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On Barreira-Valls Polynomial Stability of Evolution Operators in Banach Spaces

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Abstract

Our main objective is to consider a concept of nonuniform behavior and obtain appropriate versions of the well-known stability due to R. Datko and L. Barbashin. This concept has been considered in the works of L. Barreira and C. Valls. Our approach is based on the extension of techniques for exponential stability to the case of polynomial stability.

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1 Introduction

In the theory of differential equations both in finite-dimensional and infinite-dimensional spaces, there is a very extensive literature concerning uniform exponential stability.

For some of the most relevant early contributions in this area we refer to the books of J.L.Massera and J.J.Schäffer [9] and by J. Daletski and M.G. Krein [7].

In their notable contribution [2], L. Barreira and C. Valls obtain results in the case of a notion of nonuniform exponential dichotomy, which is motivated by ergodic theory.

A principal motivation for weakening the assumption of uniform exponential behavior is that from the point of view of ergodic theory, almost all linear variational equations in a finite-dimensional space admit a nonuniform exponential dichotomy.

In this paper we consider a concept of nonuniform stability for evolution operators in Banach spaces. This concept has been considered in the works [2] and [3] due to L. Barreira and C. Valls. This causes that the stability results discussed in the paper hold for a much larger class of differential equations than in the classical theory of uniform exponential stability.

The obtained results are generalizations of some well-known theorems in the case of uniform exponential stability given in [1], [5], [7], [8], [10], [13], [14] and in the case of nonuniform exponential stability given in [4], [6],

[11], [12] and [15]. Our approach is based on the extension of techniques for exponential stability to the case of polynomial stability.

2 Evolution operators

In this section we recall some definitions which will be used in what follows.

Let X be a real or complex Banach space and let I be the identity operator on X. The norm on X and on $\mathcal{B}$(X) the Banach algebra of all bounded linear operators on X , will be denoted by $\|\cdot\|$.

Let

$$\Delta = \left\{(t,s) \in \mathbf{R}_+^2 : t \geq s\right\}$$
$$T = \{(t,s,t_0) \in \mathbf{R}_+^3 : t \geq s \geq t_0\}$$

We recall that an operator-valued function $\Phi : \Delta \to \mathcal{B}(X)$ is called an *evolution operator* on the Banach spaces X iff:

$e_1)\Phi\left(t,t\right) = I$ for every $t \geq 0$;

$e_2)\Phi\left(t,s\right)\Phi\left(s,t_0\right) = \Phi\left(t,t_0\right)$ for all $(t,s)$and $(s,t_0) \in \Delta$.

Remark 2.1 *In the examples considered in this paper we consider evolution operators on X defined by*

$$\Phi : \Delta \to \mathcal{B}(X), \Phi(t,s)x = \frac{u(s)}{u(t)}x$$

where $u : \boldsymbol{R}_+ \to \boldsymbol{R}_+^* = (0,\infty)$.

An evolution operator $\Phi : \Delta \to \mathcal{B}(X)$ with the property
$e_3)$ there exists a nondecreasing function $\varphi : \mathbf{R}_+ \to [1,\infty)$ such that:

$$\|\Phi(t,s)\| \leq \varphi(t-s) \text{ for all } (t,s) \in \Delta$$

then Φ is called the evolution operator with *uniform growth.*

The evolution operator $\Phi : \Delta \to \mathcal{B}(X)$ is said to be *strongly measurable*, iff
$e_4)$ for all $(s,x) \in \mathbf{R}_+ \times X$ the mapping defined by $t \mapsto \|\Phi\left(t,s\right)x\|$ is measurable on $[s,\infty)$.

An evolution operator Φ is called $*$-*strongly measurable*, iff
$e_5)$ for all $(t,x^*) \in \mathbf{R}_+ \times X^*$ the mapping defined by $s \mapsto \|\Phi(t,s)^*x^*\|$ is measurable on $[0,t]$

3 Polynomial stability

Let $\Phi : \Delta \to B(X)$ be an evolution operator on Banach space X.

Definition 3.1 *The evolution operator Φ is called:*
(i) uniformly polynomially stable (and denoted as u.p.s.) iff there are $N \geq 1$ and $\alpha > 0$ such that

$$t^\alpha \|\Phi(t,s)x\| \leq N s^\alpha \|x\|$$

for all $(t,s,x) \in \Delta \times X$;
(ii) (nonuniformly) polynomially stable (and denoted as p.s.) iff there exist $\alpha > 0$ and a nondecreasing function $N : \boldsymbol{R}_+ \to [1,\infty)$ such that

$$t^\alpha \|\Phi(t,s)x\| \leq N(s)\|x\|$$

for every $(t,s,x) \in \Delta \times X$;
(iii) polynomially stable in the sense of Barreira and Valls (and denoted as B.V.p.s.) iff there are $N \geq 1$, $\alpha > 0$ and $\beta \geq \alpha$ such that:

$$t^\alpha \|\Phi(t,s)x\| \leq N s^\beta \|x\|$$

for all $(t,s,x) \in \Delta \times X$.

