




Positive kernels, fixed points, and integral equations

Dedicated to Professor László Hatvani on the occasion of his 75th birthday

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Abstract. There is substantial literature going back to 1965 showing boundedness properties of solutions of the integro-differential equation

$$x'(t) = - \int_0^t A(t-s)h(s, x(s))ds$$

when A is a positive kernel and h is a continuous function using

$$\int_0^T h(t, x(t)) \int_0^t A(t-s)h(s, x(s))dsdt \geq 0.$$

In that study there emerges the pair:

Integro-differential equation and Supremum norm.

In this paper we study qualitative properties of solutions of integral equations using the same inequality and obtain results on L^p solutions. That is, there occurs the pair:

Integral equations and L^p norm.


The paper also offers many examples showing how to use the L^p idea effectively.

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1 Introduction

In this paper we extend an idea for studying integro-differential equations ([14] and [17]) to the study of integral equations. It is a method, as opposed to a theory, and our contribution

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here is to offer a number of examples showing ways in which the method can be used. In the process at least three new ideas emerge which are very interesting.

The idea that the solution of an integral equation may follow the forcing function is brought to light in a simple way.

The idea of using the average of a solution in place of a solution as seen in much work in probability theory emerges, again, in a very simple way and the average comes from integration of the forcing function.

There is a relation between $F(t)$, x^{2n+1} , and the limit sets of the solution of

$$x(t) = F(t) - \int_0^t A(t-s)x^{2n+1}(s)ds$$

with $F \in L^p[0, \infty)$ depending on the magnitude of n .

The paper is brief but we work enough examples that the interested investigator can proceed in several different ways in the solution of many problems. We view the technique as akin to Liapunov's direct method in which the basic idea is elementary, but it has led to a hundred years of fruitful research in solving many different kinds of problems.

Several general existence theorems are found in the open access journal [4] or in the book [15, p. 30] for Riemann–Liouville fractional differential equations.

A function $A : (0, \infty) \rightarrow \mathfrak{R}$ is said to be a *positive kernel* if whenever $h : [0, \infty) \rightarrow \mathfrak{R}$ is continuous then for $T > 0$ we have

$$\int_0^T h(t) \int_0^t A(t-s)h(s)dsdt \geq 0. \quad (1.1)$$

First, we mention that there is a nice sufficient condition for this to hold found in [16, p. 217] requiring

$$(-1)^k A^{(k)}(t) \geq 0, \quad k = 0, 1, 2, \dots$$

and A not identically constant. This is particularly useful because it holds for many problems in heat transfer as well as all scalar fractional differential equations of both Riemann–Liouville and Caputo type. For our work here we will use the elementary sufficient condition for (1.1) to hold in the form of Theorem 1.1 below. However, the classical condition to ensure (1.1) can be found in [17, p. 2] or [13, p. 493] which is that the Laplace transform of A , say A^* , satisfies the real part relation

$$\operatorname{Re} s > 0 \implies \operatorname{Re} A^*(s) > 0.$$

This condition is simple to state, but verifying it can be both challenging and deceptive. Therefore we supply an alternate way of verifying (1.1) which is completely elementary. We put together two separate results of MacCamy and Wong [17, pp. 5, 17] who study equations such as

$$x'(t) = - \int_0^t A(t-s)g(x(s))ds, \quad x \neq 0 \implies xg(x) > 0. \quad (1.2)$$

Those two results on pp. 5, 17 allow us to replace the transform condition by the following.

Theorem 1.1. *Let $A(t)$ satisfy the following conditions:*

- (a₁) $A(t) \in C^1(0, \infty) \cap L^1(0, 1)$,
- (a₂) $A(t) \geq 0, A'(t) \leq 0$,
- (a₃) $A'(t)$ is nondecreasing,
- (a₄) $A(t)$ is not identically constant.

Then $A(t)$ defines a positive kernel.

That theorem will enable us to show L^p properties of solutions of integral equations provided that a solution exists. Thus, in the first approximate half of this paper we will be discussing properties of solutions if they exist. But the next result will enable us to find a resolvent for the kernel and that, in turn, will enable us to make a transformation which is fixed point friendly and will lead us to show that there is a solution. It goes beyond that, too, in allowing us to treat equations having a discontinuous forcing function which can make a solution discontinuous and that, in turn, will make the $h(t, x(t))$ discontinuous which disqualifies the conclusion of (1.1). The result is found in Miller [18, p. 209, pp. 212–213, p. 224] and certain improvements are found in Gripenberg [12, p. 381]. A study of Theorem 1.1 and the following Theorem 1.2, along with discussion of Gripenberg on p. 381, shows that the assumptions of the two results are very similar. This is interesting because the assumptions of Theorem 1.2 lie at the heart of so much of the basic linear theory of integral equations of convolution type. Thus, the assumptions of Theorem 1.1 are also very close to being basic to the theory.

Theorem 1.2. *Suppose that the kernel $A(t)$ satisfies*

- (A1) $A(t) \in C(0, \infty) \cap L^1(0, 1)$.
- (A2) $A(t)$ is positive and non-increasing for $t > 0$.
- (A3) For each $T > 0$ the function $A(t)/A(t+T)$ is non-increasing in t for $0 < t < \infty$.

Then there is a unique solution $R(t)$ of the resolvent equation

$$R(t) = A(t) - \int_0^t A(t-s)R(s)ds \quad (1.3)$$

and it is continuous on $(0, \infty)$. Moreover

$$0 < t < \infty \implies 0 < R(t) \leq A(t). \quad (1.4)$$

If $A \in L^1[0, \infty)$ and if $\alpha = \int_0^\infty A(s)ds$ then

$$\int_0^\infty R(s)ds = \frac{\alpha}{(1+\alpha)} < 1. \quad (1.5)$$

If $A \notin L^1[0, \infty)$, then

$$\int_0^\infty R(s)ds = 1. \quad (1.6)$$

Finally, if A is completely monotone, so is R .

The kernel $A(t) = t^{q-1}, 0 < q < 1$ occurs so commonly in applied mathematics that it needs special mention. We see it in virtually all heat transfer, all fractional differential equations of either Caputo or Riemann–Liouville type, and several complete packages found in [5]. The following result is very useful in proving existence of solutions.

