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Extensions of Gronwall's inequality with quadratic growth terms and applications

Dedicated to Professor László Hatvani on the occasion of his 75th birthday

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Abstract. We obtain some new Gronwall type inequalities where, instead of linear growth assumptions, we allow quadratic (or more) growth provided some additional conditions are satisfied. Applications are made to both local and nonlocal boundary value problems for some second order ordinary differential equations which have quadratic growth in the derivative terms.

Keywords: Gronwall inequality, quadratic growth, second order equation.

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1 Introduction

The Gronwall inequality is a well-known tool in the study of differential equations and Volterra integral equations, see for example [3, 6, 10], and is useful in establishing a priori bounds which help prove global existence and stability, and it can help prove uniqueness results.

There are differential and integral versions which are closely related. We shall consider integral versions of the inequality. The simplest case is: if u is a continuous non-negative function and $u(t) \le a + b\int_0^t u(s)\,ds$ for positive constants a, b and $t \in [0, T]$ then $u(t) \le a\exp(bt)$ for $t \in [0, T]$. In particular this says that u does not blow up on $[0, T]$, moreover there is no restriction on T. For an initial value problem for a first order ordinary differential equation (ODE) $u'(t) = f(t, u(t))$, $u(0) = a$ this can be applied to give an a priori bound on possible solutions u when $f(t, u)$ has at most linear growth in u.

The constants a, b can be replaced by suitable functions as in one classical version of the result which was proved by Bellman [1]. Other versions may be found in several books, for example [10, 12].

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Theorem 1.1. *Suppose that $u \in L^\infty_+[0,T]$ satisfies $u(t) \le c_0(t) + \int_0^t c_1(s)u(s)\,ds$ for almost every (a.e.) $t \in [0,T]$, where c_0 is non-negative and non-decreasing, and $c_1 \in L^1_+[0,T]$. Then*

$$u(t) \le c_0(t) \exp\left(\int_0^t c_1(s)\,ds\right) \quad \text{for a.e. } t \in [0,T]. \tag{1.1}$$

Here, $L^1_+[0,T]$ denotes the integrable functions u with $u(t) \ge 0$ a.e., similarly $L^\infty_+[0,T]$ will denote the essentially bounded functions with $u(t) \ge 0$ a.e.

For an initial value problem for a second order differential equation

$$u''(t) = f(t, u(t), u'(t)), \quad u(0) = u_0, \quad u'(0) = u_1$$

the Gronwall inequality can be used to get an a priori bound on the derivative u' provided f has at most linear growth in u, u'. The bound on u' immediately gives a bound on u so global existence results can then be proved.

Some interesting results were proved by Granas, Guenther and Lee [5], where boundary value problems (BVPs) for differential equations of the form $u'' = f(t,u,u')$ with f having quadratic growth in u' were studied. It was shown that it is possible to get a uniform L^∞ bound for u' if u' vanishes at least once and a uniform L^∞ bound for u is known, that is if $|u(t)| \le M$. Indeed, assuming that $f(t,u,u') \le Au'^2 + B$, by writing $u''u' \le u'(Au'^2 + B)$, the idea used is that the inequality $2Au''u'/(Au'^2 + B) \le 2Au'$ can be directly integrated and $\int_0^t 2Au' = 2A(u(t) - u(0)) \le 4AM$. This work was extended by Petryshyn [11] to cover the ODE $u'' = f(t,u,u',u'')$ by employing his theory of A-proper maps.

Motivated by this we shall prove a Gronwall inequality, which, when applied to second order ODEs, allows quadratic growth in u'. In fact we can prove a more complicated inequality with terms of higher order. One advantage is that, even in situations where other methods are available, such as use of a comparison principle, we can give explicit bounds. Another advantage is that we cover cases where no systematic methods are known. The result can be regarded as a possible replacement of a Nagumo condition in suitable circumstances.

The following special case of our inequality illustrates the kind of result we prove.

