

# Study of limit cycles in piecewise smooth perturbations of Hamiltonian centers via regularization method

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Received 24 February 2017, appeared 20 November 2017

Communicated by Gabriele Villari

**Abstract.** In this article we study the existence and positions of limit cycles in piecewise smooth perturbations of planar Hamiltonian centers. By using the regularization method we provide an analytical expression for the first order Melnikov function frequently used in the literature directly from the original non-smooth problem.

**Keywords:** piecewise smooth system, limit cycle, periodic orbit, regularization method, Melnikov function.

**2010 Mathematics Subject Classification:** 34C25, 37G15, 34C05.

## 1 Introduction and statement of the results

The main subjects studied in this article are the existence and positions of limit cycles in piecewise smooth perturbations of planar Hamiltonian centers. Consider

$$X' = \frac{dX}{dt} = Z(X, \varepsilon) = \begin{cases} X^-(X) = F(X) + \varepsilon G^-(X), & y \leq 0, \\ X^+(X) = F(X) + \varepsilon G^+(X), & y \geq 0, \end{cases} \quad (1.1)$$

where  $X = (x, y) \in \mathbb{R}^2$ ,  $\varepsilon \geq 0$  is a small parameter, the prime denotes derivative with respect to the independent variable  $t$ , called here the time,

$$\begin{aligned} F : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ X &\longmapsto F(X) = (-H_y(X), H_x(X)), \end{aligned} \quad (1.2)$$

is a smooth (class  $C^k$ ,  $k \geq 1$ ) Hamiltonian vector field, defined by a Hamiltonian  $H : \mathbb{R}^2 \longrightarrow \mathbb{R}$ ,

$$\begin{aligned} G : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ X &\longmapsto G(X) = \begin{cases} G^-(X) = (g_1^-(x, y), g_2^-(x, y)), & y \leq 0, \\ G^+(X) = (g_1^+(x, y), g_2^+(x, y)), & y \geq 0, \end{cases} \end{aligned} \quad (1.3)$$

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with  $G^\pm$  also smooth vector fields. Assume that the origin  $(0,0)$  is a center of the vector field  $F$  surrounded by a period annulus  $U \subset \mathbb{R}^2$ .

In this article we study the bifurcation of limit cycles from the period annulus  $U$  of system (1.1) when  $\varepsilon = 0$ . There are many articles in the literature addressing the problem of limit cycles bifurcating from a period annulus by piecewise smooth perturbations. In general in those studies the authors mainly used Melnikov functions or averaging theory. Without being exhaustive, see [6–8, 14] and references therein.

We propose to use the regularization method in order to study limit cycles of system (1.1) when  $\varepsilon > 0$ . In this sense, we obtain generalizations of the studies carried out in [3], where the authors studied limit cycles in piecewise smooth perturbations of linear centers. As far as we know, this approach was not used in problems like in (1.1). It gives a beautiful theoretical way to prove some known results on this subject.

By hypothesis there is a family of periodic orbits  $\Gamma_a \subset U$  of system (1.1) when  $\varepsilon = 0$  defined by

$$\Gamma_a = \{(x, y) \in U : H(x, y) = a, a \in I\},$$

where  $I$  is an open interval of the form  $I = (0, \alpha)$  for some  $\alpha > 0$ . For each  $a \in I$ , the periodic orbit  $\Gamma_a$  is parameterized by  $\gamma_a(s)$ , with  $s \in [0, T(a, 0)]$ ,  $T(a, 0) > 0$ , where  $T = T(a, \varepsilon)$  is the period function. Suppose also that  $\gamma_a^+(s)$ ,  $s \in [0, T^+(a, 0)]$ , and  $\gamma_a^-(s)$ ,  $s \in [T^+(a, 0), T^-(a, 0)]$ , are parameterizations of arc trajectories of  $\Gamma_a$  defined in

$$U^+ = U \cap \{(x, y) \in \mathbb{R}^2 : y \geq 0\}, \quad U^- = U \cap \{(x, y) \in \mathbb{R}^2 : y \leq 0\},$$

respectively, where  $T^+(a, 0) > 0$  and  $T^-(a, 0) = T(a, 0) > 0$ . Furthermore, assume without loss of generality that the periodic orbits are counterclockwise oriented.

In this article the approach chosen to study limit cycles of the discontinuous system (1.1) is based on the regularization method, which we describe briefly now. See Section 2 for details.

Consider  $\mathcal{H} : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $\mathcal{H}(x, y) = y$ . It is easy to see that zero is a regular value of the smooth function  $\mathcal{H}$ . Define the sets

$$\Sigma = \mathcal{H}^{-1}(0), \quad \Sigma^- = \mathcal{H}^{-1}(-\infty, 0), \quad \Sigma^+ = \mathcal{H}^{-1}(0, +\infty).$$

Then  $\mathbb{R}^2 = \Sigma^- \cup \Sigma \cup \Sigma^+$ . The set  $\Sigma$  is called the separation line between the two zones  $\Sigma^-$  and  $\Sigma^+$ .

From a  $C^k$  function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ , called a transition function, defined by  $\varphi(t) = 0$ , if  $t \leq -1$ ,  $\varphi(t) = 1$ , if  $t \geq 1$  and  $\varphi'(t) > 0$ , if  $t \in (-1, 1)$ , and a real number  $\mu > 0$ , called regularization parameter, we define the function  $\varphi^\mu$ , called a regularization function, by  $\varphi^\mu(t) = \varphi(t/\mu)$ , for all  $t \in \mathbb{R}$ .

