Electronic Journal of Qualitative Theory of Differential Equations
2018, No. 17, 1–12; https://doi.org/10.14232/ejqtde.2018.1.17 www.math.u-szeged.hu/ejqtde/

Analysis on stability and non-existence of equilibrium for a general chemical reaction

Wenbin Yang[✉ 1] and **Yan'e Wang**[2]

[1]School of Science, Xi'an University of Posts and Telecommunications, Xi'an, 710121, China
[2]School of Computer Science, Shaanxi Normal University, Xi'an, 710119, China

Received 30 April 2017, appeared 22 April 2018
Communicated by László Simon

Abstract. This paper is concerned with a general chemical reaction model with respect to Neumann boundary condition. The stability of positive equilibrium and the non-existence of non-constant positive solution are discussed rigorously, respectively in Case 1: $f(u) > 0$ and $f_u(u) < 0$ for $u > 0$ and Case 2: $f(0) = 0$ and $f_u(u) > 0$ for $u > 0$. The techniques include the spectrum analysis of operators, the maximal principle, the upper and lower solution method and the implicit function theorem.

Keywords: general chemical reaction, reaction–diffusion equations, stability, non-existence.

2010 Mathematics Subject Classification: 35K57, 35B35, 35A01.

1 Introduction

A chemical reaction is a process that leads to the transformation of one set of chemical substances to another [2]. During a chemical reaction, a new substance is formed. Chemical equations are used to graphically illustrate chemical reactions by various mathematical models, such as CIMA reaction model [5,11], the activator-inhibitor model [19,22], and the Schnakenberg model [8,10,16]. The classical Schnakenberg kinetics [16] is a system of interacting chemicals describing the relations of activation between substances, this system has been used and studied extensively, such as [1,4,6,7,9,14,15,18,24].

Based on the classical Schnakenberg equations, Wu, Ma and Guo (2013) [17] have given the following reaction-diffusion system with a general reactive function:

$$\begin{cases} u_t = d_1\Delta u + \alpha_1 - \beta u + \gamma f(u)v, & x \in \Omega,\ t > 0, \\ v_t = d_2\Delta v + \alpha_2 - \gamma f(u)v, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq\not\equiv 0,\ v(x,0) = v_0(x) \geq\not\equiv 0, & x \in \Omega. \end{cases} \tag{1.1}$$

✉Corresponding author. Email: yangwenbin-007@163.com

The reaction mechanism of system (1.1) could be expressed as follows: there is a diffusing substance u, which activates itself and consumes a substrate v at interaction ratio γ while supplying to the system at constant rate β in a bounded domain $\Omega \in \mathbb{R}^N$ with smooth boundary $\partial\Omega$.

Here, α_1 and α_2 are the feed concentrations for u and v, respectively. The reactor is assumed to be closed, thus a zero-flux boundary condition is imposed and $\partial/\partial n$ represents the outer normal derivative. Moreover, $\alpha_1, \alpha_2, \beta, \gamma$ are all positive constants. The term $f(u)v$ represents the reactive law of u and v, where $f(u)$ is smooth and satisfies $f(u) > 0$ for $u > 0$.

The system (1.1) has a unique constant equilibrium $U^* = (u^*, v^*)$, where

$$u^* = \frac{\alpha_1 + \alpha_2}{\beta}, \qquad v^* = \frac{\alpha_2}{\gamma f(u^*)}.$$

The steady-state system corresponding to (1.1) is given by

$$\begin{cases} -d_1 \Delta u = \alpha_1 - \beta u + \gamma f(u) v, & x \in \Omega, \\ -d_2 \Delta v = \alpha_2 - \gamma f(u) v, & x \in \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

In this paper, we consider the following two cases simultaneously:

Case 1: $f(u) > 0$ **and** $f_u(u) < 0$ **for** $u > 0$ *or*
Case 2: $f(0) = 0$ **and** $f_u(u) > 0$ **for** $u > 0$.

For the classical type of Case 2, take $f(u) = u^2$ for an example. Zhou (1985) [24] considered the existence, uniqueness and stability of limit cycles for the spatially homogeneous system corresponding to (1.1). When $\alpha_1 = 0$, Iron, Wei and Winter (2004) [4] showed that the stability of symmetric N-peaked steady states to system (1.1) with $\Omega = (-1, 1)$, both analytically and numerically. When $f(u) = u^2$ and $\alpha_1 > 0$ or $\alpha_1 < 0$, Ruuth (1995) [14] performed some experiments on system (1.1) by using finite differences (implicit-explicit schemes) and pseudospectral methods in one and two dimensions. Madzvamuse, Wathen and Maini (2003) [9] applied a novel moving grid finite element method to obtain the morphogenesis of system (1.1) in two-dimensional continuously deforming Euclidean domains. Shakeri and Dehghan (2011) [15] developed a hybrid finite volume spectral element method, and applied to system (1.1) for the variety of spatio-temporal patterns. Recently, for the general Case 2, [17] discussed the steady-state bifurcation both from the simple and double eigenvalues.

According to the requirement of the research, the Case 1 is not studied up to now. Besides, the stability and non-existence of equilibrium of system (1.1) is also unknown for the general Case 2. We will try to solve the two problems this paper.

The present paper is organized as follows. In Section 2 and 3, we briefly discuss the stability of positive equilibrium for system (1.1) and the non-existence of non-constant positive solution to system (1.2), respectively in Case 1: $f_u(u) < 0$ for $u > 0$ and Case 2: $f(0) = 0$, $f_u(u) > 0$ for $u > 0$. Finally, we give a conclusion in Section 4.

2 Case 1: $f(u) > 0$ and $f_u(u) < 0$ for $u > 0$

In this section, we mainly discuss the stability of positive equilibrium for system (1.1) and the non-existence of non-constant positive solution to system (1.2), under the conditions that $f(u) > 0$ and $f_u(u) < 0$ for $u > 0$.