Suppose that, for a.e. $t \in [0,T]$,

$$u(t) \le c_0 + \int_0^t c_1(s)u(s) + c_2(s)u^2(s)\,ds \tag{1.2}$$

where $u \in L^\infty_+[0,T]$, $c_0 > 0$ is a constant and $c_i \in L^1_+[0,T]$, $i \in \{1,2\}$. If it is known that

$$\text{there exists a constant } M > 0 \text{ such that } \int_0^T c_2(s)u(s)\,ds \le M, \tag{1.3}$$

then it follows that

$$u(t) \le c_0 \exp\Big(\int_0^t c_1(s)\,ds\Big) \exp(M), \quad \text{for a.e. } t \in [0,T]. \tag{1.4}$$

When u is a derivative, say $u = v'$, and c_2 is a constant our result (in the special case) would say that, for an arbitrary $T > 0$, if by some means we also know that $|v(t)| \le M_0$, then $v'(t) \le M_1$ for $t \in [0,T]$, where M_1 can be computed explicitly. Here $\int_0^T c_2 u(s)\,ds = c_2(v(T) - v(0)) \le 2c_2 M_0$ when $|v| \le M_0$. The required known bound on v occurs under an hypothesis given in [5] and recalled below, but also can occur in other situations, for example the commonly occurring case where v is constrained to lie between upper and lower solutions. Furthermore, our method is different from that of [5] and [11], and we can allow some integrable (rather than bounded) functions in the upper bounds.

Our Gronwall inequality result is perhaps surprising, because given an inequality such as

$$u(t) \le 1 + \int_0^t u^2(s)\,ds,$$

it is impossible to obtain an a priori bound on $[0, T]$ for an arbitrary T since the solution of $u'(t) = u^2(t), u(0) = 1$ is $u(t) = 1/(1-t)$ which blows up as $t \to 1-$, that is, only exists for $t < 1$.

There are some results which treat problems with higher order growth, for example, [7, 9, 15]. One result of this type is due to Perov, if

$$u(t) \le a + b\int_0^t u(s)\,ds + \int_0^t u^\alpha(s)\,ds, \quad \text{where } \alpha > 1,$$

then there exists $h > 0$, sufficiently small, with an explicit estimate, such that $u(t)$ is uniformly bounded for $t \in [0, h]$. This is stated as Theorem 1, Chapter XII in [10]), but since this is not proved in the book [10], and there are typos in the statement, and it is perhaps not well known, we will provide a proof using the suggested method. We also use our idea to prove a global estimate when this inequality holds, that is we prove that $u(t)$ is uniformly bounded for $t \in [0, T]$ for an arbitrary T provided an extra condition holds.

We then give applications to the second order ODE $u'' = f(t, u, u')$ where f is allowed to have quadratic growth in u'. The main hypothesis is the following type of sign condition as used by Granas, Guenther and Lee in [5]; a similar one is used by Petryshyn in [11].

There exists a constant $M > 0$ such that

$$uf(t, u, 0) > 0, \quad \text{for all } t \in [0, T], \quad \text{and all } |u| > M. \tag{1.5}$$

The other hypothesis is that f grows at most quadratically in u'. We first give a small improvement of the results of [5]. Then we consider a nonlocal BVP which, as far as we are aware, is studied here for the first time under these types of hypotheses. Nonlocal BVPs have been well studied in recent years with other methods, for some general methods of studying positive solutions using fixed point index theory when there is no u' dependence we refer to [13, 14], but in these cases it is usually supposed that $f(t, u) \le 0$ for all $u \ge 0$ so (1.5) does not hold. For problems with u' dependence it is often supposed that f satisfies a Nagumo condition and suitable growth conditions of a different type to the conditions imposed here, some examples are [8] and [16]. Much work on problems with derivative dependence uses the method of upper and lower solutions, but we do not discuss this here.

2 Extended Gronwall inequality

We shall prove a more general version than (1.2)–(1.4) given in the Introduction which allows higher order growth under suitable extra hypotheses. We are interested in the case when the inequalities hold a.e. since this is a commonly occurring situation. Our result is the following one.

