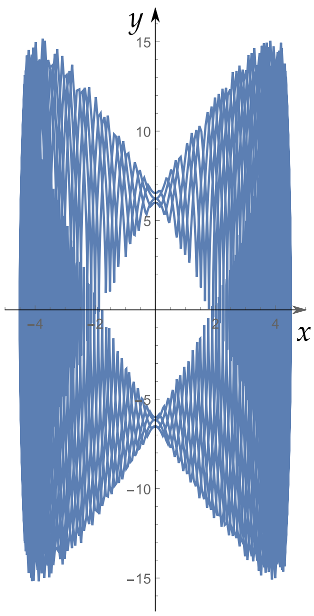

Figure 4.8: Phase curve, when $t \in [0, 20]$.

References

[1] V. I. Arnold, *Ordinary differential equations*, Springer-Verlag, Berlin, 2006. MR2242407

[2] J. A. Blackburn, H. J. T. Smith, N. Gronbeck-Jensen, Stability and Hopf bifurcation in an inverted pendulum, *Amer. J. Physics* **60**(1992), 903–908. `https://doi.org/10.1119/1.17011`; MR1181951

[3] E. I. Butikov, Parametric resonance in a linear oscillator at square-wave modulation, *Eur. J. Phys.* **26**(2005), 157–174. https://doi.org/10.1088/0143-0807/26/1/016

[4] E. I. Butikov, An improved criterion for Kapitza's pendulum, *J. Phys. A: Math. Theor.* **44**(2011), 1–16. https://doi.org/10.1088/1751-8113/44/29/295202; MR2812318

[5] C. Chicone, *Ordinary differential equations with applications*, Springer-Verlag, New York, 1999. MR1707333

[6] L. Csizmadia, L. Hatvani, An extension of the Levi–Weckesser method to the stabilization of the inverted pendulum under gravity, *Meccanica* **49**(2014), 1091–1100. https://doi.org/10.1007/s11012-013-9855-z; MR3192675

[7] L. Csizmadia, L. Hatvani, On the existence of periodic motions of the excited inverted pendulum by elementary methods, *Acta Math. Hungar.* (2018). https://doi.org/10.1007/s10474-018-0835-6

[8] A. M. Formal'skii, On the stabilization of an inverted pendulum with a fixed or moving suspension point, *Dokl. Akad. Nauk* **406**(2006), No. 2, 175–179. MR2258513

[9] J. Hale, *Ordinary differential equations*, Wiley-Interscience, New York, 1969. MR0419901

[10] L. Hatvani, An elementary method for the study of Meissner's equation and its application to proving the Oscillation Theorem, *Acta Sci. Math.* **79**(2013), No. 1–2, 87–105. MR3100431

[11] L. Hatvani, On the critical values of parametric resonance in Meissner's equation by the method of difference equations, *Electron. J. Qual. Theory Differ. Equ.* **2009**, Special Edition I, No. 13, 1–10. https://doi.org/10.14232/ejqtde.2009.4.13; MR2558838

[12] O. Haupt, Über eine Methode zum Beweise von Oszillationstheoremen (in German), *Math. Ann.* **76**(1914), 67–104. https://doi.org/10.1007/BF01458673; MR1511814

[13] G. W. Hill, On the part of the motion of the lunar perigee which is a function of the mean motions of the Sun and Moon, *Acta Mathematica* **8**(1886), 1–36. MR1554690

[14] P. L. Kapitsa, Dynamical stability of a pendulum when its point of suspension vibrates, in: *Collected papers by P. L. Kapitsa*, Vol. II, Pergamon Press, Oxford, 1965. https://doi.org/10.1016/C2013-0-01850-6

[15] M. Levi, Stability of the inverted pendulum – a topological explanation, *SIAM Rev.* **30**(1988), 639–644. https://doi.org/10.1137/1030140; MR0967966

[16] M. Levi, Geometry of Kapitsa's potentials, *Nonlinearity* **11**(1998), No. 5, 1365–1368. https://doi.org/10.1088/0951-7715/11/5/011; MR1644409

[17] M. Levi, W. Weckesser, Stabilization of the inverted, linearized pendulum by high frequency vibrations, *SIAM Rev.* **37**(1995), 219–223. https://doi.org/10.1137/1037044; MR1343212

[18] W. Magnus, S. Winkler, *Hill's equation*, Dover Publications, Inc., New York, 1979. MR0559928

[19] D. R. Merkin, *Introduction to the theory of stability*, Springer-Verlag, New York, 1997. MR1418401

[20] A. A. Seyranian, A. P. Seyranian, The stability of an inverted pendulum with a vibrating suspension point, *J. Appl. Math. Mech.* **70**(2006), 754–761. https://doi.org/10.1016/j.jappmathmech.2006.11.009; MR2319534

[21] A. Stephenson, On a new type of dynamical stability, *Manchester Memoirs* **52**(1908), 1–10. Zbl 39.0768.02