2.1 The stability of positive equilibrium

Let $\Omega = (0, l)$ with some constant $l > 0$ this subsection. Consider the following eigenvalue problem

$$\begin{cases} -\Delta\phi = \mu u, & x \in \Omega, \\ \frac{\partial \phi}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

All the eigenvalues of problem (2.1) are $\mu_n, n = 0, 1, 2, \ldots$ and the corresponding eigenfunction are $\{\phi_n\}_0^\infty$, which construct the normal orthogonal basis of $L^2(\Omega)$.

Let $H^2(\Omega) = \{u \in W^{2,p}(\Omega), \frac{\partial u}{\partial n} = 0, x \in \partial\Omega\}$, $X = \{(u, v) : u, v \in H^2(\Omega)\}, Y = L^p(\Omega) \times L^p(\Omega)$. Then X is a Banach space, and Y is a Hilbert space. Define a mapping $F : (0, \infty) \times X \to Y$ by

$$F(U) = \begin{pmatrix} d_1\Delta u + \alpha_1 - \beta u + \gamma f(u)v \\ d_2\Delta v + \alpha_2 - \gamma f(u)v \end{pmatrix},$$

where $U = (u, v)$.

At the positive constant solution U^*, the Fréchet derivative of $F(U)$ with respect to U can be characterized by

$$L = \begin{pmatrix} d_1\Delta - \beta + \frac{\alpha_2 f_u(u^*)}{f(u^*)} & \gamma f(u^*) \\ -\frac{\alpha_2 f_u(u^*)}{f(u^*)} & d_2\Delta - \gamma f(u^*) \end{pmatrix}. \tag{2.2}$$

The characteristic equation of L is $L(\xi, \eta) = \lambda(\xi, \eta)$, where $(\xi, \eta) \in X$. That is to say, (ξ, η) satisfies

$$\begin{cases} d_1\Delta\xi + (-\beta + \frac{\alpha_2 f_u(u^*)}{f(u^*)})\xi + \gamma f(u^*)\eta = \lambda\xi, & x \in \Omega, \\ d_2\Delta\eta - \gamma f(u^*)\eta - \frac{\alpha_2 f_u(u^*)}{f(u^*)}\xi = \lambda\eta, & x \in \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega. \end{cases}$$

Let $\xi = \sum_{n=0}^\infty a_n\phi_n, \eta = \sum_{n=0}^\infty b_n\phi_n$. Then the above characteristic equation translates into

$$\sum_{n=0}^\infty M_n \begin{pmatrix} a_n \\ b_n \end{pmatrix} \phi_n = 0,$$

where

$$M_n = \begin{pmatrix} -d_1\mu_n - \beta + \frac{\alpha_2 f_u(u^*)}{f(u^*)} - \lambda & \gamma f(u^*) \\ -\frac{\alpha_2 f_u(u^*)}{f(u^*)} & -d_2\mu_n - \gamma f(u^*) - \lambda \end{pmatrix}. \tag{2.3}$$

Letting $|M_n| = 0,\ n = 0, 1, 2, \ldots$, we have

$$\lambda^2 - T_n\lambda + D_n = 0, \qquad n = 0, 1, 2, \ldots, \tag{2.4}$$

where

$$\begin{aligned} T_n &= -(d_1 + d_2)\mu_n - \beta + \frac{\alpha_2 f_u(u^*)}{f(u^*)} - \gamma f(u^*), \\ D_n &= \mu_n \left[d_1 d_2\mu_n + d_2\beta - \frac{d_2\alpha_2 f_u(u^*)}{f(u^*)} + d_1\gamma f(u^*)\right] + \gamma\beta f(u^*). \end{aligned} \tag{2.5}$$

Theorem 2.1. *Suppose that $f(u) > 0$ and $f_u(u) < 0$ for $u > 0$. Then the positive equilibrium U^* is asymptotically stable for system (1.1).*

Proof. Note that $f(u) > 0$ and $f_u(u) < 0$ for $u > 0$. Then for any $n \geq 0$, we have

$$D_n \geq \gamma\beta f(u^*) > 0 > -\beta + \frac{\alpha_2 f_u(u^*)}{f(u^*)} - \gamma f(u^*) \geq T_n.$$

This implies that Eq. (2.9) have two different roots λ_n^+, λ_n^-, with $\mathrm{Re}(\lambda_n^+) < 0$ and $\mathrm{Re}(\lambda_n^-) < 0$. The proof is completed. $\square$

Example A. Let $d_1 = 0.5, d_2 = 0.6, \alpha_1 = 1.5, \alpha_2 = 5.5, \beta = 0.65$ and $\gamma = 3.75$. If $f(u) = 1/u$, then the unique constant equilibrium $U^* = (u^*, v^*) = (10.7692, 15.7949)$; if $f(u) = 1/u^2$, then $U^* = (10.7692, 170.0986)$. By Theorem 2.1, U^* is asymptotically stable for system (1.1). See Figure 2.1 and 2.2 for the numerical simulations.

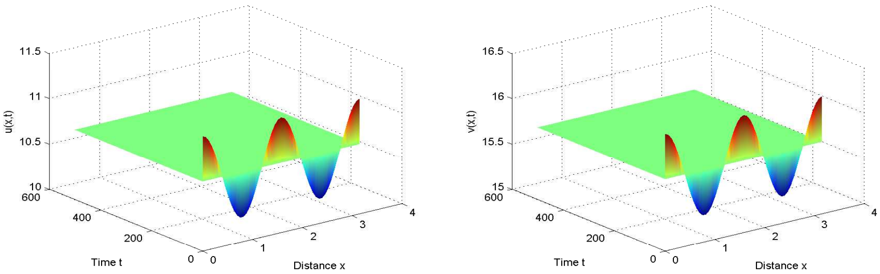

Figure 2.1: Numerical simulation of stability of $(u^*, v^*) = (10.7692, 15.7949)$: $f(u) = 1/u$. Here, $d_1 = 0.5, d_2 = 0.6, \alpha_1 = 1.5, \alpha_2 = 5.5, \beta = 0.65, \gamma = 3.75$ and the initial conditions $(u_0(x), v_0(x)) = (10.7692 + 0.5\cos 4x, 15.7949 + 0.5\cos 4x)$.