Lemma 1.3. *If $A(t) = t^{q-1}, 0 < q < 1$, then A and R are completely monotone and satisfy Theorem 1.1.*

Proof. They are both clearly completely monotone and (a1)–(a3) hold. We need to see that R is not identically constant. By way of contradiction, suppose that R is constant so that the resolvent equation is

$$R = t^{q-1} - R \int_0^t s^{q-1} ds = t^{q-1} - R \frac{t^q}{q}.$$

As $t \rightarrow \infty$ we see that $t^{q-1} \rightarrow 0$, while $t^q \rightarrow \infty$, a contradiction. \square

The following transformation will finish the preliminaries. We suppose that A satisfies the conditions of Theorem 1.2, that $F : [0, \infty) \rightarrow \mathfrak{R}$ is continuous, that $g : [0, \infty) \times \mathfrak{R} \rightarrow \mathfrak{R}$ is continuous, and consider the equation

$$x(t) = F(t) - \int_0^t A(t-s)g(s, x(s))ds. \quad (1.7)$$

In the integrand add and subtract $Jx(s)$ for J an arbitrary positive constant to obtain

$$\begin{aligned} x(t) &= F(t) - \int_0^t A(t-s)g(s, x(s))ds \\ &= F(t) - \int_0^t A(t-s)[Jx(s) - Jx(s) + g(s, x(s))]ds \\ &= F(t) - \int_0^t JA(t-s)x(s)ds + \int_0^t JA(t-s) \left[x(s) - \frac{g(s, x(s))}{J} \right] ds. \end{aligned}$$

The linear part is

$$z(t) = F(t) - \int_0^t JA(t-s)z(s)ds \quad (1.8)$$

and the resolvent equation is

$$R(t) = JA(t) - \int_0^t JA(t-s)R(s)ds \quad (1.9)$$

so that by the linear variation-of-parameters formula we have

$$z(t) = F(t) - \int_0^t R(t-s)F(s)ds. \quad (1.10)$$

Then by the nonlinear variation of parameters formula [18, pp. 191-193] we have

$$x(t) = z(t) + \int_0^t R(t-s) \left[x(s) - \frac{g(s, x(s))}{J} \right] ds. \quad (1.11)$$

We now point out to the reader just how crucial this is for global existence results. First, perhaps the most common existence result for (1.7) involves either a contraction argument or a Schauder type fixed point theorem. Consider the very common case in which $A(t) = t^{-1/2}$,

found throughout heat transfer. It is easy to see that unless g decays very rapidly in s then a global contraction is impossible. In the same way Schauder's theorem calls for a compact mapping of a closed bounded set into itself. Notice that if such a set contains a non-zero constant function, then the natural mapping defined by (1.7) will map that function into an unbounded function. By contrast, the fact that either (1.5) or (1.6) holds, both of the aforementioned difficulties vanish with the natural mapping defined by (1.11). We later give several references showing such mappings with fixed points satisfying (1.7).

Transition of integro-differential equations to integral equations

When (1.1) is applied to (1.2) we obtain boundedness. In other words, there is the pair:

$$\text{Integro-differential equation and Supremum norm.} \quad (1.12)$$

When we turn to apply (1.1) to an integral equation

$$x(t) = F(t) - \int_0^t A(t-s)g(s, x(s))ds \quad (1.7)$$

there arises in a most natural way boundedness of the solution in terms of an integral of F which needs to be bounded in order to yield qualitative properties of solutions. In other words, there is the pair:

$$\text{Integral equation and } L^p\text{-norm.} \quad (1.13)$$

2 Examples with F bounded: averages

Example of (1.12). First, we feel it is essential to put all of this in context to see how simple and effective (1.1) can be for showing boundedness of solutions of integro-differential equations of the type MacCamy and Wong considered. And this will ultimately show how much more difficult it is to use the concept of positive kernel on integral equations. We begin with

$$x'(t) = - \int_0^t A(t-s)w(x(s))ds, \quad x \neq 0 \implies xw(x) > 0, \quad (2.1)$$

and A is a positive kernel so that (1.1) holds, while w is continuous. It will avoid some niggling if we also assume that

$$|x| \rightarrow \infty \implies \int_0^x w(s)ds \rightarrow \infty \quad (2.2)$$

although it is well-known in stability theory how to avoid this. These conditions will ensure that for each initial condition $x(0)$ there is a solution which, if it remains bounded, can be continued on $[0, \infty)$. Boundedness will now follow from the Liapunov function

$$V(x(t)) = \int_0^{x(t)} w(s)ds \quad (2.3)$$

which is positive definite and radially unbounded. For a given solution we will take the derivative of (2.3) along the solution of (2.1) using the chain rule to obtain

$$\frac{dV(x(t))}{dt} = w(x(t)) \frac{dx}{dt} = -w(x(t)) \int_0^t A(t-s)w(x(s))ds \quad (2.4)$$

so that for any $T > 0$ we have upon integration of (2.4)

$$V(x(T)) - V(x(0)) = - \int_0^T w(x(t)) \int_0^t A(t-s)w(x(s))dsdt \leq 0 \quad (2.5)$$

by (1.1). Hence, $V(x(T)) \leq V(x(0))$ or

$$\int_0^{x(T)} w(s)ds \leq \int_0^{x(0)} w(s)ds \leq \max \int_0^{\pm x(0)} w(s)ds =: L^* \quad (2.6)$$

so by (2.6) there is an $L > 0$ independent of T with

$$|x(T)| \leq L. \quad (2.7)$$

All of this can be extended to the case of $w(t, x)$, but we will not take that up here.

Obviously, the sign of w is critical and the condition $xw(x) > 0$ if $x \neq 0$ is often called the ‘‘spring condition’’. It occurs frequently in many kinds of problems in applied mathematics.

We are now going to look at two informative examples which are motivated by a fractional differential equation of Caputo type, although such equations are commonly found in applied mathematics. Both of these will feature a forcing function which is bounded and continuous. In the next section we offer examples in which the forcing function is in $L^1[0, \infty)$.

To place this in perspective, consider a fractional differential equation of Caputo type

$${}^c D^q x(t) = -x^{1/3}, \quad 0 < q < 1, \quad x(0) = x^0 \neq 0. \quad (2.8)$$

This is known to invert as the common integral equation

$$x(t) = x(0) - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} x^{1/3}(s)ds \quad (2.9)$$

where t^{q-1} with $0 < q < 1$ is a positive kernel. To be more general we will consider

$$x(t) = F(t) - \int_0^t A(t-s)x^{1/3}(s)ds \quad (2.10)$$

where F is bounded and continuous while A is a positive kernel so that (1.1) will hold.

Theorem 2.1. *A solution of (2.10) is in $L^{4/3}[0, \infty)$ if F is also in $L^{4/3}[0, \infty)$.*

Proof. Multiply both sides of (2.10) by $x^{1/3}(t)$, integrate from 0 to $T > 0$, and apply (1.1) to obtain

$$\int_0^T x^{4/3}(t)dt \leq \int_0^T |F(t)||x^{1/3}(t)|dt.$$

We are now ready to use the Hölder inequality to unite $x^{4/3}$ on the left with $x^{1/3}$ on the right. For that inequality we note that

$$\frac{1}{4} + \frac{1}{4/3} = 1.$$

Hence

$$\int_0^T x^{4/3}(t)dt \leq \left(\int_0^T F^{4/3}(t)dt \right)^{3/4} \left(\int_0^T x^{4/3}(t)dt \right)^{1/4}$$

or

$$\left(\int_0^T x^{4/3}(t) dt \right)^{3/4} \leq \left(\int_0^T F^{4/3}(t) dt \right)^{3/4}$$

or

$$\int_0^T x^{4/3}(t) dt \leq \int_0^T F^{4/3}(t) dt \quad (2.11)$$

as required. \square

Inequality (2.11) reminds us of the old idea that the solution of an integral equation will “follow” the forcing function under certain general conditions on $h(x)$ in the integral. Our next step is to study such a relationship. In our choice of function $h(x)$ recall that on a closed bounded interval every continuous function can be approximated arbitrarily well by polynomials so we begin with a general term found in polynomials and see if the relation in (2.11) still holds.