Theorem 2.1. *Let $p \in \mathbb{N}$ and suppose that for a.e. $t \in [0, T]$, $u \in L^\infty_+[0, T]$ satisfies*

$$u(t) \le c_0(t) + \int_0^t \big(c_1(s)u(s) + c_2(s)u^2(s) + \cdots + c_{p+1}(s)u^{p+1}(s)\big)\,ds, \tag{2.1}$$

where $c_0 \in L^\infty_+[0,T]$ is non-decreasing, and $c_j \in L^1_+[0,T]$ for $j \in \{1,\dots,p+1\}$. Then, if

$$\int_0^T c_{j+1}(s)u^j(s)\,ds \le M_j, \qquad j \in \{1,\dots,p\},$$

it follows that for a.e. $t \in [0,T]$

$$u(t) \le c_0(t)\exp\Big(\int_0^t c_1(s)\,ds\Big)\exp(M_1+\cdots+M_p). \tag{2.2}$$

Proof. By taking an arbitrary $\tau \in [0,T]$, replacing $c_0(t)$ by $c_0(\tau)$ and considering the inequality on $[0,\tau]$ we may suppose that c_0 is a *positive* constant (add $\varepsilon > 0$ if necessary). Let

$$w(t) = c_0 + \int_0^t \big(c_1(s)u(s) + c_2(s)u^2(s) + \cdots + c_{p+1}(s)u^{p+1}(s)\big)\,ds.$$

Then w is absolutely continuous, $u(t) \le w(t)$ for a.e. t, and

$$w'(t) = c_1(t)u(t) + c_2(t)u^2(t) + \cdots + c_{p+1}(t)u^{p+1}(t) \quad \text{for a.e. } t.$$

Therefore we have

$$w'(t) \le c_1(t)w(t) + c_2(t)u(t)w(t) + \cdots + c_{p+1}(t)u^p(t)w(t).$$

Hence we obtain

$$w'(t)/w(t) \le c_1(t) + c_2(t)u(t) + \cdots + c_{p+1}(t)u(t)^p \quad \text{for a.e. } t \in [0,\tau].$$

Because $w(t) \ge c_0 > 0$ it follows that $\ln(w)$ is absolutely continuous and therefore we can integrate the previous inequality to get

$$\ln(w(t)/c_0) \le \int_0^t \big(c_1(s) + c_2(s)u(s) + \cdots + c_{p+1}(s)u(s)^p\big)\,ds \quad \text{for all } t \in [0,\tau],$$

so that

$$\ln(w(t)/c_0) \le \int_0^t c_1(s)\,ds + M_1 + \cdots + M_p, \quad \text{for all } t \in [0,\tau].$$

This gives

$$w(t) \le c_0 \exp\Big(\int_0^t c_1(s)\,ds + M_1 + \cdots + M_p\Big) \quad \text{for all } t \in [0,\tau].$$

This holds for $t = \tau$ and since τ is arbitrary the inequality holds for every t. Then the inequality $u(t) \le w(t)$ a.e. yields the conclusion. □

Remark 2.2.

(1) The interval $[0,T]$ can be replaced by any finite interval $[\alpha,\beta]$ with obvious changes.

(2) If c_j are constants, instead of the hypotheses

$$\int_0^T c_{j+1}u^j(s)\,ds \le M_j, \qquad j \in \{1,\dots,p\},$$

we could assume just one integrability condition on u namely that $\int_0^T u^p(s)\,ds \le M_p$; apply Hölders inequality.

3 Perov type inequality

We will use the same ideas as in section 2 to obtain another result that is related to a result due to Perov, in a paper published in Russian in 1959, details are given in [10]. We first recall the result of Perov and include a proof since the proof is left to the reader in [10] and, confusingly, the result is mis-stated in [10, Theorem 1, Chapter XII], some typos include an omitted minus sign.

Notation: For an integrable function b we write $B(t) := \int_0^t b(s)\,ds$.