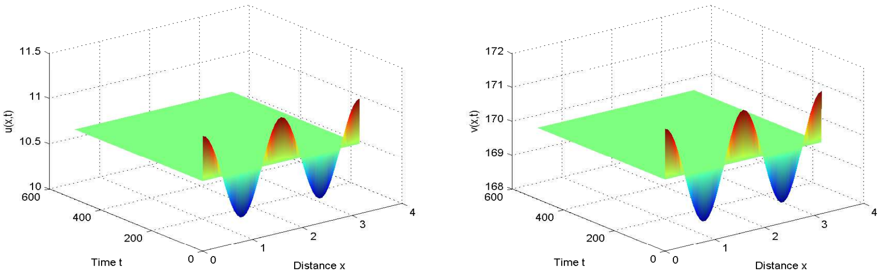

Figure 2.2: Numerical simulation of stability of $(u^*, v^*) = (10.7692, 170.0986)$: $f(u) = 1/u^2$. Here, $d_1 = 0.5, d_2 = 0.6, \alpha_1 = 1.5, \alpha_2 = 5.5, \beta = 0.65, \gamma = 3.75$ and the initial conditions $(u_0(x), v_0(x)) = (10.7692 + 0.5\cos 4x, 170.0986 + 1.5\cos 4x)$.

2.2 The non-existence of non-constant positive equilibrium

The following lemma is useful to obtain the non-existence of non-constant positive solution to system (1.2).

Lemma 2.2 ([11, Lemma 3.1]; [20,21, Lemma 2.1]). *Suppose that $F(x,w) \in C(\overline{\Omega} \times \mathbb{R})$, $q_i \in C(\overline{\Omega})$, $i = 1, 2, \ldots, N$. If $w \in C^2(\Omega) \times C^1(\overline{\Omega})$ satisfies*

$$\Delta w(x) + \sum_{i=1}^{N} q_i(x) w_{x_i}(x) + F(x, w(x)) \geq 0 \quad \text{in } \Omega, \qquad \frac{\partial w}{\partial n} \leq 0 \quad \text{on } \partial\Omega \tag{2.6}$$

and $w(x_0) = \max_{\overline{\Omega}} w$. Then $F(x_0, w(x_0)) \geq 0$. Similarly, if the two inequalities in Eq. (2.6) are reversed and $w(x_0) = \min_{\overline{\Omega}} w$, then $F(x_0, w(x_0)) \leq 0$.

Theorem 2.3. *Suppose that $f(u) > 0$ and $f_u(u) < 0$ for $u > 0$. Then U^* is the unique positive solution to system (1.2), that is, system (1.2) has no non-constant positive solutions.*

Proof. Let $(u(x), v(x))$ be a positive solution to system (1.2). Take $x_i \in \overline{\Omega}$ $(i = 1, 2, 3, 4)$ such that

$$\max_{\overline{\Omega}} u(x) = u(x_1), \qquad \max_{\overline{\Omega}} v(x) = v(x_2),$$
$$\min_{\overline{\Omega}} u(x) = u(x_3), \qquad \min_{\overline{\Omega}} v(x) = v(x_4).$$

By Lemma 2.2, we have

$$\begin{aligned} \alpha_1 - \beta u(x_1) + \gamma f(u(x_1))v(x_1) \geq 0 \Rightarrow u(x_1) \leq \frac{\alpha_1 + \gamma f(u(x_1))v(x_1)}{\beta}, \\ \alpha_2 - \gamma f(u(x_2))v(x_2) \geq 0 \Rightarrow v(x_2) \leq \frac{\alpha_2}{\gamma f(u(x_2))}. \end{aligned} \tag{2.7}$$

Note that $f(u) > 0$ and $f_u(u) < 0$ for $u > 0$. From (2.7), we have

$$u(x_1) \leq \frac{\alpha_1 + \gamma f(u(x_1))v(x_1)}{\beta} \leq \frac{\alpha_1 + \gamma f(u(x_2))v(x_2)}{\beta} \leq \frac{\alpha_1 + \alpha_2}{\beta}. \tag{2.8}$$

By Lemma 2.2 again, we have

$$\begin{aligned} \alpha_1 - \beta u(x_3) + \gamma f(u(x_3))v(x_3) \geq 0 \Rightarrow u(x_3) \geq \frac{\alpha_1 + \gamma f(u(x_3))v(x_3)}{\beta}, \\ \alpha_2 - \gamma f(u(x_4))v(x_4) \geq 0 \Rightarrow v(x_4) \geq \frac{\alpha_2}{\gamma f(u(x_4))}. \end{aligned}$$

Similarly, we have

$$u(x_3) \geq \frac{\alpha_1 + \gamma f(u(x_3))v(x_3)}{\beta} \geq \frac{\alpha_1 + \gamma f(u(x_4))v(x_4)}{\beta} \geq \frac{\alpha_1 + \alpha_2}{\beta}. \tag{2.9}$$

Inequations (2.8) and (2.9) imply $u \equiv \frac{\alpha_1 + \alpha_2}{\beta} = u^*$ in $\overline{\Omega}$. Moreover, we also have $v \equiv \frac{\alpha_2}{\gamma f(u^*)}$. The proof is completed. □

3 Case 2: $f(0) = 0$ and $f_u(u) > 0$ for $u > 0$

In this section, we mainly discuss the stability of positive equilibrium for system (1.1) and the non-existence of non-constant positive solution to system (1.2), under the conditions that $f(0) = 0$ and $f_u(u) > 0$ for $u > 0$.