Theorem 2.2. *Let $A(t-s)$ be a positive kernel, $F : [0, \infty) \rightarrow \Re$ be continuous, and let n be a nonnegative integer in the equation*

$$x(t) = F(t) - \int_0^t A(t-s)x^{2n+1}(s)ds. \quad (2.12)$$

Any solution satisfies

$$\int_0^T x^{2n+2}(t)dt \leq \int_0^T |F(t)|^{2n+2}dt. \quad (2.13)$$

Proof. If we multiply both sides of (2.12) by $x^{2n+1}(t)$, integrate from 0 to $T > t$, and apply (1.1) we obtain

$$\int_0^T x^{2n+2}(t)dt \leq \int_0^T |F(t)||x^{2n+1}(t)|dt. \quad (2.14)$$

Using Hölder’s inequality as before we take

$$\frac{1}{\frac{2n+2}{2n+1}} + \frac{1}{2n+2} = 1$$

and obtain from (2.14)

$$\int_0^T x^{2n+2}(t)dt \leq \left(\int_0^T |F(t)|^{2n+2}dt \right)^{\frac{1}{2n+2}} \left(\int_0^T x^{2n+2}(t)dt \right)^{\frac{2n+1}{2n+2}}.$$

Divide by the last term and obtain

$$\left(\int_0^T x^{2n+2}(t)dt \right)^{\frac{1}{2n+2}} \leq \left(\int_0^T |F(t)|^{2n+2}dt \right)^{\frac{1}{2n+2}}$$

so that on any interval $[0, T]$ we have (2.13), as required. \square

Relation (2.13) would be natural if F were positive and if x were a positive solution. In fact, in such a case a positive solution x would follow the behavior of F : if F is bounded or in L^1 then so is x . If F tends to zero for t tending to infinity then so does x . But neither is assumed here.

It has long been known that the average value of a solution may be very useful. In the classic paper by Feller [11, p. 244] a discussion is given. To put matters into the present

context, suppose we arrive at (2.13) and set $2n + 2 = 2m$. Then for any solution existing on $[0, \infty)$ the average of x^{2m} on $[0, T]$ is given by

$$(1/T) \int_0^T x^{2m}(t) dt \leq (1/T) \int_0^T F^{2m}(t) dt. \quad (2.15)$$

In the common case in which F is periodic of period $L > 0$, such as when F is constant, then for any positive integer k we have the average of x^{2m} on $[0, kL]$ given by

$$\begin{aligned} \frac{1}{kL} \int_0^{kL} x^{2m}(t) dt &\leq \frac{1}{kL} \int_0^{kL} F^{2m}(t) dt \\ &= \frac{k}{kL} \int_0^L F^{2m}(t) dt \\ &= \frac{1}{L} \int_0^L F^{2m}(t) dt. \end{aligned}$$

While it is unfortunate that we get the average of x^{2m} instead of the average of x , with patience we can parlay our result into the desired one and it is something of a surprise to see the simple change required in (2.15).

Theorem 2.3. *If (2.13) holds for some solution x of (2.12) on $[0, \infty)$, then the average value of the solution is bounded above by*

$$\left(\frac{1}{T} \int_0^T F^{2m}(t) dt \right)^{\frac{1}{2m}} \quad (2.16)$$

where $2m = 2n + 2$.

Proof. We begin with

$$\frac{1}{2m} + \frac{1}{p} = 1$$

so that $p = 2m/(2m - 1)$. Then

$$\begin{aligned} \int_0^T |x(t)| dt &= \int_0^T 1 |x(t)| dt \\ &\leq \left(\int_0^T 1 dt \right)^{\frac{2m-1}{2m}} \left(\int_0^T x^{2m}(t) dt \right)^{\frac{1}{2m}} \\ &= T^{\frac{2m-1}{2m}} \left(\int_0^T x^{2m}(t) dt \right)^{\frac{1}{2m}} \end{aligned}$$

so

$$\begin{aligned} \int_0^T |x(t)| dt &\leq T^{\frac{2m-1}{2m}} \left(\int_0^T x^{2m}(t) dt \right)^{\frac{1}{2m}} \\ &\leq T^{\frac{2m-1}{2m}} \left(\int_0^T F^{2m}(t) dt \right)^{\frac{1}{2m}}. \end{aligned}$$

To see an upper bound on the mean value of x , divide the last line by T to obtain (2.16). Thus,

$$\frac{1}{T} \int_0^T |x(t)| dt \leq T^{\frac{-1}{2m}} \left(\int_0^T F^{2m}(t) dt \right)^{\frac{1}{2m}} = \left(\frac{1}{T} \int_0^T F^{2m}(t) dt \right)^{\frac{1}{2m}}. \quad \square$$

Existence and the transformation

Our previous results are quite lame since they always say “if there is a solution”. The same was said of the MacCamy–Wong work and here we can offer some help. The kernel $A(t-s) = (t-s)^{q-1}, 0 < q < 1$ occurs in so many real-world problems that it is a good place to start.

In a series of papers over the last ten years we have studied equations of the general form of (1.7) and have shown that this is not a suitable form for a variety of fixed point theorems aimed at showing existence, uniqueness, and boundedness of solutions. There is an annotated bibliography giving a sketch of the transformation and then references for the full papers applying that transformation and giving many examples of mappings by the integral equations which have a fixed point. That bibliography is available as a free download at

<https://www.researchgate.net/publication/3138385>

and is entitled “An annotated bibliography on fixed points for integral and fractional equations: a uniting transformation and the Brouwer–Schauder theorem”.

Here we encounter technical problems when we try to proceed from (1.7) and apply a fixed point theorem, but it turns out that there is a very happy ending. We have to obtain an existence theorem (see [2, Thm. 2.5, Thm. 2.7], [5, Thm. 3.1] for example) on a short interval $(0, T]$ and then translate (1.7) by setting

$$y(t) = x(t + T),$$

then finishing with an equation

$$y(t) = F(t) + \int_0^t R(t-s) \left[y(s) - \frac{g(s+T, y(s))}{J} \right] ds. \quad (2.17)$$

The function F turns out to be very nice. It is uniformly continuous, bounded, converges to zero, and is in $L^1[0, \infty)$ [2, Thm. 4.2].

Here is a list of five items showing the use of the transformation. We will apply all of these to subsequent examples so that the reader can see the details.

1. The details of the transformation can be found at the free online journal [5, p. 436] for a forcing function $t^{\alpha-1}$ or at [8, p. 319] for a constant forcing function. Under that transformation (1.7) would transform as (2.17).

2. In these cases both A and R will satisfy the conditions of Theorem 1.1 and so both are positive kernels and the properties of R enable the natural mapping defined by (2.17) to map bounded sets of bounded continuous functions on $[0, \infty)$ into bounded sets, a property failing when $A(t) = t^{-1/2}$, for example, as well as all fractional differential equations of either Riemann–Liouville or Caputo type. This is fundamental for fixed point theorems yielding global solutions by mapping closed convex bounded sets into themselves.