We have our first result.

**Proposition 1.1.** *A regularization of the one parameter family of piecewise smooth vector fields  $Z$  in (1.1) produces a two parameter family of smooth vector fields*

$$Z^\mu(X, \varepsilon) = (1 - \varphi^\mu(y))X^-(x, y) + \varphi^\mu(y)X^+(x, y) = F(X) + \varepsilon R(X, \mu), \quad (1.4)$$

where

$$R(X, \mu) = (r_1(x, y, \mu), r_2(x, y, \mu))$$

with

$$r_1(x, y, \mu) = g_1^-(x, y) + \varphi^\mu(y)(g_1^+(x, y) - g_1^-(x, y)), \quad (1.5)$$

$$r_2(x, y, \mu) = g_2^-(x, y) + \varphi^\mu(y)(g_2^+(x, y) - g_2^-(x, y)), \quad (1.6)$$

is a one parameter family of smooth vector fields.

From Proposition 1.1, it is worth to mention that we are able to study the existence and positions of limit cycles of smooth perturbations of planar Hamiltonian centers. See equation (1.4). In order to do this we can use classical Melnikov functions according to the following theorem.

**Theorem 1.2.** For each  $\mu > 0$ , if  $a_0 > 0$  is a simple zero of the function

$$\mathcal{R}(a, \mu) = \mathcal{R}^+(a, \mu) + \mathcal{R}^-(a, \mu), \quad (1.7)$$

where

$$\mathcal{R}^+(a, \mu) = \int_{\gamma_a^+} r_1^+ dy - r_2^+ dx$$

and

$$\mathcal{R}^-(a, \mu) = \int_{\gamma_a^-} r_1^- dy - r_2^- dx,$$

then for  $\varepsilon > 0$  sufficiently small there exists a limit cycle  $X^{\varepsilon, \mu}$  of (1.4) such that  $X^{\varepsilon, \mu}$  tends to  $\Gamma_{a_0}$  when  $\varepsilon$  goes to 0. The limit cycle is stable if  $\mathcal{R}_a(a_0, \mu) < 0$  and unstable if  $\mathcal{R}_a(a_0, \mu) > 0$ , where  $\mathcal{R}_a(a_0, \mu) = \partial \mathcal{R}(a_0, \mu) / \partial a$ .

A natural question can be formulated about the existence of the limit of the Melnikov function (1.7) when the regularization parameter goes to zero. The next two results give answers in this direction.

**Theorem 1.3.** The function  $\mathcal{R}$  in (1.7) satisfies for each  $a > 0$  the following relation

$$\lim_{\mu \rightarrow 0} \mathcal{R}(a, \mu) = \mathcal{M}(a), \quad (1.8)$$

where

$$\mathcal{M}(a) = \mathcal{M}^+(a) + \mathcal{M}^-(a), \quad (1.9)$$

with

$$\mathcal{M}^+(a) = \int_{\gamma_a^+} g_1^+ dy - g_2^+ dx$$

and

$$\mathcal{M}^-(a) = \int_{\gamma_a^-} g_1^- dy - g_2^- dx.$$

In fact, as a consequence of the proof of Theorem 1.3 we have the following theorem.

**Theorem 1.4.** Consider the hypotheses of Theorem 1.2. For each simple zero  $a > 0$  of the function  $\mathcal{M}$  in (1.9) there exists  $\mu_0 > 0$  such that, for every transition function  $\varphi$ , the function  $\mathcal{R}$  in (1.7) has a simple zero  $a(\mu) > 0$ , for each  $0 < \mu < \mu_0$ .

So we can summarize our study about limit cycles of system (1.1) with the following theorem.

**Theorem 1.5.** If  $a_0 > 0$  is a simple zero of the function  $\mathcal{M}$  in (1.9) then for  $\varepsilon > 0$  sufficiently small there exists a limit cycle  $X^\varepsilon$  of (1.1) such that  $X^\varepsilon$  tends to  $\Gamma_{a_0}$  when  $\varepsilon$  goes to 0. The limit cycle is stable if  $\mathcal{M}'(a_0) < 0$  and unstable if  $\mathcal{M}'(a_0) > 0$ .

Note that the function  $\mathcal{M}$  in (1.9) depends only on the components of the smooth vector fields  $G^\pm$  of (1.1). So we define the function  $\mathcal{M}$  in (1.9) as the Melnikov function associated to the discontinuous system (1.1). In fact, this function coincides with ones obtained in some articles without analytical proofs. See [6] among others.

The article is organized as follows. The proofs of Proposition 1.1 and Theorem 1.2 are given in Section 2. In Section 3 we give the proofs of Theorems 1.3 and 1.4. We present some applications in Section 4 and give some concluding remarks in Section 5.

## 2 Proofs of Proposition 1.1 and Theorem 1.2

In this section we prove Proposition 1.1 and Theorem 1.2. In order to do so, we use the following theorem whose proof follows directly from Theorem 5.17 (page 339) of [4].