3.1 The stability of positive equilibrium

Let us consider the spatially homogeneous system corresponding to (1.1):

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = \alpha_1 - \beta u + \gamma f(u)v, & t > 0, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = \alpha_2 - \gamma f(u)v, & t > 0. \end{cases} \tag{3.1}$$

Theorem 3.1. *Suppose that $\gamma f(u^*) > \frac{\alpha_2 f_u(u^*)}{f(u^*)} - \beta > 0$. If there exist $d_1^*, d_2^* > 0$ such that $d_1 < d_1^*$ and $d_2 > d_2^*$, then the positive equilibrium U^* is asymptotically stable for system (3.1) and unstable for system (1.1) with $\Omega = (0, l)$ as that in Subsection 2.1.*

Proof. Note that $\gamma f(u^*) > \frac{\alpha_2 f_u(u^*)}{f(u^*)} - \beta$. Recall that

$$\begin{aligned} T_n &= -(d_1+d_2)\mu_n + \left[\frac{\alpha_2 f_u(u^*)}{f(u^*)} - \beta - \gamma f(u^*)\right], \\ D_n &= \mu_n \left[d_1 d_2 \mu_n - d_2\left(\frac{\alpha_2 f_u(u^*)}{f(u^*)} - \beta\right) + d_1 \gamma f(u^*)\right] + \gamma\beta f(u^*). \end{aligned} \tag{3.2}$$

Since $\gamma f(u^*) > \frac{\alpha_2 f_u(u^*)}{f(u^*)} - \beta$, similarly as the proof of Theorem 2.1, the positive equilibrium U^* is asymptotically stable for system (3.1).

Since $\frac{\alpha_2 f_u(u^*)}{f(u^*)} - \beta > 0$, there exists a $d_2^* > 0$ such that $\mu_1 d_2^* \left(\frac{\alpha_2 f_u(u^*)}{f(u^*)} - \beta\right) > \gamma\beta f(u^*)$. By (3.2), for any $d_2 > d_2^*$, we have

$$\begin{aligned} D_1 &= \mu_1 \left[d_1 d_2 \mu_1 - d_2\left(\frac{\alpha_2 f_u(u^*)}{f(u^*)} - \beta\right) + d_1 \gamma f(u^*)\right] + \gamma\beta f(u^*) \\ &< \mu_1 \left[d_1 d_2 \mu_1 - d_2^*\left(\frac{\alpha_2 f_u(u^*)}{f(u^*)} - \beta\right) + d_1 \gamma f(u^*)\right] + \gamma\beta f(u^*) \end{aligned}$$

Letting $d_1 \to 0$, we have

$$\lim_{d_1\to 0} D_1 \le \gamma\beta f(u^*) - \mu_1 d_2^* \left(\frac{\alpha_2 f_u(u^*)}{f(u^*)} - \beta\right) < 0.$$

Thus, there exists a $d_1^* > 0$ such that $D_1 < 0$ for any $d_1 < d_1^*$, $d_2 > d_2^*$. This implies that M_1, and also the operator L has at least one positive eigenvalue, where M_1 and L are defined by (2.3) and (2.4). By Corollary 5.1.1 in [3], the positive equilibrium U^* is asymptotically instable for system (1.1). The proof is completed. □

Remark 3.2. Note that Subesection 3.1 is subject to Case 2: $f(u) > 0$ and $f_u(u) > 0$ for $u > 0$. Thus the condition $\frac{\alpha_2 f_u(u^*)}{f(u^*)} - \beta > 0$ in Theorem 3.1 may be tenable, and the instability is called Turing instability or diffusion-driven instability.

Example B. If $f(u) = u$, letting $\alpha_1 = 1, \alpha_2 = 5.5, \beta = 0.65$ and $\gamma = 0.05$, then the unique constant equilibrium $U^* = (u^*, v^*) = (10, 11)$ and $\gamma f(u^*) = 0.5000 > 0 > \frac{\alpha_2 f_u(u^*)}{f(u^*)} - \beta = -0.1000$. If $f(u) = u^2$, letting $\alpha_1 = 1, \alpha_2 = 5.5, \beta = 0.65$ and $\gamma = 0.005$, then $U^* = (10, 11)$ and $\gamma f(u^*) = 0.5000 > \frac{\alpha_2 f_u(u^*)}{f(u^*)} - \beta = 0.4500 > 0$. By Theorem 3.1, U^* is asymptotically stable for system (3.1). See Figure 3.1 for the numerical simulations.

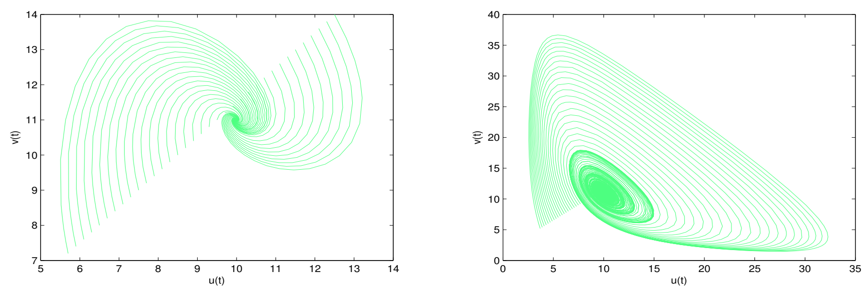

Figure 3.1: Numerical simulation of stability of (u^*, v^*). Left: $f(u) = u$, $\alpha_1 = 1, \alpha_2 = 5.5, \beta = 0.65, \gamma = 0.05$ and $(u^*, v^*) = (10, 11)$; Right: $f(u) = u^2$, $\alpha_1 = 1, \alpha_2 = 5.5, \beta = 0.65, \gamma = 0.005$ and $(u^*, v^*) = (10, 11)$.

Theorem 3.3. *Suppose that $f(0) = 0$ and $f_u(u) > 0$ is continuous for $u > 0$ and set $d_1 = d_2 = d > 0$. Then there exists a $\alpha_2^* > 0$ such that the positive equilibrium U^* is global asymptotically stable for system (1.1) when $0 < \alpha_2 < \alpha_2^*$.*

Proof. By the hypotheses, it follows from Theorem 8.3.3 in [12] that system (1.1) has a unique globally defined solution $(u(x,t), v(x,t))$. It is well known that if $a, b > 0$ and $z(t) > 0$ satisfies the equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = a - bz, \quad t > 0, \quad w(0) = w_0 > 0,$$

then $z(t) \to a/b$ as $t \to \infty$.

Note that $u(x,t)$ satisfies

$$u_t - d_1 \Delta u = \alpha_1 - \beta u + \gamma f(u) v \geq \alpha_1 - \beta u.$$

By the comparison principle for parabolic equations, there exists $T_1 > 0$ and $\epsilon_0 > 0$ such that $u(x,t) \geq \underline{u} = \alpha_1/\beta + \epsilon_0$ for any $t > T_1$. Then $v(x,t)$ satisfies

$$v_t - d_2 \Delta v = \alpha_2 - \gamma f(u) v \leq \alpha_2 - \gamma f(\underline{u}) v.$$

And also, there exists a $T_2 > 0$ such that $v(x,t) \leq \overline{v} = \frac{\alpha_2}{\gamma f(\underline{u})} + \epsilon_0$ for any $t > T_2$.