3. Something even more important happens. We show in [8, p. 322] and in [9] how bounded sets of continuous functions are mapped by the natural mapping defined by (2.17) when F is uniformly continuous into bounded and equicontinuous sets. Then using a weighted norm we show that using (2.17) self mapping balls have a fixed point which is in an equicontinuous set and, hence, is uniformly continuous. That solution of (2.17) also solves (1.7). Specializing (1.7) to (2.12) we conclude that the solution x is satisfying (2.13) and is bounded and uniformly continuous. We will now drive that solution to zero.

4. Now, assume that F in (2.13) is in $L^{2n+2}[0, \infty)$ so that the same holds for x . In Lemma 2.4 below we argue that x^{2n+2} is also uniformly continuous. Thus, x^{2n+2} is in L^1 , it is bounded (since x is) and uniformly continuous, so by Lemma 2.5 we have that x^{2n+2} tends to zero, and hence so does x .

5. We would point out that in the fixed point result in Item 3 above it was critical that the self mapping set be a ball. For that reason we could not add the condition that all functions in it tend to zero at infinity and get our conclusion in Item 4 above in a much simpler way.

Lemma 2.4. *Suppose that $\phi, \psi : [0, \infty) \rightarrow \mathfrak{R}$ are both uniformly continuous and there is an $M > 0$ such that $|\phi(t)| + |\psi(t)| \leq M$. Then $\Phi(t) := \phi(t)\psi(t)$ is uniformly continuous on $[0, \infty)$.*

Proof. For a given $\epsilon > 0$ there is a $\delta > 0$ such that $|t_1 - t_2| < \delta$ implies $|\phi(t_1) - \phi(t_2)| < \epsilon/(2M)$ and $|\psi(t_1) - \psi(t_2)| < \epsilon/(2M)$. Then

$$\begin{aligned} & |\phi(t_1)\psi(t_1) - \phi(t_2)\psi(t_2)| \\ &= |\phi(t_1)\psi(t_1) - \phi(t_2)\psi(t_1) + \phi(t_2)\psi(t_1) - \phi(t_2)\psi(t_2)| \\ &\leq |\psi(t_1)||\phi(t_1) - \phi(t_2)| + |\phi(t_2)||\psi(t_1) - \psi(t_2)| \\ &\leq M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \square \end{aligned}$$

Lemma 2.5. *If $\Phi : [0, \infty) \rightarrow \mathfrak{R}$ is bounded and uniformly continuous with $\int_0^\infty |\Phi(t)|dt < \infty$, then $\Phi(t) \rightarrow 0$ as $t \rightarrow \infty$.*

Proof. By way of contradiction, if the lemma is false then there exists $\epsilon > 0$ and $\{t_n\} \uparrow \infty$ such that $|\Phi(t_n)| \geq \epsilon$ and there is a $\delta > 0$ independent of n such that $|\Phi(t)| \geq \epsilon/2$ on $[t_n, t_n + \delta]$ for any n . An integral of Φ on that interval is at least $\delta\epsilon/2$. This gives a contradiction. \square

2.1 Bounded solutions

We are now going to give conditions to ensure that there is a bounded solution which, in turn, will allow us to show that not only is a solution in L^p but it also converges to zero. Brouwer's fixed point theorem states that if M is a closed ball in \mathfrak{R}^n and if $P : M \rightarrow M$ is continuous then P has a fixed point. Schauder's theorem requires considerably more in a Banach space, but if M is a ball in the Banach space then for a large class of problems there is a fixed point. We outline the main points as shown in [10] and collect the conditions as follows.

- a) Let $C : (0, \infty) \times (0, \infty)$ be continuous for $t > s > 0$.
- b) $0 < s < t_2 < t_1 \implies C(t_2, s) \geq C(t_1, s)$.
- c) Assume that for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$0 \leq t_2 \leq t_1, \quad t_1 - t_2 < \delta \implies \int_{t_2}^{t_1} C(t_1, s)ds < \epsilon.$$

Let $(\mathcal{B}, \|\cdot\|)$ be the Banach space of bounded continuous functions $\phi : [0, \infty) \rightarrow \mathfrak{R}$ with the supremum norm.

Theorem 2.6. *Let M be any bounded subset of \mathcal{B} and let Q be defined by $\phi \in M$ implies that*

$$(Q\phi)(t) = \int_0^t C(t, s)\phi(s)ds,$$

let a), b), c) hold and let $\int_0^t C(t, s)ds$ be uniformly continuous on $[0, \infty)$. Then QM is an equicontinuous set.

Regarding equation (2.12) assume that A is a positive kernel satisfying a), b), c) above and F is uniformly continuous with $F \in L^{2n+2}$. If we know that there is a bounded solution x , then by Theorem 2.6 we have that $x(t)$ is uniformly continuous so by Lemma 2.4 x^{2n+2} is uniformly continuous. Since from Theorem 2.2 we have that $x^{2n+2} \in L^1$, we see that x^{2n+2} satisfies the assumptions of Lemma 2.5 so we conclude that $x^{2n+2} \rightarrow 0$ as $t \rightarrow \infty$ and so does x . In [10] we give an extensive treatment of uniform continuity and boundedness of solutions of some integral equations.

We will now give two very simple but informative examples of existence of a global solution which is uniformly continuous. One example will be unique and the other will not address that question.

Theorem 2.7. *There is a unique uniformly continuous solution of*

$$x(t) = x(0) - \int_0^t A(t-s)x^3(s)ds \quad (2.18)$$

on $[0, \infty)$ satisfying $|x(t)| \leq 1$ when A satisfies Theorem 1.2 and $|x(0)| \leq 1$.

Proof. The transformation (1.11) will map (2.18) into

$$x(t) = x(0) \left[1 - \int_0^t R(s)ds \right] + \int_0^t R(t-s) \left[x(s) - \frac{x^3(s)}{J} \right] ds \quad (2.19)$$

with $J \geq 3$. Let $M = \{\phi \in \mathcal{B}: \|\phi\| \leq 1\}$.

If P is the natural mapping defined on M then

$$\begin{aligned} |(P\phi)(t)| &\leq \left[1 - \int_0^t R(s)ds \right] + \|\phi - \frac{\phi^3}{J}\| \int_0^t R(s)ds \\ &\leq 1 - \int_0^t R(s)ds + \int_0^t R(s)ds = 1 \end{aligned}$$

so $P : M \rightarrow M$.

Notice that $F(t) := x(0) \left[1 - \int_0^t R(s)ds \right]$ is uniformly continuous since it is continuous, bounded, and converges to zero as $t \rightarrow \infty$.

We now show that PM is an equicontinuous set so any solution residing in it is uniformly continuous. In fact, referring to Theorem 2.6 with $C(t, s) = R(t-s)$ we have

$$\int_0^t C(t, s)ds = \int_0^t R(t-s)ds$$

is continuous, bounded, and converges to one so it is uniformly continuous. By Theorem 2.6 we then note that any solution lying in M is uniformly continuous.