**Theorem 2.1.** *Consider system (1.1) with*

$$G^+(X) = G^-(X) = (g_1(x, y), g_2(x, y)).$$

*If  $a_0 > 0$  is a simple zero of the function*

$$\mathcal{M}(a) = \oint_{\gamma_a} g_1 dy - g_2 dx, \quad (2.1)$$

*then for  $\varepsilon > 0$  sufficiently small there exists a limit cycle  $X^\varepsilon$  of this system such that  $X^\varepsilon$  tends to  $\Gamma_{a_0}$  when  $\varepsilon$  goes to 0. The limit cycle is stable if  $\mathcal{M}'(a_0) < 0$  and unstable if  $\mathcal{M}'(a_0) > 0$ .*

Since the vector field  $G$  in (1.3) is not necessarily smooth we can not apply Theorem 2.1 directly in system (1.1). In fact, in this article we are interested in the case where  $G$  is discontinuous. So, in order to use Theorem 2.1 it is necessary to transform the vector field  $Z$  in (1.1) in a smooth vector field. One way to do this is by the regularization method introduced by Sotomayor and Teixeira in [13].

Discontinuous differential systems with two zones in the plane are generally defined by

$$X' = \frac{dX}{dt} = \begin{cases} X^-(X), & \mathcal{H}(X) \leq 0, \\ X^+(X), & \mathcal{H}(X) \geq 0, \end{cases} \quad (2.2)$$

where  $X = (x, y) \in \mathbb{R}^2$ ,  $X^\pm$  are smooth vector fields, the function  $\mathcal{H} : \mathbb{R}^2 \rightarrow \mathbb{R}$  is smooth having zero as a regular value and the set  $\Sigma = \mathcal{H}^{-1}(0)$  divides the plane in two unbounded components (zones)  $\Sigma^+$  and  $\Sigma^-$  where  $\mathcal{H}$  is positive and negative, respectively. Thus  $\mathbb{R}^2 = \Sigma^+ \cup \Sigma \cup \Sigma^-$ .

There are a lot of published articles about discontinuous differential systems with two zones in the plane addressing theoretical issues as well as applied questions. We recommend the seminal books of Andronov et al. [1] and Filippov [5].

Usually, the points of discontinuity on the separation boundary  $\Sigma$  are classified as crossing (sewing), sliding, escaping or tangency points [9]. In particular, a point  $X_0 = (x_0, y_0) \in \Sigma = \mathcal{H}^{-1}(0)$  is a crossing point if

$$(X^-(X_0) \cdot \nabla \mathcal{H}(X_0)) (X^+(X_0) \cdot \nabla \mathcal{H}(X_0)) > 0.$$

A  $C^k$  function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is a *transition function* if

$$\varphi(t) = 0, \text{ if } t \leq -1; \quad \varphi(t) = 1, \text{ if } t \geq 1; \quad \varphi'(t) > 0 \text{ if } t \in (-1, 1).$$

Given a real number  $\mu > 0$  we define the function  $\varphi^\mu$  by  $\varphi^\mu(t) = \varphi(t/\mu)$ , for all  $t \in \mathbb{R}$ .

A  $\varphi$ -regularization, or simply a *regularization* of the vector field  $Y = (X^-, X^+)$  in (2.2) is defined by the one parameter family of smooth vector fields

$$Y^\mu(x, y) = (1 - \varphi^\mu(\mathcal{H}(x, y)))X^-(x, y) + \varphi^\mu(\mathcal{H}(x, y))X^+(x, y). \quad (2.3)$$

The idea behind the process of regularization is to create a one parameter family of smooth vector fields  $Y^\mu$  which agrees with the original vector fields  $X^-$  and  $X^+$  outside a strip around the separation line. In this strip, the transition function is used to average the vector fields  $X^-$  and  $X^+$ . The expectation is that by using classical analytic tools in the smooth systems  $Y^\mu$  we could obtain some pieces of information about the behavior of the non-smooth system  $Y = (X^-, X^+)$  when the regularization parameter goes to zero. See interesting results in [12].

If we apply the above construction to the discontinuous vector field  $Z$  given in (1.1), using the function  $\mathcal{H} : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $\mathcal{H}(x, y) = y$ , we obtain the two parameter family of smooth vector fields given by (1.4). So, Proposition 1.1 is proved.

We emphasize that the two parameter family of smooth vector fields (1.4) obtained from the piecewise smooth perturbations of a Hamiltonian center (1.1) via the regularization method is a smooth perturbation of the same Hamiltonian center. So we can apply Theorem 2.1 in order to study the continuation of periodic solutions of (1.4). Thus, for each  $\mu > 0$  we can write equation (2.1) as

$$\mathcal{R}(a, \mu) = \oint_{\gamma_a} r_1 dy - r_2 dx,$$

where the functions  $r_1$  and  $r_2$  are given in (1.5) and (1.6), respectively. Equivalently the above function  $\mathcal{R}$  can be written as in (1.7) simply writing it as the sum of two integrals  $\mathcal{R}^-$  and  $\mathcal{R}^+$ .

In short, Theorem 1.2 is proved.

### 3 Proofs of Theorems 1.3 and 1.4

In this section we prove Theorems 1.3 and 1.4. We start the proof of Theorem 1.3 analyzing the function  $\mathcal{R}$  given in (1.7).