Since $d_1 = d_2 = d > 0$ and let $w = u + v$, adding the two equations of (1.1), we have

$$w_t - d\Delta w = \alpha_1 + \alpha_2 - \beta u \leq \alpha_1 + \alpha_2 + \beta \overline{v} - \beta w$$

for any $t > T_2$. And also, there exists a $T_3 > 0$ such that $w(x,t) \leq (\alpha_1 + \alpha_2 + \beta\overline{v})/\beta + \epsilon_0$ for any $t > T_3$. Thus, $u(x,t) \leq \overline{u} = (\alpha_1 + \alpha_2 + \beta\overline{v})/\beta + \epsilon_0$ for any $t > T_3$. Then $v(x,t)$ satisfies

$$v_t - d_2 \Delta v = \alpha_2 - \gamma f(u) v \geq \alpha_2 - \gamma f(\overline{u}) v.$$

And also, there exists a $T_4 > 0$ such that $v(x,t) \geq \underline{v} = \frac{\alpha_2}{\gamma f(\overline{u})} + \epsilon_0$ for any $t > T_4$. Take $T_0 = \max\{T_1, T_2, T_3, T_4\}$. Then for $t > T_0$, we have

$$\underline{u} \leq u(x,t) \leq \overline{u}, \quad \underline{v} \leq v(x,t) \leq \overline{v}.$$

We could verify that $(\overline{u}, \overline{v})$ and $(\underline{u}, \underline{v})$ are a pair of coupled upper and lower solutions of system (1.1) by the definition in [12, 23], and $(\alpha_1 - \beta u + \gamma f(u) v, \alpha_2 - \gamma f(u) v)$ satisfies the Lipschitz conditions.

Let $\overline{\mathbf{c}}^{(0)} = (\overline{u}, \overline{v})$, $\underline{\mathbf{c}}^{(0)} = (\underline{u}, \underline{v})$. Then we construct the iterative sequences of upper and lower solutions $\{\overline{\mathbf{c}}^{(m)}\}$, $\{\underline{\mathbf{c}}^{(m)}\}$, where $\overline{\mathbf{c}}^{(m)} \equiv (\overline{c}_1^{(m)}, \overline{c}_2^{(m)})$, $\underline{\mathbf{c}}^{(m)} \equiv (\underline{c}_1^{(m)}, \underline{c}_2^{(m)})$, $m = 1, 2, \ldots$, which satisfy

$$\begin{cases} \overline{c}_1^{(m)} = \overline{c}_1^{(m-1)} + \dfrac{1}{K}\left[\alpha_1 - \beta \overline{c}_1^{(m-1)} + \gamma f(\overline{c}_1^{(m-1)}) \overline{c}_2^{(m-1)}\right], \\ \overline{c}_2^{(m)} = \overline{c}_2^{(m-1)} + \dfrac{1}{K}\left[\alpha_2 - \gamma f(\underline{c}_1^{(m-1)}) \overline{c}_2^{(m-1)}\right], \\ \underline{c}_1^{(m)} = \underline{c}_1^{(m-1)} + \dfrac{1}{K}\left[\alpha_1 - \beta \underline{c}_1^{(m-1)} + \gamma f(\underline{c}_1^{(m-1)}) \underline{c}_2^{(m-1)}\right], \\ \underline{c}_2^{(m)} = \underline{c}_2^{(m-1)} + \dfrac{1}{K}\left[\alpha_2 - \gamma f(\overline{c}_1^{(m-1)}) \underline{c}_2^{(m-1)}\right], \end{cases}$$

and

$$(\underline{u}, \underline{v}) \leq \underline{\mathbf{c}}^{(1)} \leq \underline{\mathbf{c}}^{(2)} \leq \cdots \leq \underline{\mathbf{c}}^{(m)} \leq \cdots \leq \overline{\mathbf{c}}^{(m)} \leq \cdots \leq \overline{\mathbf{c}}^{(2)} \leq \overline{\mathbf{c}}^{(1)} \leq (\overline{u}, \overline{v}).$$

Here, K is the Lipschitz coefficients.

Let $\underline{\mathbf{c}} = \lim_{m\to\infty} \underline{\mathbf{c}}^{(m)} = (\widetilde{u}, \widetilde{v})$ and $\overline{\mathbf{c}} = \lim_{m\to\infty} \overline{\mathbf{c}}^{(m)} = (\widehat{u}, \widehat{v})$. Then $(\underline{u}, \underline{v}) \le (\widetilde{u}, \widetilde{v}) \le (\widehat{u}, \widehat{v}) \le (\overline{u}, \overline{v})$, satisfying

$$
\begin{aligned}
&\alpha_1 - \beta\widehat{u} + \gamma f(\widehat{u})\widehat{v} = 0, \qquad \alpha_2 - \gamma f(\widetilde{u})\widehat{v} = 0,\\
&\alpha_1 - \beta\widetilde{u} + \gamma f(\widetilde{u})\widetilde{v} = 0, \qquad \alpha_2 - \gamma f(\widehat{u})\widetilde{v} = 0.
\end{aligned} \tag{3.3}
$$

From (3.3), we have

$$
\begin{aligned}
&c\alpha_1 - \beta\widehat{u} + \alpha_2 f(\widehat{u})/f(\widetilde{u}) = 0,\\
&\alpha_1 - \beta\widetilde{u} + \alpha_2 f(\widetilde{u})/f(\widehat{u}) = 0.
\end{aligned} \tag{3.4}
$$