Remark 3.1 *The evolution operator Φ is polynomially stable in the sense of Barreira and Valls if and only if there are $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that*

$$t^\alpha \|\Phi(t,s)x\| \leq N s^{(\alpha+\beta)} \|x\|$$

for all $(t,s,x) \in \Delta \times X$.

Remark 3.2 *The evolution operator Φ is :*
(i) uniformly polynomially stable iff there are $N \geq 1$ and $\alpha > 0$ such that

$$t^\alpha \|\Phi(t,t_0)x_0\| \leq N s^\alpha \|\Phi(s,t_0)x_0\|$$

for all $(t,s,t_0,x_0) \in T \times X$;
(ii) (nonuniformly) polynomially stable iff there exist $\alpha > 0$ and a nondecreasing function $N : \boldsymbol{R}_+ \to [1,\infty)$ such that

$$t^\alpha \|\Phi(t,t_0)x_0\| \leq N(s)\|\Phi(s,t_0)x_0\|$$

for every $(t,s,t_0,x_0) \in T \times X$;
(iii) polynomially stable in the sense of Barreira and Valls iff there are $N \geq 1$, $\alpha > 0$ and $\beta \geq \alpha$ such that:

$$t^\alpha \|\Phi(t,t_0)x_0\| \leq N s^\beta \|\Phi(s,t_0)x_0\|$$

for all $(t,s,t_0,x) \in T \times X$.

Remark 3.3 *It is obvious that*

$$u.p.s. \quad \Rightarrow \quad B.V.p.s. \quad \Rightarrow \quad p.s.$$

The converse implications between these stability concepts are not valid. This is proved in the following two examples.

The following example shows an evolution operator that is B.V.p.s which is not u.p.s.

Example 3.1 *Let $u : \boldsymbol{R}_+ \to \boldsymbol{R}_+^*$ be the function defined by*

$$u(t) = (t+2)^{5-sinln(t+2)}$$

. Then $\Phi : \Delta \to \mathcal{B}(X), \Phi(t,s)x = \frac{u(s)}{u(t)}x$ is an evolution operator on X with:

$$\|\Phi(t,s)x\| \leq \frac{(s+2)^6}{(t+2)^4}\|x\| \leq \frac{s^2(s+2)^2}{t^2}\|x\| \leq 9s^4t^{-2}\|x\|$$

for all $(t,s,x) \in \Delta \times X$ with $s \geq t_0 = 2$ and hence Φ is B.V.p.s.

If we suppose that Φ is u.p.s then there exist $N \geq 1$ and $\alpha > 0$ such that:

$$(s+2)^5(t+2)^{sinln(t+2)} \leq Nt^{-\alpha}s^{\alpha}(t+2)^5(s+2)^{sinln(s+2)}$$

for all $(t,s) \in \Delta$.

From here, for $t = exp(2n\pi + \frac{\pi}{2}) - 2$ and $s = exp(2n\pi - \frac{\pi}{2}) - 2$ we obtain

$$e^{4n\pi} \leq Ne^{5\pi}\left(\frac{e^{-\frac{\pi}{2}} - 2e^{-2n\pi}}{e^{\frac{\pi}{2}} - 2e^{-2n\pi}}\right)^{\alpha}$$

which for $n \to \infty$ yields to a contradiction.

Example 3.2 *(Evolution operator which is p.s. and is not B.V.p.s.)*
Let $u : \boldsymbol{R}_+ \to [1,\infty)$ be a function with

$$u(n) = e^{n^2} \text{ and } u\left(n + \tfrac{1}{n^2}\right) = e^4$$

for every $n \in \boldsymbol{N}^*$. Then

$$\Phi : \Delta \to \mathcal{B}(X), \Phi(t,s)x = \frac{su(s)}{tu(t)}x$$

is an evolution operator on X with the property

$$t\|\Phi(t,s)x\| = \frac{su(s)\|x\|}{u(t)} \leq N(s)\|x\|$$

for all $(t,s,x) \in \Delta \times X$*, where* $N(s) = 1 + su(s)$*. This shows that* Φ *is p.s.*