Let  $\gamma_a^+(s) = (x_a(s), y_a(s))$ ,  $s \in [0, T^+(a, 0)]$ . If  $0 < s < T^+(a, 0)$  then the component  $y_a$  is positive and bounded. By the continuity of the transition function  $\varphi$  we have

$$\lim_{\mu \rightarrow 0} \varphi^\mu(y_a(s)) = \lim_{\mu \rightarrow 0} \varphi\left(\frac{y_a(s)}{\mu}\right) = \varphi\left(\lim_{\mu \rightarrow 0} \frac{y_a(s)}{\mu}\right) = 1. \quad (3.1)$$

Therefore

$$\lim_{\mu \rightarrow 0} r_1(\gamma_a(s), \mu) = g_1^+(\gamma_a(s)) \quad (3.2)$$

and

$$\lim_{\mu \rightarrow 0} r_2(\gamma_a(s), \mu) = g_2^+(\gamma_a(s)). \quad (3.3)$$

Now, if  $T^+(a, 0) < s < T^-(a, 0)$  then the component  $y_a$  is negative and bounded, from which we have

$$\lim_{\mu \rightarrow 0} \varphi^\mu(y_a(s)) = \lim_{\mu \rightarrow 0} \varphi\left(\frac{y_a(s)}{\mu}\right) = \varphi\left(\lim_{\mu \rightarrow 0} \frac{y_a(s)}{\mu}\right) = 0. \quad (3.4)$$

Thus

$$\lim_{\mu \rightarrow 0} r_1(\gamma_a(s), \mu) = g_1^-(\gamma_a(s)) \quad (3.5)$$

and

$$\lim_{\mu \rightarrow 0} r_2(\gamma_a(s), \mu) = g_2^-(\gamma_a(s)). \quad (3.6)$$

For  $s = 0$ ,  $s = T^+(a, 0)$  or  $s = T^-(a, 0)$  we have  $y_a(s) = 0$ . Thus, for  $s = 0$ ,  $s = T^+(a, 0)$  or  $s = T^-(a, 0)$  we have

$$\lim_{\mu \rightarrow 0} r_1(\gamma_a(s), \mu) = g_1^-(x_a(s), 0) + \varphi(0)(g_1^+(x_a(s), 0) - g_1^-(x_a(s), 0)),$$

$$\lim_{\mu \rightarrow 0} r_2(\gamma_a(s), \mu) = g_2^-(x_a(s), 0) + \varphi(0)(g_2^+(x_a(s), 0) - g_2^-(x_a(s), 0)).$$

For each  $a \in I$  and  $\mu > 0$  fixed, define the function  $F^+ : [0, T^+(a, 0)] \rightarrow \mathbb{R}$  by

$$F^+(a, s, \mu) = r_1(\gamma_a(s), \mu)\dot{y}_a(s) - r_2(\gamma_a(s), \mu)\dot{x}_a(s).$$

As  $r_1$ ,  $r_2$ ,  $\dot{x}_a$  and  $\dot{y}_a$  are continuous functions in the interval  $[0, T^+(a, 0)]$  for all  $a \in I$  and  $\mu > 0$ , then  $F^+$  is continuous in this interval.

Consider the sequence

$$F_n^+(a, s) = F^+(a, s, \mu_n) \quad (3.7)$$

where  $(\mu_n)_{n \in \mathbb{N}}$  is a sequence of positive real numbers that tends to zero when  $n$  goes to infinity.

For each  $n \in \mathbb{N}$  the function  $F_n^+$  is Riemann integrable on  $[0, T^+(a, 0)]$ . We have  $F_n^+(a, s) \rightarrow F_0^+(a, s)$  almost everywhere on  $[0, T^+(a, 0)]$  when  $n \rightarrow \infty$ , where

$$F_0^+(a, s) = g_1^+(\gamma_a(s))\dot{y}_a(s) - g_2^+(\gamma_a(s))\dot{x}_a(s).$$

Furthermore,  $F_0^+$  is Riemann integrable on  $[0, T^+(a, 0)]$ . The sequence  $\{F_n^+\}$  is bounded, since their components are continuous functions on  $[0, T^+(a, 0)]$  and are defined in the compact set  $K = \gamma^+(a, [0, T^+(a, 0)])$ .

By the above analysis, we are under the hypotheses of the bounded convergence theorem. See [2] and [10]. Thus,

$$\lim_{n \rightarrow +\infty} \int_0^{T^+(a, 0)} F_n^+(a, s) ds = \int_0^{T^+(a, 0)} F_0^+(a, s) ds. \quad (3.8)$$

By the same way, define the function  $F^- : [T^+(a, 0), T^-(a, 0)] \rightarrow \mathbb{R}$  given by

$$F^-(a, s, \mu) = r_1(\gamma_a(s), \mu)\dot{y}_a(s) - r_2(\gamma_a(s), \mu)\dot{x}_a(s)$$

and the sequence

$$F_n^-(a, s) = F^-(a, s, \mu_n), \quad (3.9)$$

where  $(\mu_n)_{n \in \mathbb{N}}$  is a sequence of positive real numbers that tends to zero when  $n$  goes to infinity. With the same above arguments,  $F_n^-(a, s) \rightarrow F_0^-(a, s)$  almost everywhere on  $[T^+(a, 0), T^-(a, 0)]$  when  $n \rightarrow \infty$ , where

$$F_0^-(a, s) = g_1^-(\gamma_a(s))\dot{y}_a(s) - g_2^-(\gamma_a(s))\dot{x}_a(s).$$