Theorem 3.1 (Perov). *Let $\alpha > 1$. Suppose that there are a constant $a > 0$ and functions $b, c \in L^1_+[0,h]$ such that $u \in L^\infty_+[0,h]$ satisfies*

$$u(t) \le a + \int_0^t b(s)u(s)\,ds + \int_0^t c(s)u^\alpha(s)\,ds, \quad \text{for a.e. } t \in [0,h], \tag{3.1}$$

where h is such that

$$(\alpha-1)a^{\alpha-1}\int_0^h c(s)\exp\big((\alpha-1)B(s)\big)\,ds < 1. \tag{3.2}$$

Then, for a.e. $t \in [0,h]$ we have

$$u(t) \le \frac{a\exp(B(t))}{\big[1-(\alpha-1)a^{\alpha-1}\int_0^t c(s)\exp\big((\alpha-1)B(s)\big)\,ds\big]^{1/(\alpha-1)}}. \tag{3.3}$$

When $c \equiv 0$ we recover the standard Gronwall inequality, cf. Theorem 1.1, as expected. If $b \equiv 0$ the Bihari inequality can also be applied, see [2]. The result of Perov is also proved in [15, Theorem 2] but the authors do not mention the restriction required on h. Similar inequalities with power nonlinearities can also be found in the papers [7, 9].

The special case where a, b, c are positive constants is most likely to occur in which case the result takes a simpler form.

Corollary 3.2. *Suppose that a, b, c are positive constants and $u \in L^\infty_+[0,h]$ satisfies*

$$u(t) \le a + b\int_0^t u(s)\,ds + c\int_0^t u^\alpha(s)\,ds, \quad \text{for a.e. } t \in [0,h], \tag{3.4}$$

where h is such that

$$\exp(b(\alpha-1)h) < 1 + \frac{b}{a^{\alpha-1}c}. \tag{3.5}$$

Then, for a.e. $t \in [0,h]$ we have

$$u(t) \le \frac{a\exp(bt)}{\big[1-a^{\alpha-1}\frac{c}{b}(\exp(b(\alpha-1)t)-1)\big]^{1/(\alpha-1)}}. \tag{3.6}$$

The case $b = 0$ can be obtained by taking the limit as $b \to 0+$ and is

$$u(t) \le \frac{a}{[1-(\alpha-1)a^{\alpha-1}ct]^{1/(\alpha-1)}} \quad \text{for a.e. } t \in [0,h], \quad \text{where } (\alpha-1)h < \frac{1}{a^{\alpha-1}c}.$$

Proof of Theorem 3.1. Let

$$v(t) := a + b\int_0^t u(s)\,ds + c\int_0^t u^\alpha(s)\,ds$$

so that v is absolutely continuous, $v(0) = a$, and $u(t) \le v(t)$ for a.e. $t \in [0, h]$. Then for for a.e. $t \in [0, h]$ we have
$$v'(t) \le b(t)v(t) + c(t)v^{\alpha}(t).$$
For t in an interval on which v remains finite we set $w(t) = v(t)^{1-\alpha}$. Then we obtain, using the integrating factor $\exp\big((\alpha-1)B(t)\big)$,
$$\begin{aligned} w'(t) &\ge (1-\alpha)\big(b(t)w(t) + c(t)\big), \\ \big(w(t)\exp\big((\alpha-1)B(t)\big)\big)' &\ge -(\alpha-1)c(t)\exp\big((\alpha-1)B(t)\big) \\ w(t)\exp\big((\alpha-1)B(t)\big) &\ge 1/a^{\alpha-1} - (\alpha-1)\int_0^t c(s)\exp\big((\alpha-1)B(s)\big)\,ds. \end{aligned}$$
Note that w remains positive for $t \le h$ provided that (3.2) holds, w can become zero and v can blow up as soon as (3.2) fails. The above gives
$$\frac{1}{w(t)} \le \frac{a^{\alpha-1}\exp((\alpha-1)B(t)}{1 - a^{\alpha-1}(\alpha-1)\int_0^t c(s)\exp\big((\alpha-1)B(s)\big)\,ds}.$$
Using $v(t) = (1/w(t))^{1/(\alpha-1)}$ this gives (3.3). □

Remark 3.3. The inequalities are sharp since equality could hold at every step. The above results are valid for real values of $\alpha > 1$ but the conclusion holds on intervals whose length decreases as α increases. The constants should be chosen as small as possible to obtain h as large as possible.