Now, we do not have a contraction, but we will make one out of what we have. On any interval $[0, n]$ for n a positive integer we see that $\phi, \psi \in M$ and $0 \leq t \leq n$ implies that

$$\begin{aligned} |(P\phi)(t) - (P\psi)(t)| &\leq \int_0^n R(n-s) |\phi(s) - \psi(s)| \left| 1 - \frac{\phi^2(s) + \phi(s)\psi(s) + \psi^2(s)}{3} \right| ds \\ &\leq \|\phi - \psi\|_{[0, n]} \int_0^n R(s)ds \leq \alpha_n \|\phi - \psi\|_{[0, n]} \end{aligned}$$

where the subscript denotes the interval over which the supremum is taken and $\alpha_n = \int_0^n R(s)ds < 1$ since $R(t) > 0$. This means that P is a contraction on $[0, n]$ and there is indeed a unique solution on that interval which we denote by x_n . We then define a sequence

$\Phi_n(t) = x_n(t)$ on $[0, n]$ and $\Phi_n(n)$ if $t > n$. Then $\Phi_n(t)$ converges uniformly on compact sets to a continuous function $\Phi(t)$ agreeing with x_n on $[0, n]$ and, hence, it is a solution on the entire interval $[0, \infty)$ lying entirely in M and is uniformly continuous, as required. \square

Theorem 2.8. *Let A satisfy Theorem 1.2 and let $|x(0)|$ be an arbitrary positive number larger than one. Then there is a solution $x : [0, \infty) \rightarrow \mathfrak{R}$ in $M = \{\phi \in \mathcal{B} : \|\phi\| \leq |x(0)|\}$ of*

$$x(t) = x(0) - \int_0^t A(t-s)x^{1/3}(s)ds$$

which is uniformly continuous.

Proof. We transform the equation as before to

$$x(t) = x(0) \left[1 - \int_0^t R(s)ds \right] + \int_0^t R(t-s) \left[x(s) - \frac{x^{1/3}(s)}{J} \right] ds.$$

Let $J = 1$. If $\phi \in M$ then $\phi(t)$ and $\phi^{1/3}(t)$ have the same sign, while $|\phi^{1/3}(t)| > |\phi(t)|$ implies that $|\phi(t)| < 1$. Hence $\phi \in M$ implies that

$$|(P\phi)(t)| \leq |x(0)| \left[1 - \int_0^t R(s)ds \right] + |x(0)| \int_0^t R(s)ds = |x(0)|.$$

Just as in the last proof, PM is equicontinuous. Finally, M is a ball. It then follows from [6] and [7] that P has a fixed point in M which is uniformly continuous. \square

3 A main lemma and examples

Very seldom can we expect to encounter an integral equation from applied mathematics in which the forcing function is in $L^1[0, \infty)$ or the natural mapping defined by the integral equation will map a bounded set into itself. The transformation of (1.7) to (1.11) changes the equation so that both will happen in a very simple way. For example, if the kernel takes the form $(t-s)^{q-1}$, $0 < q < 1$, and if the forcing function is a constant, $x(0)$, as happens so often then the transformation of

$$x(t) = x(0) - \int_0^t (t-s)^{q-1} [x(s) + x^{2n+1}(s)] ds, \quad n \geq 1$$

into the form (2.11) yields a new forcing function

$$z(t) = x(0) \left[1 - \int_0^t R(s)ds \right]$$

where R is the resolvent and z is in $L^p[0, \infty)$ if and only if $p > 1/q$. This result is found in Lemma 4.2 and 4.3 of [9, p. 317] and it also discusses the transformation in detail. This tells us something very **interesting and surprising** concerning Theorem 2.2 and inequality (2.13). Consider the transformation of (1.7) yielding (1.11) with $F(t) = x(0)$, $g(t, x) = x + x^{2n+1}$, and $J = 1$. By Theorem 2.2 we would get the relation (2.13) so that $x \in L^{2n+2}$ provided that $2n+2 > 1/q$. This is a sufficient condition, but it would be so interesting to discover how close it is to a necessary condition. We have never seen even a hint of this in the literature.

In a series of papers over the last several years [1–5] we have studied Riemann–Liouville fractional differential equations and have shown under general conditions [3, p. 257] that there

is an existence result, followed by a translation, followed by a transformation which always gives a forcing function F which is perfect: it is bounded, uniformly continuous on $[0, \infty)$, and is $L^1[0, \infty)$. Of course, there are other integral equations which also have that property so they are also under discussion here.

The following lemma will simplify the calculations of the type used on the right-hand-side of (2.10) so that our g here is our solution x to some power.

Lemma 3.1. *Let $F, g : [0, \infty) \rightarrow \mathfrak{R}$ be continuous, let $|F(t)| \leq K$ for $t \geq 0$, and let $F \in L^1[0, \infty)$. Then for given numbers $p > 1, r > 1$ with*

$$\frac{1}{p} + \frac{1}{r} = 1$$

and for any $T > 0$ it is true that

$$\int_0^T |F(s)g(s)|ds \leq B \left(\int_0^T |g(s)|^p ds \right)^{1/p},$$

where

$$B = K^{1-\frac{1}{r}} \left(\int_0^{+\infty} |F(s)|ds \right)^{1/r}.$$

Proof. We have

$$\begin{aligned} \int_0^T |F(s)g(s)|ds &\leq \left(\int_0^T |F(s)|^r ds \right)^{1/r} \left(\int_0^T |g(s)|^p ds \right)^{1/p} \\ &\leq K^{(r-1)(1/r)} \left(\int_0^T |F(s)|ds \right)^{1/r} \left(\int_0^T |g(s)|^p ds \right)^{1/p} \\ &\leq K^{(r-1)(1/r)} \left(\int_0^{+\infty} |F(s)|ds \right)^{1/r} \left(\int_0^T |g(s)|^p ds \right)^{1/p}. \quad \square \end{aligned}$$

A main building block for both integral equations and fractional differential equations is given in the following example.

Example 3.2. In the equation

$$y(t) = F(t) - \int_0^t A(t-s)y^{2n+1}(s) ds \quad (3.1)$$

let $F : [0, \infty) \rightarrow \mathfrak{R}$ be continuous, let $A(t-s)$ satisfy (1.1), and let n be a non-negative integer. Also, let $(2n+1)p = 2n+2$, $(1/p) + (1/r) = 1$, and suppose that there is a $D > 0$ with $(\int_0^\infty |F(t)|^r dt)^{1/r} = D$.

Theorem 3.3. *Any solution y of (3.1) satisfies $y \in L^{2n+2}[0, \infty)$.*

Proof. Because A is a positive kernel we let T be an arbitrary positive number, multiply by $y^{2n+1}(t)$, integrate from 0 to T , and apply the positive kernel result to obtain

$$\begin{aligned} \int_0^T y^{2n+2}(t)dt &= \int_0^T y^{2n+1}(t)F(t)dt - \int_0^T y^{2n+1}(t) \int_0^t A(t-s)y^{2n+1}(s)dsdt \\ &\leq \int_0^T y^{2n+1}(t)|F(t)|dt \\ &\leq \left(\int_0^T (|y^{2n+1}(t)|)^p dt \right)^{1/p} \left(\int_0^T |F(t)|^r dt \right)^{1/r} \end{aligned}$$

where $(2n + 1)p = 2n + 2$ and $(1/p) + (1/r) = 1$.