Thus, using again the bounded convergence theorem, we have

$$\lim_{n \rightarrow +\infty} \int_{T^+(a, 0)}^{T^-(a, 0)} F_n^-(a, s) ds = \int_{T^+(a, 0)}^{T^-(a, 0)} F_0^-(a, s) ds. \quad (3.10)$$

Therefore, by using a classical theorem from analysis (see [11], page 84), we obtain

$$\begin{aligned} \lim_{\mu \rightarrow 0} \mathcal{R}(a, \mu) &= \lim_{n \rightarrow +\infty} \mathcal{R}(a, \mu_n) \\ &= \lim_{n \rightarrow +\infty} \int_0^{T^+(a, 0)} F_n^+(a, s) ds + \lim_{n \rightarrow +\infty} \int_{T^+(a, 0)}^{T^-(a, 0)} F_n^-(a, s) ds \\ &= \int_0^{T^+(a, 0)} F_0^+(a, s) ds + \int_{T^+(a, 0)}^{T^-(a, 0)} F_0^-(a, s) ds \\ &= \mathcal{M}^+(a) + \mathcal{M}^-(a) = \mathcal{M}(a). \end{aligned}$$

In short, Theorem 1.3 is proved. The proof of Theorem 1.4 follows from Theorem 1.3 and the transversality condition at a simple zero of the Melnikov function  $\mathcal{M}$ .

## 4 Applications

In this section we use the previous constructions in order to study lower bounds for the number of limit cycles that can appear from piecewise smooth perturbations of an isochronous Hamiltonian center.

From (1.1), consider

$$X' = \begin{cases} F(X) + \varepsilon G^-(X), & y \leq 0, \\ F(X) + \varepsilon G^+(X), & y \geq 0, \end{cases} \quad (4.1)$$

with the Hamiltonian vector field  $F$  given by

$$F(x, y) = \left( \frac{2}{(1-x^2)^2}(-y + x^2y), \frac{2}{(1-x^2)^2}(x + xy^2) \right)$$

and the vector fields  $G^\pm$  defined by

$$\begin{aligned} G^+(x, y) &= (g_1^+(x, y), g_2^+(x, y)), \\ G^-(x, y) &= (g_1^-(x, y), g_2^-(x, y)). \end{aligned}$$

The Hamiltonian function  $H$  associated to  $F$  is given by

$$H(x, y) = \frac{x^2 + y^2}{1 - x^2}.$$

The periodic orbits in  $U = \{(x, y) \in \mathbb{R}^2 : |x| < 1, y \in \mathbb{R}\} \setminus \{(0, 0)\}$  are given by

$$\Gamma_a : \{H(x, y) = a^2, a > 0\}$$

and a parametrization of these periodic orbits can be written as

$$\gamma_a(s) = \left( \frac{a}{\sqrt{1+a^2}} \cos(s), a \sin(s) \right), \quad s \in [0, 2\pi].$$

Also consider the displacement function

$$a \mapsto \Delta_\varepsilon(a) = \mathcal{P}_\varepsilon(a) - a,$$

where  $\mathcal{P}_\varepsilon$  is the Poincaré map associated with system (4.1) when the transversal section is  $\Sigma = \{(0, a) : a > \beta\}$  for some  $\beta > 0$ .

**Application 1.** The first case discussed here is one in which the components of the vector fields  $G^\pm$  are given by

$$\begin{aligned} g_1^+(x, y) &= \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 x^2 + \alpha_4 xy + \alpha_5 y^2, \\ g_2^+(x, y) &= \beta_0 + \beta_1 x + \beta_2 y + \beta_3 x^2 + \beta_4 xy + \beta_5 y^2, \\ g_1^-(x, y) &= \chi_0 + \chi_1 x + \chi_2 y + \chi_3 x^2 + \chi_4 xy + \chi_5 y^2, \\ g_2^-(x, y) &= \delta_0 + \delta_1 x + \delta_2 y + \delta_3 x^2 + \delta_4 xy + \delta_5 y^2, \end{aligned} \quad (4.2)$$

and  $\alpha_i, \beta_i, \chi_i, \delta_i \in \mathbb{R}$  for  $i \in \{0, 1, 2, 3, 4, 5\}$ .

From Theorem 1.5 we obtain the Melnikov function for (4.1)

$$\mathcal{M}(a) = \frac{a}{6(1+a^2)^{\frac{3}{2}}} P(a),$$

where

$$P(a) = p_4 a^4 + p_3 a^3 + p_2 a^2 + p_1 a + p_0$$

with

$$\begin{aligned} p_0 &= 12(\beta_0 - \delta_0), \\ p_1 &= 3\pi(\alpha_1 + \beta_2 + \delta_2 + \chi_1), \\ p_2 &= 4(\alpha_4 + 3\beta_0 + \beta_3 + 2\beta_5 - 3\delta_0 - \delta_3 - 2\delta_5 - \chi_4), \\ p_3 &= p_1, \\ p_4 &= 4(\alpha_4 + 2\beta_5 - 2\delta_5 - \chi_4). \end{aligned}$$