We now use our previous method to prove a result for an interval $[0, T]$ where $T > 0$ can be arbitrary; of course an extra condition is necessary.

Theorem 3.4. *Let $\alpha > 1$. Suppose that there are a constant $a > 0$ and functions $b, c \in L^1_+[0, T]$ such that $u \in L^{\infty}_+[0, T]$ satisfies*
$$u(t) \le a + \int_0^t b(s)u(s)\,ds + \int_0^t c(s)u^{\alpha}(s)\,ds, \quad \textit{for a.e. } t \in [0, T], \tag{3.7}$$
and suppose it is known that there is a constant $M > 0$ such that
$$\int_0^T c(s)u^{\alpha-1}(s)\,ds \le M. \tag{3.8}$$
Then we have
$$u(t) \le a\exp(B(t))\exp(M), \quad \textit{for a.e } t \in [0, T]. \tag{3.9}$$

Proof. Let $v(t) := a + \int_0^t b(s)u(s)\,ds + \int_0^t c(s)u^{\alpha}(s)\,ds$. Then v is absolutely continuous, $v(t) \ge a > 0$, and $u(t) \le v(t)$ for a.e. t. Moreover we have $v'(t) \le b(t)v(t) + c(t)u(t)^{\alpha-1}v(t)$. Then $v'(t)/v(t) \le b(t) + c(t)u(t)^{\alpha-1}$ which can be integrated to give
$$\begin{aligned} \ln\big(v(t)/v(0)\big) &\le B(t) + \int_0^t c(s)u(s)^{\alpha-1}\,ds, \\ v(t) &\le a\exp(B(t))\exp(M), \quad \text{for } t \in [0, T], \end{aligned}$$
which gives the conclusion. □

4 Applications to some second order ODEs

We will improve slightly on the problems studied in [5] and then treat in detail the following nonlocal BVP that is not covered by the results in [5],

$$\begin{gathered} u''(t) = f(t, u(t), u'(t)) \quad t \in [0, T], \\ u(0) - b_0 u'(0) = \beta_0[u], \quad u(T) + b_1 u'(T) = \beta_1[u], \end{gathered} \tag{4.1}$$

where $b_i > 0$ and $\beta_i[u] := \int_0^T u(s)dB_i(s)$ are Riemann–Stieltjes integrals, B_i are non-decreasing functions, that is dB_i are (positive) Stieltjes measures.

We consider classical solutions, that is $u \in C^2([0, T])$ which satisfies the equation at all points of $[0, T]$. It would appear to be more natural to seek solutions $u \in C^2(0, T) \cap C^1[0, T]$. However, as remarked in [5], the assumptions on f given below imply, a priori, that u must be in $C^2[0, T]$ so no generality would be gained.

For a continuous function u we shall use the norm $\|u\|_\infty := \max\{|u(t)|, t \in [0, T]\}$.

Firstly we will consider problems similar to those studied in [5], that is second order ODEs of the form

$$u''(t) = f(t, u(t), u'(t)), \qquad t \in [0, T], \tag{4.2}$$

where f is continuous on $[0, T] \times \mathbb{R} \times \mathbb{R}$, together with one of the following boundary conditions which were considered in [5] and in [11].

(I) $u(0) = 0, u(T) = 0$; Dirichlet BCs,

(II) $u'(0) = 0, u'(T) = 0$; Neumann BCs,

(III) $u(0) = u(T), u'(0) = u'(T)$; periodic BCs

(IV) $a_0 u(0) - b_0 u'(0) = 0$, $a_1 u(T) + b_1 u'(T) = 0$, where $a_0^2 + b_0^2 > 0$, $a_1^2 + b_1^2 > 0$, and $a_0^2 + a_1^2 > 0$; Sturm–Liouville BCs

(V) $u(0) = -u(T), u'(0) = -u'(T)$; antiperiodic BCs.

The problem of solving the differential equation (4.2) subject to one of the boundary conditions such as (I) will be referred to as problem (I), etc.

The key assumption made in [5] is a type of sign assumption.