By assumption, that last term in the display is bounded by some number D leaving

$$\int_0^T y^{2n+2}(t)dt \leq D \left(\int_0^T y^{2n+2}(t)dt \right)^{1/p}.$$

As $p > 1$ this says that

$$\int_0^T y^{2n+2}(t)dt \leq D^{\frac{p}{p-1}}$$

for $0 < T < \infty$. □

For fractional equations we will see that the existence of D is automatic and we will also find that if a solution is bounded then it converges to zero.

This example is quite simple because $h(y)$ consists of a single term. But the next example is very instructive concerning the problem of $h(y)$ being the sum of two very different terms and a change in sign. The following lemma sets up the problem for Theorem 3.5.

Lemma 3.4. *Consider the equation*

$$y(t) = F(t) + \int_0^t A(t-s)[y(s) - 2y^{1/3}(s) - 2y^3(s)]ds \quad (3.2)$$

in which $F : [0, \infty) \rightarrow \mathfrak{R}$ is continuous and $A(t-s)$ satisfies (1.1). If y is any solution of (3.2) then it satisfies

$$\int_0^T [y^{4/3}(t) + y^4(t)]dt \leq 3 \int_0^T |F(t)|[|y^{1/3}(t)| + |y^3(t)|]dt, T \geq 0. \quad (3.3)$$

Proof. Here, we have

$$-h(y(s)) = -[2y^{1/3}(s) + 2y^3(s) - y(s)] \leq 0$$

so that by (1.1) we obtain

$$\int_0^T y(t)h(y(t))dt \leq \int_0^T |F(t)||h(y(t))|dt.$$

The following inequality is instructive. Notice that for $T > 0$ we have

$$\begin{aligned} \int_0^T y(t)h(y(t))dt &= \int_0^T [2y^{4/3}(t) + 2y^4(t) - y^2(t)]dt \\ &\leq \int_0^T |F(t)||h(y(t))|dt \leq \int_0^T |F(t)|[2|y^{1/3}(t)| + 2|y^3(t)| + |y(t)|]dt. \end{aligned}$$

We must work with the second and last terms to make them comparable in terms of L^p . That is done as follows:

$$\begin{aligned} \int_0^T [y^{4/3}(t) + y^4(t)]dt &\leq \int_0^T [2y^{4/3}(t) + 2y^4(t) - y^2(t)]dt \\ &= \int_0^T y(t)h(y(t))dt \\ &\leq \int_0^T |F(t)||h(y(t))|dt \\ &\leq \int_0^T |F(t)|[2|y^{1/3}(t)| + 2|y^3(t)| + |y(t)|]dt \\ &\leq 3 \int_0^T |F(t)|[|y^{1/3}(t)| + |y^3(t)|]dt. \end{aligned}$$

We have completely left $h(y(t))$ and are now focused entirely on the first and last terms which give us exactly (3.3). \square

It would be more convenient to assume F in $L^1[0, \infty)$ and bounded, but we can get by without the boundedness as follows. Notice in the next result that the relation (2.15) extends to sums.

Theorem 3.5. *Suppose that A satisfies (1.1), that*

$$\int_0^\infty |F(t)|^{4/3} dt =: b < \infty, \text{ and that } \int_0^\infty |F(t)|^4 dt =: c < \infty.$$

Then any solution of (3.2) satisfies

$$\int_0^\infty [y^{4/3}(t) + y^4(t)] dt =: a < \infty. \quad (3.4)$$

Proof. By way of contradiction, suppose that $a = \infty$. We will show that if either of the terms in the last integral are infinite, then we get a contradiction. Use Lemma 3.4.

Take $p = 4$ and $p' = 4/3$ so that starting from (3.3) we obtain

$$\begin{aligned} \int_0^T [y^{4/3}(t) + y^4(t)] dt &\leq 3 \int_0^T |F(t)| |y^{1/3}(t)| dt + 3 \int_0^T |F(t)| |y^3(t)| dt \\ &\leq 3b^{3/4} \left(\int_0^T y^{4/3}(t) dt \right)^{1/4} + 3 \left(\int_0^T |F(t)|^4 dt \right)^{1/4} \left(\int_0^T y^4(t) dt \right)^{3/4} \\ &\leq 3b^{3/4} \left(\int_0^T y^{4/3}(t) dt \right)^{1/4} + 3c^{1/4} \left(\int_0^T y^4(t) dt \right)^{3/4} \end{aligned}$$

and so

$$\left[\int_0^T y^{4/3}(t) dt - 3b^{3/4} \left(\int_0^T y^{4/3}(t) dt \right)^{1/4} \right] + \left[\int_0^T y^4(t) dt - 3c^{1/4} \left(\int_0^T y^4(t) dt \right)^{3/4} \right] \leq 0.$$

Consider the last two brackets. First, they cannot both be infinite as $T \rightarrow \infty$ because for large enough T the left-hand side is positive. Next, if only one is infinite as $T \rightarrow \infty$ then one bracket is bounded. In this case the other bracket becomes unbounded as $T \rightarrow \infty$ and it is positive. This is a contradiction to the fact that the sum of the brackets is not positive. This proves (3.4). \square

Example 3.6. Let A satisfy (1.1), let a_i be nonnegative constants at least one of which is positive, and for some positive integer n let

$$h(y) = a_1 y + a_3 y^3 + a_5 y^5 + \cdots + a_{2n+1} y^{2n+1}. \quad (3.5)$$

Theorem 3.7. *In (3.5) let n be a fixed positive integer and consider*

$$y(t) = F(t) - \int_0^t A(t-s) h(y(s)) ds \quad (3.6)$$

where $|F(t)| \leq K$ for some $K > 0$ with F continuous and in $L^1[0, \infty)$. Then $a_i y^{2i+2} \in L^1[0, \infty)$ for $i = 1, \dots, n$ with $a_i > 0$.