So if  $p_4 \neq 0$  then the function  $\mathcal{M}$  has at most four positive zeros and there are explicit choices of parameters for which this number is reached. For instance, if we take in (4.2)

$$\begin{aligned} \alpha_1 &= \frac{2}{\pi}, & \alpha_4 &= \frac{3}{2}, & \beta_0 &= \frac{5}{34}, & \beta_5 &= \frac{3}{2}, \\ \chi_1 &= -\frac{223}{17\pi}, & \chi_4 &= \frac{3}{2}, & \delta_3 &= -\frac{435}{34}, & \delta_5 &= \frac{3}{4}, \end{aligned} \quad (4.3)$$

and the other coefficients equal to zero, we obtain

$$a_0 = \frac{1}{17}, \quad a_1 = 1, \quad a_2 = 2, \quad a_3 = \frac{5}{2}$$

as zeros of the function  $\mathcal{M}$ . Moreover, since

$$\mathcal{M}'(a_0) = -\frac{5478\sqrt{\frac{2}{145}}}{2465}, \quad \mathcal{M}'(a_1) = \frac{6\sqrt{2}}{17},$$

$$\mathcal{M}'(a_2) = -\frac{33}{85\sqrt{5}}, \quad \mathcal{M}'(a_3) = \frac{1245}{986\sqrt{29}},$$

then for  $\varepsilon > 0$  sufficiently small there are two stable and two unstable limit cycles in the phase portrait of system (4.1). Figure 4.1 shows the graph of the function  $\mathcal{M}$  and its zeros. In Figure 4.2 we present the graph of the displacement function  $\Delta_\varepsilon$  obtained numerically for  $\varepsilon = 0.05$ . In this case the zeros of the function  $\Delta_\varepsilon$  are

$$a_0^{0.05} = 0.058630, \quad a_1^{0.05} = 1.040895, \quad a_2^{0.05} = 1.893110, \quad a_3^{0.05} = 2.573276,$$

with only six decimals.

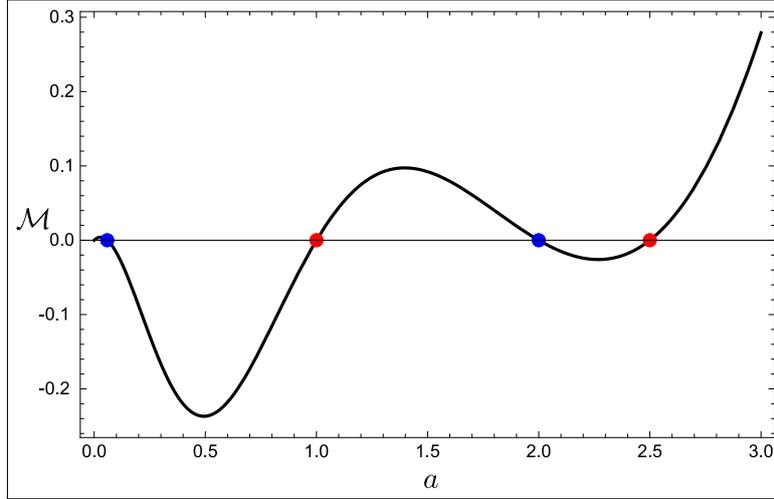


Figure 4.1: The black line represents the graph of the function  $\mathcal{M}$  associated with system (4.1) when  $g_1^+$ ,  $g_2^+$ ,  $g_1^-$  and  $g_2^-$  are such as in (4.2). The values of the coefficients in (4.2) are those given in (4.3). The blue dots correspond to the zeros  $a_0 = 1/17$  and  $a_2 = 2$  of the function  $\mathcal{M}$  and they are associated with the stable limit cycles of system (4.1) for  $\varepsilon > 0$  sufficiently small. The red dots correspond to the zeros  $a_1 = 1$  and  $a_3 = 5/2$  of the function  $\mathcal{M}$  and they are associated with the unstable limit cycles of system (4.1) for  $\varepsilon > 0$  sufficiently small.

**Application 2.** Consider now the case in which the coefficients of the quadratic part of (4.2) are all null and the others may be chosen arbitrarily. It is easy to see that there is a unique positive zero of  $\mathcal{M}$  given by

$$a_0 = \frac{4(\delta_0 - \beta_0)}{\pi(\alpha_1 + \beta_2 + \chi_1 + \delta_2)},$$

provided that  $\delta_0 - \beta_0 \neq 0$  and  $\alpha_1 + \beta_2 + \chi_1 + \delta_2$  have the same sign. Furthermore, this zero is simple since

$$\mathcal{M}'(a_0) = \frac{2}{\sqrt{1 + (a_0)^2}}(\delta_0 - \beta_0) \neq 0.$$

Therefore for  $\varepsilon > 0$  sufficiently small there is a stable (if  $\delta_0 - \beta_0 < 0$ ) or an unstable (if  $\delta_0 - \beta_0 > 0$ ) limit cycle  $X^\varepsilon$  of (4.1) such that  $X^\varepsilon$  tends to  $\Gamma_{a_0}$  when  $\varepsilon$  goes to 0.