If we suppose that Φ *is B.V.p.s then there are* $N \geq 1$ *and* $\beta \geq \alpha > 0$ *such that*

$$t^{\alpha} su(s) \leq N t s^{\beta} u(t)$$

for all $(t,s) \in \Delta$*.*

Then for $s = n$ *and* $t = n + \frac{1}{n^2}$ *we obtain*

$$\left(1 + \frac{1}{n^3}\right)^{\alpha-1} \leq N \frac{n^{\beta-\alpha}}{e^{n^2}} e^4$$

which for $n \to \infty$ *gives a contradiction and hence* Φ *is not B.V.p.s.*

Theorem 3.1 *Let* $\Phi : \Delta \to \mathcal{B}(X)$ *be a strongly measurable evolution operator with uniform growth. If there are* $D \geq 1$*,* $\gamma > 0$ *and* $\delta \geq 0$ *such that:*

$$\int_s^t \left(\frac{\tau}{s}\right)^{\gamma} \|\Phi(\tau, t_0)x_0\| \, d\tau \leq D s^{\delta} \|\Phi(s, t_0)x_0\|$$

for all $(t,s,x) \in \Delta \times X$*, then* Φ *is polynomially stable in the sense of Barreira and Valls.*

Proof

If $t \geq s+1$ then

$$\left(\frac{t}{s}\right)^{\gamma} \|\Phi(t,s)x\| = \int_{t-1}^{t} \left(\frac{t}{s}\right)^{\gamma} \|\Phi(t,s)x\| \, d\tau =$$

$$= \int_{t-1}^{t} \left(\frac{t}{\tau}\right)^{\gamma} \left(\frac{\tau}{s}\right)^{\gamma} \|\Phi(t,\tau)\Phi(\tau,s)x\| \, d\tau \leq$$

$$\leq 2^{\gamma}\varphi(1) \int_s^t \left(\frac{\tau}{s}\right)^{\gamma} \|\Phi(\tau,s)x\| \, d\tau \leq D\varphi(1) 2^{\gamma} s^{\delta s} \|x\|$$

for all $x \in X$.

If $t \in [s, s+1)$ then

$$\left(\frac{t}{s}\right)^{\gamma} \|\Phi(t,s)x\| = 2^{\gamma} \|\Phi(t,s)x\| \leq 2^{\gamma}\varphi(1) \|x\| \leq D\varphi(1) 2^{\gamma} s^{\delta} \|x\|$$

for all $x \in X$.

Finally, we obtain that

$$t^{\gamma} \|\Phi(t,s)x\| \leq N s^{(\gamma+\delta)} \|x\|$$

for all $(t,s,x) \in \Delta \times X$, where $N = D\varphi(1)2^{\gamma}$.

Remark 3.4 *Theorem 3.1 is a generalization for the case of polynomial stability in the sense of Barreira and Valls of the classic result proved by R.Datko in Theorem 11 of [8] for the case of uniform exponential stability. The case of exponential stability has been considered by Buse in [4].*

Remark 3.5 *The converse of the preceding theorem is valid in the case when the constant α given by Definition 3.1 (iii) satisfies the condition $\alpha > 1$ and we consider $0 < \gamma < \alpha - 1$ and $\delta = \beta + 1$.*

For the case when $\alpha \in (0,1)$ and $\beta > 0$ the converse of Theorem 3.1 is not valid, result illustrated by

Example 3.3 *The evolution operator*

$$\Phi : \Delta \to \mathcal{B}(X), \Phi(t,s)x = \frac{s+1}{t+1}x$$

satifies the condition

$$\sqrt[3]{t}\, \|\Phi(t,s)x\| \leq \sqrt{s}\, \|x\|$$

for all $t \geq s \geq t_0 = 1$ and all $x \in X$. Hence Φ is B.V.p.s. with $\alpha = \frac{1}{3} \in (0,1)$ and $\beta = \frac{1}{2}$.

We observe that

$$\int_s^{\infty} \tau^{\gamma} \|\Phi(\tau,s)x\| \, d\tau \geq \|x\| (s+1) \int_s^{\infty} \frac{d\tau}{\tau+1} = \infty$$

Some immediate characterizations of the polynomial stability in the sense of Barreira and Valls are given by:

Proposition 3.1 *Let $\Phi : \Delta \to \mathcal{B}(X)$ be an evolution operator.*
The following statements are equivalent:
(i) Φ is polynomially stable in the sense of Barreira and Valls;
(ii) there are $N \geq 1$, $\nu > 0$ and $\beta \in [0,\nu)$ such that:

$$t^{\nu} s^{-\nu} \|\Phi(t,s)x\| \leq N t^{\beta} \|x\| \tag{1}$$

for all $(t,s,x) \in \Delta \times X$;
(iii) there are $N \geq 1$, $a, b > 0$ and $b \geq a$ such that:

$$t^{a} \|\Phi(t,s)x\| \leq N s^{b} \|x\| \tag{2}$$

for all $(t,s,x) \in \Delta \times X$.