Proof. Multiply (3.6) by $h(y(t))$ and integrate from 0 to $T < \infty$ obtaining

$$\int_0^T y(s)h(y(s))ds \leq \int_0^T |F(s)||h(y(s))|ds$$

or

$$\begin{aligned} & \int_0^T [a_1 y^2(s) + a_3 y^4(s) + a_5 y^6(s) + \cdots + a_{2n+1} y^{2n+2}(s)] ds \\ & \leq \int_0^T a_1 |F(s)| |y(s)| + \int_0^T a_3 |F(s)| |y^3(s)| ds + \cdots + \int_0^T a_{2n+1} |F(s)| |y^{2n+1}(s)| ds. \end{aligned}$$

With Lemma 3.1 in mind, for the i^{th} integral in the right-hand side choose a $p_i > 1$ so that

$$(2i + 1)p_i = 2i + 2$$

and have

$$\int_0^T a_{2i+1} |F(s)| |y^{2i+1}(s)| ds \leq B_i a_{2i+1} \left(\int_0^T y^{2i+2}(s) ds \right)^{\frac{1}{p_i}} \quad (3.7)$$

for some positive constants B_i . Transpose each of these terms on the right-hand side and pair each one with the corresponding term having the same power of y . If any of the original terms on the left-hand side tends to infinity with T then for large T that term dominates the term with which it is paired. This will give us a contradiction to the fact that the right-hand side is now zero. \square

Example 3.8. The hyperbolic sine is a classic example of the behavior in the last example. For

$$f(x) := \sinh x = (1/2)(e^x - e^{-x})$$

there are three properties of interest here:

- (i) $f(-x) = (1/2)(e^{-x} - e^x) = -f(x)$,
- (ii) $f'(x) = (1/2)(e^x + e^{-x}) > 0$,
- (iii) $f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$.

If $g_n(x)$ denotes the n^{th} partial sum then

$$y(t) = F(t) - \int_0^t A(t-s)g_n(y)ds$$

and $g_n(y)$ satisfies the conditions of the last example and so the conclusion there obtains.

4 A general result

The ideas presented in the preceding sections are the guide for obtaining L^p results for a class of functions h wider than odd polynomials in y . Below we present a result concerning solutions of the equation

$$y(t) = F(t) - \int_0^t A(t-s)h(s, y(s))ds, \quad t \geq 0, \quad (4.1)$$

with $F : [0, \infty) \rightarrow \mathfrak{R}$ continuous, A satisfying (1.1), and $h : [0, \infty) \times \mathfrak{R} \rightarrow \mathfrak{R}$ continuous and depending on both variables y and t .

Theorem 4.1. Assume that there exist a continuous $\phi : [0, \infty) \rightarrow [0, \infty)$, real numbers $k \geq m \geq 1$, $p, q \in [0, 1)$, positive constants a, b, A , and $B \geq 0$ such that for $y \in \mathfrak{R}$ and $t \geq 0$ we have

$$a|y|^k + b|y|^m \leq yh(t, y) \leq \phi(t) \left[A|y|^{k+p} + B|y|^{m+q} \right]. \quad (4.2)$$

If $F\phi \in L^{\frac{k}{1-p}} \cap L^{\frac{m}{1-q}}$, then for any solution of (4.1) it holds $y \in L^k \cap L^m$.

Proof. Clearly we may assume that $B > 0$ (otherwise we may take $B = b$). Multiplying (4.1) by $h(t, y)$ and using the fact that A is a positive kernel we have for arbitrary $T > 0$

$$\begin{aligned} & \int_0^T a|y(t)|^k + b|y(t)|^m dt \\ & \leq \int_0^T y(t) h(t, y(t)) dt \\ & = \int_0^T h(t, y(t)) F(t) dt - \int_0^T h(t, y(t)) \int_0^t A(t-s) h(s, y(s)) ds dt \\ & \leq \int_0^T h(t, y(t)) F(t) dt \\ & = \int_0^T |h(t, y(t))| |F(t)| dt \\ & \leq \int_0^T |F(t)| \phi(t) \left[A|y|^{k-1+p} + B|y|^{m-1+q} \right] dt \\ & = \int_0^T |F(t)| \phi(t) A|y|^{k-1+p} dt + \int_0^T |F(t)| \phi(t) \left[B|y|^{m-1+q} \right] dt \\ & \leq A \left[\int_0^T |F(t)| \phi(t) dt \right]^{\frac{1-p}{k}} \left[\int_0^T \left(|y|^{k-1+p} \right)^{\frac{k}{k-1+p}} dt \right]^{\frac{k-1+p}{k}} \\ & \quad + B \left[\int_0^T |F(t)| \phi(t) dt \right]^{\frac{1-q}{m}} \left[\int_0^T \left(|y|^{m-1+q} \right)^{\frac{m}{m-1+q}} dt \right]^{\frac{m-1+q}{m}} \end{aligned}$$

so

$$\left\{ a \int_0^T |y(t)|^k dt - F_1 \left[\int_0^T |y(t)|^k dt \right]^{\frac{k-1+p}{k}} \right\} + \left\{ b \int_0^T |y(t)|^m dt - F_2 \left[\int_0^T |y(t)|^m dt \right]^{\frac{m-1+q}{m}} \right\} \leq 0$$

with

$$F_1 := A \left[\int_0^\infty |F(t)| \phi(t) dt \right]^{\frac{1-p}{k}}, \quad F_2 := B \left[\int_0^\infty |F(t)| \phi(t) dt \right]^{\frac{1-q}{m}}.$$

If either $y \notin L^k$ or $y \notin L^m$, in view of $\frac{k-1+p}{k}, \frac{m-1+q}{m} < 1$ we reach a contradiction by observing that letting $T \rightarrow \infty$ the left-hand side of the last inequality tends to infinity. Hence $y \in L^k \cap L^m$. \square

Observe that from (4.2) it follows that the function ϕ must be bounded below by a positive number as for $y = 1$ we have that $\frac{a+b}{A+B} \leq \phi(t)$ for all $t \geq 0$. However, this positive bound can be arbitrarily small as A and B may be taken arbitrarily large (also a, b may be arbitrarily small). In fact, the conclusion of Theorem 4.1 does not depend on the magnitude of the positive constants a, b and A .

The process in this proof can be repeated so that S can consist of any finite number of terms.

Corollary 4.2. Assume that for ϕ of Theorem 4.1 there exist real constants $k_1 \geq \dots \geq k_s \geq 1$, positive numbers a_1, \dots, a_s, A_1 , nonnegative numbers A_2, \dots, A_s and $p_1, \dots, p_s \in [0, 1)$ such that for $y \in \mathbb{R}$ we have

$$\sum_{i=1}^{i=s} a_i |y|^{k_i} \leq yh(t, y) \leq \phi(t) \sum_{i=1}^{i=s} A_i |y|^{k_i + p_i}. \quad (4.3)$$

If $F\phi \in \bigcap_{i=1, \dots, s} L^{\frac{k_i}{1-p_i}}$, then for any solution of (4.1) we have $y \in \bigcap_{i=1, \dots, s} L^{k_i}$.

In case that the function F is bounded and in L^1 , combining Theorem 4.1 and Corollary 4.2 we get the following result.

Corollary 4.3. Assume that for ϕ of Theorem 4.1 there exist $k \geq m \geq 1$, $p, q \in [0, 1)$, real constants a, b, A , and a nonnegative number B such that (4.2) holds true. If $F\phi$ is bounded and $F\phi \in L^1$, then for any solution y of (4.1) we have $y \in L^r$ for any $r \in [m, k]$.