Taking, for example, the following coefficients in (4.2)

$$\begin{aligned} \alpha_0 &= 1, & \alpha_1 &= -1, & \alpha_2 &= 1, & \beta_0 &= -\frac{\pi}{2}, & \beta_1 &= 2, & \beta_2 &= -1, \\ \chi_0 &= -1, & \chi_1 &= -1, & \chi_2 &= \frac{1}{2}, & \delta_0 &= -\pi, & \delta_1 &= -2, & \delta_2 &= -1 \end{aligned}$$

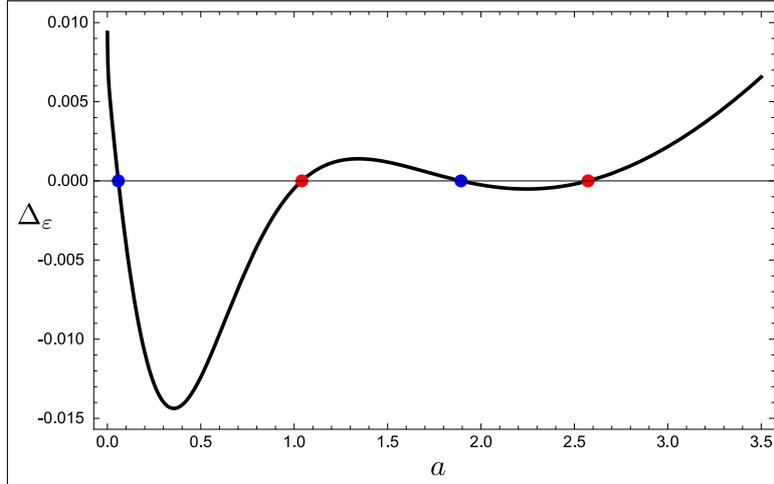


Figure 4.2: The black line represents the graph of the displacement function  $\Delta_\varepsilon$  associated with system (4.1) when  $g_1^+$ ,  $g_2^+$ ,  $g_1^-$  and  $g_2^-$  are such as in (4.2) and for  $\varepsilon = 0.05$ . The values of the coefficients in (4.2) are those given in (4.3). The blue dots correspond to the zeros  $a_0^\varepsilon = 0.058630$  and  $a_2^\varepsilon = 1.893110$  of the function  $\Delta_\varepsilon$  and they are associated with the stable limit cycles of system (4.1). The red dots correspond to the zeros  $a_1^\varepsilon = 1.040895$  and  $a_3^\varepsilon = 2.573276$  of the function  $\Delta_\varepsilon$  and they are associated with the unstable limit cycles of system (4.1).

and  $\alpha_i = \beta_i = \chi_i = 0$  for  $i = 3, 4, 5$  we obtain

$$a_0 = \frac{1}{2}, \quad \mathcal{M}'\left(\frac{1}{2}\right) = -\frac{2\pi\sqrt{5}}{5} < 0.$$

Thus for  $\varepsilon > 0$  sufficiently small there is a stable limit cycle  $X^\varepsilon$  of (4.1) such that  $X^\varepsilon$  tends to the ellipse  $\Gamma_{1/2}$  when  $\varepsilon$  goes to 0. The phase portrait of system (4.1) for  $\varepsilon = 0.2$  is illustrated in Figure 4.3. The stable limit cycle  $X^{0.2}$  is depicted in black.

**Application 3.** Another interesting case occurs when the components of the vector fields  $G^\pm$  are of the form

$$\begin{aligned} g_1^+(x, y) &= 0, \\ g_2^+(x, y) &= \sin(y), \\ g_1^-(x, y) &= \cos(x), \\ g_2^-(x, y) &= 0. \end{aligned} \tag{4.4}$$

From Theorem 1.5 the Melnikov function for (4.1) is

$$\mathcal{M}(a) = \frac{\pi a}{\sqrt{1+a^2}} J_1(a)$$

where  $J_1$  is the Bessel function of first kind. Therefore in this case the function  $\mathcal{M}$  has a countable number of positive zeros. Figure 4.4 shows the graph of the function  $\mathcal{M}$  and the first five positive zeros displayed with six decimals

$$\begin{aligned} a_0 &= 3.831705, & a_1 &= 7.015586, & a_2 &= 10.173468, \\ a_3 &= 13.323691, & a_4 &= 16.470630. \end{aligned}$$