There exists a constant $M > 0$ such that

$$uf(t, u, 0) > 0, \quad \text{for all } t \in [0, T], \quad \text{and all } |u| > M. \tag{4.3}$$

Lemma 4.1 ([5, Lemma 2.1]). *Let f be continuous and satisfy (4.3). Then if u is a solution of the equation (4.2) and $|u|$ does not achieve its maximum at $t = 0$ or $t = T$ then $|u(t)| \le M$ for $t \in [0, T]$.*

Proof. If u has a positive maximum at $t_0 \in (0, T)$ with $u(t_0) > M$ then $u'(t_0) = 0$ and $u''(t_0) \le 0$. Since $u''(t_0) = f(t_0, u(t_0), 0)$ has the same sign as $u(t_0)$ by (4.3) this is impossible. The case of a negative minimum less than $-M$ is exactly similar. □

Remark 4.2. It was shown in [5, Lemma 2.2] that for each of the problems (I)–(III), if f and M satisfy the hypothesis of Lemma 4.1, then, for any solution u of (4.2), the maximum of $|u(t)|$ cannot occur at $t = 0$ or at $t = T$, hence $|u(t)| \le M$ for $t \in [0, T]$. Later remarks in [5] deal with problems (IV), (V).

We give a small extension of Lemma 3.1 of [5] which gives a bound on the derivatives of potential solutions.

Lemma 4.3.

(*i*) *Suppose there is a constant $M > 0$ such that every solution $u \in C^2[0,T]$ of (4.2) satisfies $|u(t)| \le M$ for $0 \le t \le T$.*

(*ii*) *Suppose there exist non-negative constants c_0, c_3 and functions $c_1, c_2 \in L^1_+[0,T]$ such that*

$$|f(t,u,p)| \le c_0 + c_1(t)|u| + c_2(t)|p| + c_3|p|^2 \text{ for all } (t,u,p) \in [0,T] \times [-M,M] \times \mathbb{R}.$$

Then, for each solution u of (4.2) whose derivative vanishes at least once in $[0,T]$, there is an explicit constant M_1 depending only on M, c_i, T such that $|u'(t)| \le M_1$, for $t \in [0,T]$.

Proof. Let $u \in C^2[0,T]$ be a solution of the differential equation (4.2) whose derivative vanishes at least once in $[0,T]$. Each point $t \in [0,T]$ belongs to an interval $[\alpha,\beta]$ on which u' has a fixed sign and either $u'(\alpha) = 0$ or $u'(\beta) = 0$. If $v := u' \ge 0$ on $[\alpha,\beta]$ and $v(\alpha) = 0$ we have

$$\begin{aligned} v(t) = \int_\alpha^t f(s,u(s),v(s))\,ds &\le \int_\alpha^t c_0 + c_1(s)|u(s)| + c_2(s)|v(s)| + c_3 v(s)^2\,ds, \\ &\le c_0 T + M\int_\alpha^\beta c_1(s)\,ds + \int_\alpha^t c_2(s)v(s)\,ds + \int_\alpha^t c_3 v(s)^2\,ds. \end{aligned}$$

Setting $C_0 = c_0 T + M\int_0^T c_1(s)\,ds$ we have the situation of Theorem 2.1 since $\int_\alpha^\beta c_3 v(s)\,ds = c_3(u(\beta) - u(\alpha)) \le 2c_3 M$. Therefore we obtain

$$u'(t) = v(t) \le C_0 \exp\Big(\int_0^T c_2(s)\,ds\Big)\exp(2c_3 M) =: M_1.$$

For the case when $u'(\alpha) = 0$ and $u' \le 0$ we can put $v = -u'$ and apply the same argument. For the cases where $u'(\beta) = 0$ we can make a change of variable ('reverse time') to reduce to the previous cases. □

Remark 4.4. In [5] the following condition is supposed in place of (ii).

(iii) Suppose there exist constants A, B such that $|f(t,u,p)| \le B + Ap^2$ for all $(t,u,p) \in [0,T] \times [-M,M] \times \mathbb{R}$.

Thus our hypothesis (ii) is slightly more general.

We now state a result on existence which is a small improvement on the results in [5]. Since we will discuss another boundary value problem in Theorem 4.8 below, and the proof can be done in the same way as there, we omit this proof.