Proof. Clearly if $F\phi$ is bounded and $F\phi \in L^1$ then we have that $F\phi \in L^r$ for any $r \in [m, k]$. Note that

$$\frac{a}{2} |y|^k + \left(\frac{a}{2} + \frac{b}{2}\right) |y|^r + \frac{b}{2} |y|^m \leq a |y|^k + b |y|^m$$

so from (4.2) we have that for $c = \frac{a}{2} + \frac{b}{2}$

$$\frac{a}{2} |y|^k + c |y|^r + \frac{b}{2} |y|^m \leq yh(t, y) \leq \phi(t) \left[A |y|^{k+p} + c |y|^r + B |y|^{m+q} \right] \quad y \in \mathfrak{R},$$

and the conclusion follows from Corollary 4.2. \square

Example 4.4. Consider the equation

$$y(t) = F(t) - \int_0^t A(t-s) \left[y^3 \sqrt{|y|} + 2y \sqrt{y^2 + 1} - \sin y + 3 \frac{y^{10/3}}{y^2 + 1} \right] ds, \quad t \geq 0,$$

with A satisfying (1.1). Here $\phi(t) = 1$ and

$$h(t, y) := H(y) = y^3 \sqrt{|y|} + 2y \sqrt{y^2 + 1} - \sin y + 3 \frac{y^3 \sqrt[3]{|y|}}{y^2 + 1}.$$

We have

$$\begin{aligned} yh(t, y) &= y^4 \sqrt{|y|} + 2y^2 \sqrt{y^2 + 1} - y \sin y + 3 \frac{y^4 \sqrt[3]{|y|}}{y^2 + 1} \\ &\leq y^4 \sqrt{|y|} + 3y^2 (|y| + 1) + 3y^2 \sqrt[3]{|y|} \\ &= |y|^{9/2} + 3|y|^3 + 3y^2 + 3|y|^{7/3} \\ &\leq |y|^{9/2} + 3|y|^3 + 3y^2 + 3(|y|^{9/2} + y^2) \\ &= 4|y|^{9/2} + 3|y|^3 + 6y^2 \end{aligned}$$

and

$$\begin{aligned} yh(t, y) &= y^4 \sqrt{|y|} + 2y^2 \sqrt{y^2 + 1} - y \sin y + 3 \frac{y^4 \sqrt[3]{|y|}}{y^2 + 1} \\ &\geq y^4 \sqrt{|y|} + y^2 (|y| + 1) + 3 \frac{y^4 \sqrt[3]{|y|}}{y^2 + 1} \\ &\geq |y|^{9/2} + |y|^3 + |y|^2, \end{aligned}$$

since

$$y^2 \sqrt{y^2 + 1} - y \sin y \geq y^2 \sqrt{y^2 + 1} - |y| |\sin y| \geq y^2 \sqrt{y^2 + 1} - y^2 \geq 0.$$

From Corollary 4.2 we have that if F is in $L^{9/2} \cap L^3 \cap L^2$ then so is any solution y . Furthermore, since

$$\begin{aligned} |y|^{9/2} + |y|^2 &\leq |y|^{9/2} + |y|^3 + |y|^2 \leq yh(t, y) \\ &\leq 4|y|^{9/2} + 3|y|^3 + 6y^2 \leq 7|y|^{9/2} + 9y^2 \end{aligned}$$

we see that condition (4.3) is satisfied with $\phi(t) = 1$ so from Corollary 4.3 we have that if F is bounded and absolutely integrable then any solution y is in L^r for any $r \in [2, 9/2]$.

The next example concerns an equation where h involves a function ϕ of t which is unbounded for $t \rightarrow +\infty$ while $F\phi \in L^1$.

Example 4.5. For the equation

$$y(t) = \frac{t+1}{t^3+2} - \int_0^t A(t-s) \sqrt{s+1} H(y) ds, \quad t \geq 0, \quad (4.4)$$

with H as in Example 4.4 we have $h(t, y) = \phi(t) H(y)$ with $\phi(t) = \sqrt{t+1} \geq 1$, and hence condition (4.2) is satisfied. Moreover,

$$0 \leq F(t) \phi(t) = \frac{t+1}{t^3+2} \sqrt{t+1} \leq \frac{1}{(t+1)^{3/2}}, \quad t \geq 0,$$

so $F\phi \in L^1$ and is bounded. Thus, from Corollary (4.3) we have that if A satisfies (1.1) all solutions of (4.4) are in L^r for any $r \in [2, 9/2]$.

5 Extreme singularities

There are many examples from applied mathematics in which the forcing function has a singularity at $t = 0$ and the kernel has a singularity at $s = t$. In such cases the solution develops a singularity at $t = 0$ and the integrand may have the form $g(s, x(s))$ which picks up the singularity and combines with the kernel to produce a very strong singularity. The exciting part is that we can greatly reduce the strength of the singularity. In earlier work we have considered this in a fractional differential equation of Riemann–Liouville type. There are then three steps. First, we offer several existence theorems ([3] and [4]) so that we can get a solution on an arbitrarily short interval $(0, L]$. Then we use our transformation from Section 1 to write the equation with the new kernel $R(t-s)$. Finally, we translate the equation by setting $y(t) = x(t+L)$ which now has a marvellous forcing function. All of this is given in [3, p. 266, Thm. 4.2]. The new forcing function $F : [0, \infty) \rightarrow \mathfrak{R}$ is uniformly continuous on $[0, \infty)$, tends to zero as $t \rightarrow \infty$, is bounded, and is in $L^1[0, \infty)$. For our work here it is perfect. We now give a sketch of how this proceeds.

Consider the Riemann–Liouville fractional differential equation

$$D^q x(t) = -x^{1/3} - 2x =: -w(x) \quad (5.1)$$

where $\lim_{t \downarrow 0} t^{1-q} x(t) = x^0 \in \mathfrak{R}, x^0 \neq 0, 0 < q < 1$. This inverts as

$$x(t) = t^{q-1} - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} w(y(s)) ds. \quad (5.2)$$

The kernel satisfies (A1)–(A3) of Theorem 1.2. We now use our transformation to write this as

$$y(t) = F(t) + \int_0^t R(t-s) \left[y(s) - \frac{w(y(s))}{J} \right] ds \quad (5.3)$$

shown in [3, p. 263] with $y(t) = x(t+T)$. This is a brief sketch of the path from (5.1) with its singularity problems to (5.3) with a perfect forcing function. Lengthy details have been left out and any reader wishing to perform this in a given problem should consult the full paper. It is given here to emphasize that there is a definite path to a nice forcing function.

Theorem 5.1. *If x is a solution of (5.1) and $y = x(t+T)$ solves (5.3) then the function y satisfies $y \in L^r[0, \infty)$ for any $r \in [4/3, 2]$. In particular, if y is bounded then it converges to zero for $t \rightarrow \infty$.*

Proof. Taking $J = 1$ we see that (5.3) is equation (4.1) with $h(t, y) = y^{1/3} + y$, so $yh(t, y) = y^{4/3} + y^2$ and condition (4.2) is satisfied. As the forcing function F is bounded and in L^1 , while R is a positive kernel, the conclusion follows from Corollary 4.3 with $\phi(t) = 1$. If y is bounded then in view of Theorem 2.6 and the fact that F is uniformly continuous we have that y is uniformly continuous. It follows that y^2 is bounded, uniformly continuous and in L^1 so Lemma 2.5 applies. \square

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