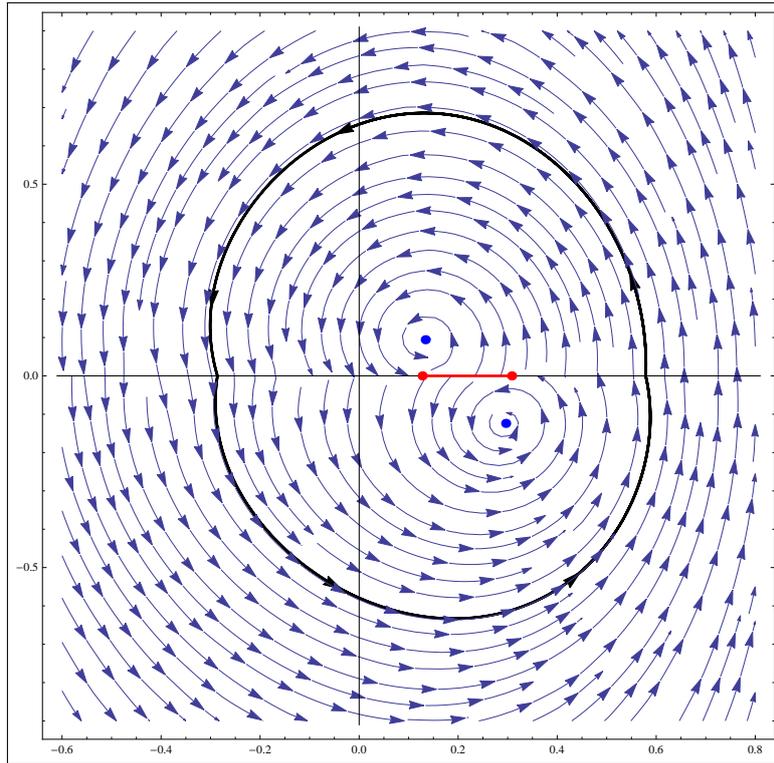


Figure 4.3: Phase portrait of system (4.1) for  $\varepsilon = 0.2$ ,  $\alpha_0 = 1$ ,  $\alpha_1 = -1$ ,  $\alpha_2 = 1$ ,  $\beta_0 = -\pi/2$ ,  $\beta_1 = 2$ ,  $\beta_2 = -1$ ,  $\chi_0 = -1$ ,  $\chi_1 = -1$ ,  $\chi_2 = 1/2$ ,  $\delta_0 = -\pi$ ,  $\delta_1 = -2$ ,  $\delta_2 = -1$  and  $\alpha_i = \beta_i = \chi_i = 0$  for  $i = 3, 4, 5$ . The stable limit cycle  $X^{0.2}$  is depicted in black while the sliding segment on  $\Sigma$  is depicted in red.

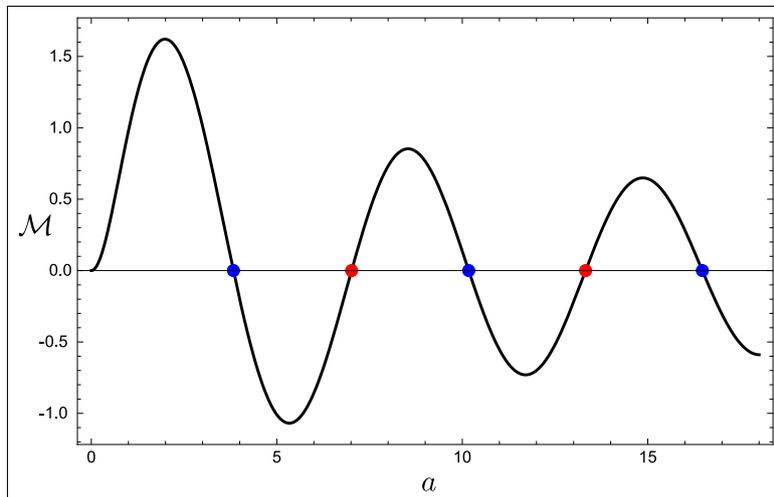


Figure 4.4: The black line represents the graph of the function  $\mathcal{M}$  associated with system (4.1) when  $g_1^+$ ,  $g_2^+$ ,  $g_1^-$  and  $g_2^-$  are such as in (4.4). The blue dots are associated with stable limit cycles and the red dots correspond to unstable limit cycles of system (4.1) for  $\varepsilon > 0$  sufficiently small.

The graph of the displacement function  $\Delta_\varepsilon$  is shown in Figure 4.5 for  $\varepsilon = 0.2$ . The first five positive zeros displayed with six decimals are

$$a_0^{0.2} = 3.831467, \quad a_1^{0.2} = 7.016595, \quad a_2^{0.2} = 10.170321,$$

$$a_3^{0.2} = 13.332004, \quad a_4^{0.2} = 16.457765.$$

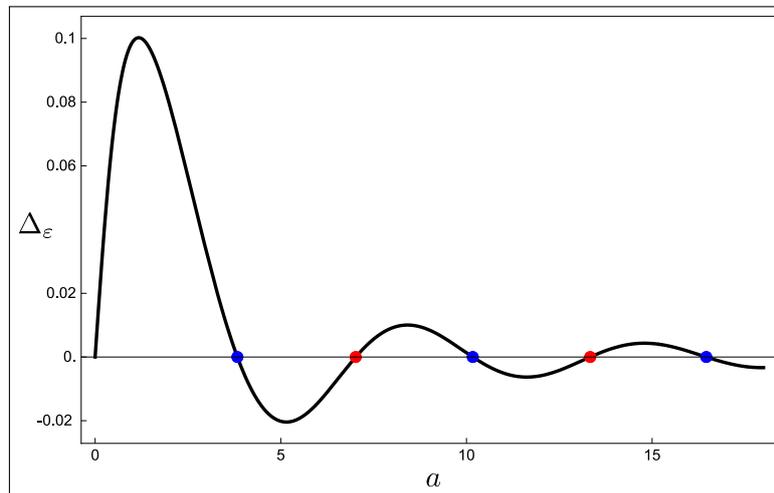


Figure 4.5: The black line represents the graph of the displacement function  $\Delta_\varepsilon$  associated with system (4.1) when  $g_1^+$ ,  $g_2^+$ ,  $g_1^-$  and  $g_2^-$  are such as in (4.4). The blue dots are associated with the stable limit cycles and the red dots correspond to the unstable limit cycles of system (4.1).

## 5 Concluding remarks

In this article we study the existence and positions of limit cycles in piecewise smooth perturbations of planar Hamiltonian centers. By using the regularization method we obtain analytically an expression for the first order Melnikov function related to the original non-smooth problem. Our study extends the previous one in [3] where the authors studied piecewise smooth perturbations of planar linear centers.

The study of piecewise smooth perturbations of piecewise planar Hamiltonian centers is a possible line of research in this context.

## Acknowledgements

The third author is partially supported by Capes/Estágio Sênior no Exterior grant number 88881.119020/2016-01. In the course of this work he was a visitor at Texas Christian University and gratefully acknowledges its warm hospitality.

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