By subtracting the first equation of (3.4) form the second one, we have

$$
\begin{aligned}
&\beta(\widehat{u} - \widetilde{u}) + \alpha_2 \left[\frac{f(\widetilde{u})}{f(\widehat{u})} - \frac{f(\widehat{u})}{f(\widetilde{u})}\right] = 0\\
\Rightarrow &\beta(\widehat{u} - \widetilde{u}) + \frac{\alpha_2}{f(\widetilde{u})f(\widehat{u})}\left[f^2(\widetilde{u}) - f^2(\widehat{u})\right] = 0\\
\Rightarrow &\beta(\widehat{u} - \widetilde{u}) + \frac{\alpha_2}{f(\widetilde{u})f(\widehat{u})}\left[f(\widetilde{u}) + f(\widehat{u})\right]\left[f(\widetilde{u}) - f(\widehat{u})\right] = 0\\
\Rightarrow &\beta(\widehat{u} - \widetilde{u}) - \frac{\alpha_2}{f(\widetilde{u})f(\widehat{u})}\left[f(\widetilde{u}) + f(\widehat{u})\right] f'(\theta)(\widehat{u} - \widetilde{u}) = 0\\
\Rightarrow &(\widehat{u} - \widetilde{u})\{\beta - \frac{\alpha_2}{f(\widetilde{u})f(\widehat{u})}\left[f(\widetilde{u}) + f(\widehat{u})\right] f'(\theta)\} = 0.
\end{aligned} \tag{3.5}
$$

Now let us determine the sign of the term $\beta - \frac{\alpha_2}{f(\widetilde{u})f(\widehat{u})}\left[f(\widetilde{u}) + f(\widehat{u})\right] f'(\theta)$. Recall that $\underline{u} = \alpha_1/\beta + \epsilon_0$, $\overline{v} = \frac{\alpha_2}{\gamma f(\underline{u})} + \epsilon_0$, $\overline{u} = (\alpha_1 + \alpha_2 + \beta\overline{v})/\beta + \epsilon_0$ and $\underline{u} \le \widetilde{u} \le \widehat{u} \le \overline{u}$. By the monotonicity of $f(u)$, we have

$$
\beta - \frac{\alpha_2}{f(\widetilde{u})f(\widehat{u})}\left[f(\widetilde{u}) + f(\widehat{u})\right] f'(\theta) = \beta - \alpha_2\left[\frac{f'(\theta)}{f(\widehat{u})} + \frac{f'(\theta)}{f(\widetilde{u})}\right] > \beta - \frac{2\alpha_2 f'(\theta)}{f(\widetilde{u})}.
$$

Note that $\underline{u}$ is independent of α_2. We obtain that $\overline{v} = \frac{\alpha_2}{\gamma f(\underline{u})} + \epsilon_0 \to \epsilon_0$ as $\alpha_2 \to 0$, and then $\overline{u} = (\alpha_1 + \alpha_2 + \beta\overline{v})/\beta + \epsilon_0 \to (\alpha_1 + \beta\epsilon_0)/\beta + \epsilon_0$ as $\alpha_2 \to 0$. Since $\underline{u} \le \widetilde{u}, \theta \le \overline{u}$, it is obviously that $f(\widetilde{u})(> 0)$ is bounded for small $\alpha_2 > 0$. Note that $f_u(u)(> 0)$ is continuous for $u > 0$. Accordingly, there exists a $\alpha_2^* > 0$ such that, $\beta - \frac{2\alpha_2 f'(\theta)}{f(\widetilde{u})} > 0$ if $\alpha_2 < \alpha_2^*$. This yields that $\beta - \frac{\alpha_2}{f(\widetilde{u})f(\widehat{u})}\left[f(\widetilde{u}) + f(\widehat{u})\right] f'(\theta) > 0$ if $\alpha_2 < \alpha_2^*$.

It follows from (3.5) that $\widehat{u} = \widetilde{u}$ if $\alpha_2 < \alpha_2^*$, and then $\widehat{v} = \widetilde{v}$ from (3.3). That is, $\widehat{u} = \widetilde{u} = \frac{\alpha_1+\alpha_2}{\beta} = u^*$, $\widehat{v} = \widetilde{v} = \frac{\alpha_2}{\gamma f(u^*)} = v^*$. By Theorem 2.2 of [13], the solution $(u(x,t), v(x,t))$ to system (1.1) satisfies

$$
\lim_{t\to\infty} u(x,t) = u^*, \quad \lim_{t\to\infty} v(x,t) = v^*, \quad \text{uniformly on } \overline{\Omega}.
$$

Thus, we obtain that the positive equilibrium $U^* = (u^*, v^*)$ is global asymptotically stable for system (1.1) when $\alpha_2 < \alpha_2^*$. The proof is completed. □

3.2 The non-existence of non-constant positive equilibrium

By Lemma 2.2, Wu, Ma and Guo (2013) [17] have given the following boundedness of positive solution to system (1.2).

Lemma 3.4 ([17, Lemma 1.2]). *Suppose that $(u(x), v(x))$ is a positive solution to system (1.2), then*

$$\frac{\alpha_1}{\beta} \leq u(x) \leq \widetilde{M}, \qquad \frac{\alpha_2}{\gamma f(\widetilde{M})} \leq v(x) \leq \frac{\alpha_2}{\gamma f(\alpha_1/\beta)},$$

where $\widetilde{M} = \frac{1}{\beta}\big(\alpha_1 + \alpha_2 + \frac{d_2\alpha_2\beta}{d_1\gamma f(\alpha_1/\beta)}\big)$.

By Lemma 3.4, we will arrive at the following corollary easily and omit the proof.

Corollary 3.5. *Suppose that $(u(x), v(x))$ is a positive solution to system (1.2) and take two constants $\overline{D}_1, \overline{D}_2 > 0$ fixed. Then there exist two constants $C_1, C_2 > 0$ such that for any $d_1 \geq \overline{D}_1, d_2 \leq \overline{D}_2$, $(u(x), v(x))$ satisfies*

$$C_1 \leq u(x), v(x) \leq C_2, \qquad x \in \overline{\Omega},$$

where $C_1 = C_1(\alpha_1, \alpha_2, \beta, \gamma, \overline{D}_1, \overline{D}_2), C_2 = C_2(\alpha_1, \alpha_2, \beta, \gamma, \overline{D}_1, \overline{D}_2)$.