Theorem 4.5. *Suppose that f is continuous on $[0,T] \times \mathbb{R} \times \mathbb{R}$ and that (4.3) and (ii) of Lemma 4.3 hold. Then each of problems (I)–(V) has at least one solution $u \in C^2[0,T]$.*

From the above and Remark 4.2 both (i) and (ii) of Lemma 4.3 hold so the necessary a priori bound holds; see the proof of Theorem 4.8 below.

We now turn our attention to the following nonlocal BVP.

$$\begin{aligned} u''(t) &= f(t,u(t),u'(t)), \quad t \in [0,T], \\ u(0) - b_0 u'(0) = \beta_0[u], &\quad u(T) + b_1 u'(T) = \beta_1[u], \end{aligned} \tag{4.4}$$

where $b_i > 0$ and $\beta_i[u] := \int_0^T u(s)dB_i(s)$ are Riemann–Stieltjes integrals, B_i are non-decreasing functions, that is dB_i are (positive) Stieltjes measures. We will assume that

$$\beta_i[1] := \int_0^T dB_i(s) \leq 1, \quad \text{for } i \in \{0,1\}. \tag{4.5}$$

Note that if $\int_0^T dB_i(s) = 1$, for $i \in \{0,1\}$ then the problem is at resonance, the constant function 1 is an eigenfunction with eigenvalue 0. We deal with both the resonant and non-resonant cases.

The Riemann–Stieltjes BCs include multipoint BCs where B is a step function, equivalently dB consists of point masses at points $\eta_j \in (0,T)$ and $\beta[u] = \sum_{j=1}^m \beta_j u(\eta_j)$ with $\beta_j \geq 0$, and also includes integral BCs where $\beta[u] = \int_0^T b(s)u(s)\,ds$. Here we would be assuming that $\sum_{j=1}^m \beta_j \leq 1$, or that $b(s) \geq 0$ and $\int_0^T b(s)\,ds \leq 1$.

Lemma 4.6. *Let $u \in C^2[0,T]$ be a solution of problem (4.4) with $b_0 > 0$ (or $b_1 > 0$) and suppose that (4.5) holds. Then if u attains a positive maximum or negative minimum at 0 (or T) we have $u'(0) = 0$ (or $u'(T) = 0$).*

Proof. If $u(0)$ is a positive maximum then $u'(0) \leq 0$ and since, by the assumption (4.5),

$$\beta_0[u] \leq \int_0^T u(s)dB_0(s) \leq u(0)\int_0^T dB_0(s) \leq u(0)$$

we also have $b_0u'(0) = u(0) - \beta_0[u] \geq 0$, hence $u'(0) = 0$. The case of negative minimum is similar. □

Lemma 4.7. *Suppose that f satisfies (4.3). If $u \in C^2[0,T]$ is a solution of problem (4.4) then $|u(t)| \leq M$ for $t \in [0,T]$.*

Proof. As in Lemma 4.1 this holds if $|u|$ does not achieve its maximum at $t = 0$ or $t = T$. So suppose u has a positive maximum $u_{\max} > M > 0$ at $t = 0$. By Lemma 4.6 we have $u'(0) = 0$. Since $u''(0) = f(t,u(0),u'(0)) = f(t,u_{\max},0)$ and $u_{\max}f(t,u_{\max},0) > 0$ we obtain $u''(0) > 0$. Hence u' is strictly increasing on a neighbourhood $(0,\delta)$ of 0, and since $u'(0) = 0$ we obtain $u'(t) > 0$ for $t \in (0,\delta)$ so that u is increasing on $(0,\delta)$, which contradicts $u(0)$ being a positive maximum. Thus $u(0) \leq M$. The other cases are similar. □

We are now able to prove an existence theorem.