Proof
(*i*) $\Rightarrow$ (*ii*) If Φ is B.V.p.s. then there are $N \geq 1$, $\alpha > 0$ and $\beta \geq 0$ such that, for all $(t, s, x) \in \Delta \times X$, we have that:

$$Nt^\beta \|x\| = Nt^\beta s^{-\beta} s^\beta \|x\| \geq t^\beta s^{-\beta} t^\alpha s^{-\alpha} \|\Phi(t,s)x\| = t^\nu s^{-\nu} \|\Phi(t,s)x\|$$

where $\nu = \alpha + \beta > \beta$.
(*ii*) $\Rightarrow$ (*iii*) We have that:

$$\|\Phi(t,s)x\| \leq Nt^{-\nu} s^\nu t^\beta \|x\| = Nt^{(\beta-\nu)} s^\nu \|x\| = Nt^{-a} s^b \|x\|$$

with $b = \nu \geq a = \nu - \beta$.
(*iii*) $\Rightarrow$ (*i*) If we denote by $c = b - a \geq 0$ then:

$$\|\Phi(t,s)x\| \geq Ns^{(c+a)} t^{-a} \|x\|$$

for all $(t, s, x) \in \Delta \times X$, where $c \geq 0$ and $a > 0$. Hence Φ is B.V.p.s.

Corollary 3.1 *Let $\Phi : \Delta \to \mathcal{B}(X)$ be an evolution operator. The following statements are equivalent:*
(i) Φ is polynomially stable in the sense of Barreira and Valls;
(ii) there are $N \geq 1$, $\nu > 0$ and $\beta \in [0, \nu)$ such that:

$$t^\nu s^{-\nu} \|\Phi(t, t_0)x_0\| \leq Nt^\beta \|\Phi(s, t_0)x_0\| \tag{3}$$

for all $(t, s, x_0) \in \Delta \times X$;
(iii) there are $N \geq 1$, $a, b > 0$ and $b \geq a$ such that:

$$t^a \|\Phi(t, t_0)x_0\| \leq Ns^b \|\Phi(s, t_0)x_0\| \tag{4}$$

for all $(t, s, x_0) \in \Delta \times X$.

Proof. It results from Proposition 3.1 and Remark 3.2.

An integral characterization for B.V.p.s. is given by

Theorem 3.2 *Let $\Phi : \Delta \to \mathcal{B}(X)$ be a $*$-strongly measurable evolution operator with uniform growth. Then Φ is polynomially stable in the sense of Barreira and Valls if and only if there are $B \geq 1$, $b > 0$ and $\delta \in [0, b)$ such that*

$$\int_0^t \left(\frac{t}{\tau}\right)^b \|\Phi(t,\tau)^* x^*\| \, d\tau \leq Bt^\delta \|x^*\|$$

for all $(t, x^*) \in \boldsymbol{R}_+ \times X^*$.

Proof
Necessity. If Φ is exponentially stable in the sense of Barreira and Valls there exist $N \geq 1,\ t_1 \geq 1,\ \alpha > 0$ si $\beta \in [0, \alpha)$ such that for all $(b, x^*) \in (\beta+1, \alpha+1) \times X^*$ we have:

$$\int_0^t \left(\frac{t}{\tau}\right)^b \|\Phi(t,\tau)^* x^*\| \, d\tau \leq N t^\beta \int_0^t \left(\frac{t}{\tau}\right)^{(b-\alpha)} \|x^*\| \, d\tau \leq \frac{N}{\alpha - b + 1} t^{\beta+1} \|x^*\| \leq$$
$$\leq B t^\delta \|x^*\|$$

where $B = 1 + \frac{N}{\alpha - b + 1}$ and $\delta = \beta + 1$.
Sufficiency.
If $t \geq s+1$ and $s \geq 0$ then:

$$\left(\frac{t}{s}\right)^b |\langle x^*, \Phi(t,s)x\rangle| = \int_s^{s+1} \left(\frac{t}{s}\right)^b |\langle \Phi(t,\tau)^* x^*, \Phi(\tau,s)x\rangle| \, d\tau \leq$$
$$\leq \int_s^{s+1} \left(\frac{t}{\tau}\right)^b \left(\frac{\tau}{s}\right)^b \|\Phi(t,\tau)^* x^*\| \, \|\Phi(\tau,s)x\| \, d\tau \leq$$
$$\leq \varphi(1) 2^b \int_s^{s+1} \left(\frac{t}{\tau}\right)^b \|\Phi(t,\tau)^* x^*\| \, \|x\| \, d\tau \leq$$
$$\leq \varphi(1) 2^b \int_0^t \left(\frac{t}{\tau}\right)^b \|\Phi(t,\tau)^* x^*\| \, \|x\| \, d\tau \leq B\varphi(1) 2^b t^\delta \|x\| \, \|x^*\|$$

for all $(x, x^*) \in X \times X^*$. Taking supremum in raport with $\|x^*\| \leq 1$ we have:

$$\|\Phi(t,s)x\| \leq B\varphi(1) 2^b s^b t^{\delta - b} \|x\|$$

Taking $\beta = b - \delta$ we have

$$\|\Phi(t,s)x\| \leq B\varphi(1) 2^b s^{\delta+\beta} t^{-\beta} \|x\|$$

for all $t \geq s+1$ and all $x \in X$.
If $t \in [s, s+1)$ then

$$\|\Phi(t,s)x\| \, t^\beta \leq \varphi(1) t^\beta \|x\| = \left(\frac{t}{s}\right)^\beta s^\beta \varphi(1) \|x\| \leq B\varphi(1) 2^b s^{\delta+\beta} \|x\|$$

for all $x \in X$.
Finally, it results that:

$$\|\Phi(t,s)x\| \leq N t^{-\beta} s^{\beta+\delta} \|x\|$$

for all $(t, s, x) \in \Delta \times X$, where $N = B\varphi(1)2^b + 1$, $b > 0$ and $\delta \in [0, b)$.
In conclusion, Φ is B.V.p.s.

Remark 3.6 *Theorem 3.2 is a generalization for the case of polynomial stability in the sense Barreira-Valls of a classic result due to Barbashin [1](see also [5] and [13]) for uniform exponential stability. The case of exponential stability has been considered by Buse in [6].*

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References

[1] E. A. Barbashin, Introduction to the Theory of Stability, Wolters-Noordhoff Publishing, Groningen, The Netherlands 1970.

[2] L. Barreira, C. Valls, Stability of Nonautonomous Differential Equations, Lecture Notes in Math., 1926, Springer, (2008).

[3] L. Barreira, C. Valls, Polynomial growth rates, Nonlinear Analysis, 7(2009), 5208-5219.

[4] C. Buse, On nonuniform exponential stability of evolutionary processes, Rend.Sem. Mat. Univ. Pol. Torino, 52, (1994), 395-406.

[5] C. Buse, M. Megan, M. Prajea, P. Preda, The strong variant of a Barbashin theorem on stability of solutions for nonautonomous differential equations in Banach spaces, Integral Equations and Operator Theory, 59, Nr 4, (2007), 491-500.

[6] C. Buse, Real Integrability Conditions for the Nonuniform Exponential Stability of Evolution Families on Banach Spaces, International Series of Numerical Mathematics, 2009, Volume 157, Part I, 31-42.

[7] J. D. Daleckii and M. G. Krein, Stability of Differential Equations in Banach Spaces, Providence, 1974.

[8] R. Datko, Uniform asymptotic stability of evolutionary processes in a Banach space, SIAM J. Math. Anal., 3(1973), 428-445.

[9] J. L. Massera and J. J. Schäffer, Linear Differential Equations and Function Spaces, Academic Press, New York, 1966.

[10] M. Megan, A. L. Sasu and B. Sasu, Asymptotic Behaviour of Evolution Families, Editura Mirton, 2003.

[11] A. A. Minda, M. Megan, On (h,k)- exponential stability of evolution operators in Banach spaces, J. Adv. Math. Studies, Vol 3(2010), No. 2, 65-70.

[12] A. A. Minda, M. Megan, On (h,k)-stability of evolution operators in Banach spaces, Appl. Math. Lett. 24(2011)44-48.

[13] P. Preda, M. Megan, Exponential dichotomy of evolutionary processes in Banach spaces, Czechoslovak Math. J. 35 (1985) 312-323.

[14] C. Stoica, M. Megan, On uniform exponential stability for skew-evolution semiflows on Banach spaces, Nonlinear Analysis: Theory, Methods & Applications, 72, (2010), 1305-1313.

[15] C. Stoica, M. Megan, Nonuniform behaviors for skew-evolution semiflows on Banach spaces, Operator Theory Live, Theta(2010), 203-211.

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