Proposition 3.6. *Take $D = D(\alpha_1, \alpha_2, \beta, \gamma, d_2) > 0$ fixed and suppose $\{d_{1i}\}_{i=1}^{\infty} \subset (0, \infty)$ such that $d_{1i} > D, d_{1i} \to \infty$ as $i \to \infty$, and $(u_i(x), v_i(x))$ is a positive solution to system (1.2) with $d_1 = d_{1i}$, then*

$$(u_i(x), v_i(x)) \to (u^*, v^*),$$

in $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ *as* $i \to \infty$.

Proof. If $(u_i(x), v_i(x))$ is a positive solution to system (1.2) with $d_1 = d_{1i}$, then integrating and adding the equations in (1.2), we have

$$\frac{1}{|\Omega|}\int_\Omega u_i(x)\mathrm{d}x = \frac{\alpha_1 + \alpha_2}{\beta} = u^*. \tag{3.6}$$

By Corollary 3.5, we know the sequence $\{(u_i(x), v_i(x))\}$ is bounded in $L^\infty(\Omega) \times L^\infty(\Omega)$. By the L^p estimation and the Sobolev imbedding theorems, $\{(u_i(x), v_i(x))\}$ converges to some (u, v) in $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ as $i \to \infty$. Thus, (u, v) satisfies

$$\begin{cases} -\Delta u = 0, & x \in \Omega, \\ -d_2\Delta v = \alpha_2 - \gamma f(u)v, & x \in \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \tag{3.7}$$

The first equation in (3.7) implies that u is a constant. Also u satisfies (3.6), which yields $u \equiv \frac{\alpha_1+\alpha_2}{\beta} = u^*$. Moreover, v satisfies

$$-d_2\Delta v = \alpha_2 - \gamma f(u^*)v, \quad x \in \Omega, \quad \frac{\partial v}{\partial n} = 0, \quad x \in \partial\Omega. \tag{3.8}$$

Multiplying (3.8) by $\alpha_2 - \gamma f(u^*)v$ and integrating over Ω, we have

$$0 \leq \gamma f(u^*)\int_\Omega |\nabla v|^2\mathrm{d}x \leq \int_\Omega [\alpha_2 - \gamma f(u^*)]^2\mathrm{d}x \leq 0,$$

which yields $v \equiv \frac{\alpha_2}{\gamma f(u^*)} = v^*$. The proof is completed. □

Theorem 3.7. *There exists a $D = D(\alpha_1, \alpha_2, \beta, \gamma, d_2) > 0$ such that system (1.2) has no non-constant positive solution for any $d_1 > D$.*

Proof. Define the function space $L_0^2(\Omega) = \{u \in L^2(\Omega) : \int_\Omega u(x)\mathrm{d}x = 0\}$. Let $w = u - u^*$, where $u^* = \frac{\alpha_1+\alpha_2}{\beta}$. It is easy to see that $w \in H^2(\Omega) \bigcap L_0^2(\Omega)$ and system (1.2) is equivalent to

$$\begin{cases} -\Delta w = \delta[\alpha_1 - \beta(w+u^*) + \gamma f(w+u^*)v], & x \in \Omega, \\ -d_2\Delta v = \alpha_2 - \gamma f(u)v, & x \in \Omega, \end{cases} \tag{3.9}$$

where $\delta = 1/d_1$, $w \in H^2(\Omega) \bigcap L_0^2(\Omega)$ and $v \in H^2(\Omega)$. Define a projection $P : L^2(\Omega) \to L_0^2(\Omega)$ by

$$P(u) = u - \frac{1}{|\Omega|}\int_\Omega u(x)\mathrm{d}x, \quad u \in L^2(\Omega).$$

We can derive that system (3.9) is is equivalent to

$$\begin{pmatrix} \Delta w + \delta P(\alpha_1 - \beta(w+u^*) + \gamma f(w+u^*)v) \\ d_2\Delta v + \alpha_2 - \gamma f(w+u^*)v \end{pmatrix} \doteq \widetilde{F}(\delta, w, v) = 0.$$

Similarly as the proof of Proposition 3.6, it follows that $\widetilde{F}(0, w, v) = 0$ has a unique silution $(w, v) = (0, v^*)$, where $v^* = \frac{\alpha_2}{\gamma f(u^*)}$. At $(\delta, w, v) = (0,(0, v^*))$, the Fréchet derivative of $\widetilde{F}(\delta, w, v)$ with respect to (w, v) can be characterized by

$$\widetilde{F}_{(w,v)}(0,(0,v^*)) = \begin{pmatrix} \Delta & 0 \\ -\frac{\alpha_2 f_u(u^*)}{f(u^*)} & d_2\Delta - \gamma f(u^*) \end{pmatrix}.$$

Therefore, $\widetilde{F}_{(w,v)}(0,(0,v^*))$ is invertible form $(H^2(\Omega) \bigcap L_0^2(\Omega)) \times H^2(\Omega)$ to $L_0^2(\Omega) \times L^2(\Omega)$ and the implicit function theorem establishes. So there exists a $r > 0$ such that $(0,(0,v^*))$ is a unique solution of $\widetilde{F}(\delta, w, v) = 0$ in $B_r(0,(0,v^*))$.

To do this, we proceed with a contradiction argument and assume that there exist $d_{1i} > D$ and $d_{1i} \to \infty$ such that system (1.2) has a non-constant positive solution $(u_i(x), v_i(x))$ with $d_1 = d_{1i}$. Let $w_i(x) = u_i(x) - u^*$ and $\delta_i = 1/d_{1i}$. Then $\widetilde{F}(\delta_i,(w_i, v_i)) = 0$.

By Proposition 3.6, it follows that $(w_i, v_i) \to (0, v^*)$ in $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$ as $i \to \infty$. Thus, there exists some $i_0 \geq 1$ such that $(\delta_i,(w_i, v_i)) \in B_r(0,(0,v^*))$ when $i > i_0$. This is a contradiction, which completes the proof. □

4 Conclusion

In this paper, the stability of positive equilibrium and the non-existence of non-constant positive solution are investigated under the Neumann boundary conditions. The local and global stability of the positive equilibrium is obtained by analyzing the characteristic equations and the upper and lower solution method. By the maximal principle and the implicit function theorem, the non-existence of non-constant positive solution are derived.