Theorem 4.8. *Suppose that f is continuous on $[0,T] \times \mathbb{R} \times \mathbb{R}$ and that*

(1) there exists a constant $M > 0$ such that $uf(t,u,0) > 0$, for all $t \in [0,T]$ and all $|u| > M$,

(2) there exist non-negative constants c_0, c_3 and functions $c_1, c_2 \in L^1_+[0,T]$ such that

$$|f(t,u,p)| \leq c_0 + c_1(t)|u| + c_2(t)|p| + c_3|p|^2, \quad \text{for all } (t,u,p) \in [0,T] \times [-M,M] \times \mathbb{R}.$$

Then the BVP

$$\begin{gathered} u''(t) = f(t,u(t),u'(t)), \quad t \in [0,T], \\ u(0) - b_0u'(0) = \beta_0[u], \quad u(T) + b_1u'(T) = \beta_1[u], \end{gathered} \tag{4.6}$$

where $b_i > 0$ and $\beta_i[u] \leq 1$, for $i \in \{0,1\}$, has at least one solution in $C^2[0,T]$.

Proof. If $\beta_0[1] = 1$ and $\beta_1[1] = 1$ the problem (4.6) is at resonance so we consider the problem in the equivalent form

$$u'' - \varepsilon u = f_\varepsilon(t, u, u') := f(t, u, u') - \varepsilon u, \tag{4.7}$$

with the same BCs, for a suitable (fixed) $\varepsilon \in (0, 1)$ for which the problem is non-resonant; in the non-resonant case we can take $\varepsilon = 0$. The problem (4.7) then has a Green's function G and u is a solution of the BVP (4.6) if and only if

$$u(t) = \int_0^T G(t,s) f_\varepsilon(s, u(s), u'(s))\, ds.$$

The methods of [13, 14] allow the Green's function to be determined explicitly when the Green's function for the corresponding *local* BVP (where $\beta_i[u]$ are replaced by 0) is known. This could be a quite complicated expression, especially for $\varepsilon > 0$, but here we do not need to know this. We define a nonlinear operator N on $C^1[0, T]$ by

$$Nu(t) = \int_0^T G(t,s) f_\varepsilon(s, u(s), u'(s))\, ds.$$

Then $N : C^1 \to C^1$ is a continuous compact map (also called completely continuous) and u is a fixed point of N if and only if u is classical solution of (4.6).

We shall apply Leray–Schauder degree theory; details can be found in many texts, for example Deimling [4]. To do this we need to find a bounded open set Ω containing 0 and, in order to apply the homotopy property, show that $u - \lambda N u \neq 0$ for all $u \in \partial\Omega$ and all $\lambda \in [0, 1]$. Obviously this is true for $\lambda = 0$.

For $0 < \lambda \leq 1$, if u is a solution of the equation $u = \lambda N u$, then u satisfies the ODE $u'' - \varepsilon u = \lambda f_\varepsilon(t, u, u')$ together with the BCs. Thus u satisfies the ODE

$$u''(t) = F_\lambda(t, u(t), u'(t)) := \lambda f(t, u(t), u'(t)) + \varepsilon(1 - \lambda) u(t).$$

Clearly F_λ satisfies the hypothesis (1) with the same given M for every $\lambda \in (0, 1]$. Also F_λ satisfies the hypothesis (2) with $c_1(t)$ replaced by $c_1(t) + 1$, again for every $\lambda \in (0, 1]$. By Lemmas 4.3, 4.6 and 4.7, $|u(t)| \leq M$ and there is a constant M_1 depending only on M, c_i, T such that $|u'(t)| \leq M_1$, for all $t \in [0, T]$.

We define Ω to be $\Omega := \{u \in C^1[0, T] : \|u\|_\infty < 1 + M, \|u'\|_\infty < 1 + M_1\}$. From the above a priori bounds we see that $u \neq \lambda N u$ for all $u \in \partial\Omega$ and all $0 \leq \lambda \leq 1$. By the homotopy property of Leray–Schauder degree we have

$$\deg_{LS}(I - N, \Omega, 0) = \deg_{LS}(I, \Omega, 0) = 1,$$

so, by the existence property of degree, there exists $u \in \Omega$ such that $u = Nu$, and u is classical solution of (4.4). □

Remark 4.9. In [5] the authors use a topological transversality theorem and the notion of essential map, here we prefer the more familiar Leray–Schauder degree. In [11] the generalized degree for A-proper mappings is used.

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