Acknowledgements

The authors gratefully acknowledge the referee for his/her comments and suggestions, which have significantly improved not only the presentation but the overall quality of the paper.

The work has been partially supported by the Natural Science Foundation of China (Grant No: 61672021), the Special Fund of Education Department of Shaanxi Province (Grant No: 16JK1710) and the Natural Science Basic Research Plan in Shaanxi Province of China (Grant No: 2015JQ1023).

References

[1] D. A. Garzon-Alvarado, C. Galeano, J. Mantilla, Numerical tests on pattern formation in 2d heterogeneous mediums: an approach using the Schnakenberg model, *Dyna* **79**(2012), No. 172, 56–66.

[2] V. Gold, *Compendium of chemical terminology: IUPAC recommendations*, Blackwell Science, Oxford, 1987.

[3] D. Henry, *Geometric theory of semilinear parabolic equations*, Vol. 840, Springer-Verlag, New York 1981. https://doi.org/10.1007/BFb0089647

[4] D. Iron, J. Wei, M. Winter, Stability analysis of Turing patterns generated by the Schnakenberg model, *J. Math. Biol.* **49**(2004), No. 4, 358–390. https://doi.org/10.1007/s00285-003-0258-y; MR2105836

[5] J. Jin, J. Shi, J. Wei, F. Yi, Bifurcations of patterned solutions in the diffusive Lengyel–Epstein system of CIMA chemical reactions, *Rocky Mountain J. Math.* **43**(2013), No. 5, 1637–1674. https://doi.org/10.1216/RMJ-2013-43-5-1637; MR3127841

[6] C. Landsberg, A. Voigt, A multigrid finite element method for reaction–diffusion systems on surfaces, *Comput. Vis. Sci.* **13**(2010), No. 4, 177–185. https://doi.org/10.1007/s00791-010-0136-2; MR2645018

[7] Y. Li, Steady-state solution for a general Schnakenberg model, *Nonlinear Anal. Real World Appl.* **12**(2011), No. 4, 1985–1990. https://doi.org/10.1016/j.nonrwa.2010.12.014; MR2800993

[8] P. Liu, J. Shi, Y. Wang, X. Feng, Bifurcation analysis of reaction–diffusion Schnakenberg model, *J. Math. Chem.* **51**(2013), No. 8, 2001–2019. https://doi.org/10.1007/s10910-013-0196-x; MR3103426

[9] A. Madzvamuse, A. J. Wathen, P. K. Maini, A moving grid finite element method applied to a model biological pattern generator, *J. Computat. Phys.* **190**(2003), No. 2, 478–500. https://doi.org/10.1016/S0021-9991(03)00294-8; MR2013026

[10] J. D. Meiss, *Differential dynamical systems*, SIAM, Philadelphia, 2017. MR3614477

[11] W.-M. Ni, M. Tang, Turing patterns in the Lengyel–Epstein system for the CIMA reaction, *Trans. Mosc. Math. Soc.* **357**(2005), No. 10, 3953–3969. https://doi.org/10.1090/S0002-9947-05-04010-9; MR2159695

[12] C. V. Pao, *Nonlinear parabolic and elliptic equations*, Springer-Verlag, Boston, 1992. https://doi.org/10.1007/978-1-4615-3034-3

[13] C. V. Pao, Convergence of solutions of reaction-diffusion systems with time delays, *Nonlinear Anal. Real World Appl.* **48**(2002), No. 3, 349–362. https://doi.org/10.1016/S0362-546X(00)00189-9; MR1869516

[14] S. J. Ruuth, Implicit-explicit methods for reaction–diffusion problems in pattern formation, *J. Math. Biol.* **34**(1995), No. 2, 148–176. https://doi.org/10.1007/BF00178771; MR1366356

[15] F. Shakeri, M. Dehghan, The finite volume spectral element method to solve Turing models in the biological pattern formation, *Comput. Math. Appl.* **62**(2011), No. 12, 4322–4336. https://doi.org//10.1016/j.camwa.2011.09.049; MR2855577

[16] J. Schnakenberg, Simple chemical reaction systems with limit cycle behaviour, *J. Theor. Biol.* **81**(1979), No. 3, 389–400. https://doi.org/10.1016/0022-5193(79)90042-0; MR0558661

[17] J. Wu, C. Ma, G. Guo, The effect of interaction ratio in a chemical reaction, *IMA J. Appl. Math.* **78**(2013), 1265–1289. https://doi.org/10.1093/imamat/hxs020; MR3142329

[18] C. Xu, J. Wei, Hopf bifurcation analysis in a one-dimensional Schnakenberg reaction–diffusion model, *Nonlinear Anal. Real World Appl.* **13**(2012), No. 4, 1961–1977. https://doi.org/10.1016/j.nonrwa.2012.01.001; MR2891023

[19] A. Yagi, Activator–inhibitor models, in: *Abstract parabolic evolution equations and their applications*, Springer-Verlag, Berlin, 2010, pp. 357–371. https://doi.org/10.1007/978-3-642-04631-5; MR2573296

[20] W. Yang, Existence and asymptotic behavior of solutions for a predator–prey system with a nonlinear growth rate, *Acta Appl. Math.* **152**(2017), No. 1, 57–72. https://doi.org/10.1007/s10440-017-0111-8; MR3721657

[21] W. Yang, Analysis on existence of bifurcation solutions for a predator–prey model with herd behavior, *Appl. Math. Modelling* **53**(2018), 433–446. https://doi.org/10.1016/j.apm.2017.09.020; MR3735067

[22] W. Yang, J. Wu, Some dynamics in spatial homogeneous and inhomogeneous activator–inhibitor model, *Acta Math. Sci. (Chin. Ed.)* **37**(2017), No. 2, 390–400.

[23] Q. Ye, Z. Li, Introduction to reaction–diffusion equations (in Chinese), Science Press, Beijing, 1990. MR1113267

[24] K. Zhou, On the problem of limit cycles in biochemical reactions, *J. Sichuan Univ. Nat. Sci. Ed.* **1985**, No. 2, 31–35. MR841475