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Multiple positive solutions for (n-1, 1)-type semipositone conjugate boundary value problems for coupled systems of nonlinear fractional differential equations*

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Abstract. In this paper, we consider (n-1, 1)-type conjugate boundary value problem for coupled systems of the nonlinear fractional differential equation

$$\begin{cases} \mathbf{D}_{0+}^{\alpha} u + \lambda f(t,v) = 0, \quad 0 < t < 1, \lambda > 0, \\ \mathbf{D}_{0+}^{\alpha} v + \lambda g(t,u) = 0, \\ u^{(i)}(0) = v^{(i)}(0) = 0, \quad 0 \le i \le n-2, \\ u(1) = v(1) = 0, \end{cases}$$

where λ is a parameter, $\alpha \in (n-1, n]$ is a real number and $n \ge 3$, and $\mathbf{D}_{0+}^{\alpha}$ is the Riemann-Liouville's fractional derivative, and f, g are continuous and semipositone. We give properties of Green's function of the boundary value problem, and derive an interval on λ such that for any λ lying in this interval, the semipositone boundary value problem has multiple positive solutions.

Key words. Riemann-Liouville's fractional derivative; fractional differential equation; boundary value problem; positive solution; fractional Green's function; fixed-point theorem.

MR(2000) Subject Classifications: 34B15

1 Introduction

We consider the (n-1, 1)-type conjugate boundary value problem for nonlinear fractional differential equation involving Riemann-Liouville's derivative

$$\begin{cases} \mathbf{D}_{0+}^{\alpha} u + \lambda f(t,v) = 0, \quad 0 < t < 1, \lambda > 0, \\ \mathbf{D}_{0+}^{\alpha} v + \lambda g(t,u) = 0, \\ u^{(i)}(0) = v^{(i)}(0) = 0, \quad 0 \le i \le n-2, \\ u(1) = v(1) = 0, \end{cases} \tag{1.1}$$

where λ is a parameter, $\alpha \in (n-1, n]$ is a real number, $n \ge 3$, $\mathbf{D}_{0+}^{\alpha}$ is the Riemann-Liouville's fractional derivative, and f, g are sign-changing continuous functions. As far as we know, there are few papers which deal with the boundary value problem for nonlinear fractional differential equation.

Because of fractional differential equation's modeling capabilities in engineering, science, economics, and other fields, the last few decades has resulted in a rapid development of the theory of fractional differential equations, see [1]-[7] for a good overview. Within this development, a fair amount of the theory has been devoted to initial and boundary value problems problems (see [9]-[20]). In most papers, the definition of fractional derivative is the Riemann-Liouville's fractional derivative or the Caputo's fractional derivative. For details, see the references.

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In this paper, we give sufficient conditions for the existence of positive solution of the semipositone boundary value problems (1.1) for a sufficiently small $\lambda > 0$ where f, g may change sign. Our analysis relies on nonlinear alternative of Leray-Schauder type and Krasnosel'skii's fixed-point theorems.

2 Preliminaries

For completeness, in this section, we will demonstrate and study the definitions and some fundamental facts of Riemann-Liouville's derivatives of fractional order which can been founded in [3].

Definition 2.1 *[3]* The integral

$$I_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x > 0,$$

where $\alpha > 0$, is called Riemann-Liouville fractional integral of order α.

Definition 2.2 *[3]* For a function $f(x)$ given in the interval $[0, \infty)$, the expression

$$D_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dx})^n \int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt,$$

where $n = [\alpha] + 1, [\alpha]$ denotes the integer part of number α, is called the Riemann-Liouville fractional derivative of order s.

From the definition of the Riemann-Liouville derivative, we can obtain the statement.

As examples, for $\mu > -1$, we have

$$\mathbf{D}_{0+}^{\alpha} x^{\mu} = \frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\alpha)} x^{\mu-\alpha}$$

giving in particular $\mathbf{D}_{0+}^{\alpha} x^{\alpha-m}, \ m = i, 2, 3, \cdots, N$, where N is the smallest integer greater than or equal to α.

Lemma 2.1 *Let $\alpha > 0$; then the differential equation*

$$\mathbf{D}_{0+}^{\alpha} u(t) = 0$$

has solutions $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, , 2 \ldots, n$, as unique solutions, where n is the smallest integer greater than or equal to α.

As $\mathbf{D}_{0+}^{\alpha} I_{0+}^{\alpha} u = u$. From the lemma 2.1, we deduce the following statement.

Lemma 2.2 *Let $\alpha > 0$, then*

$$I_{0+}^{\alpha} \mathbf{D}_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, n is the smallest integer greater than or equal to α.

Lemma 2.3 *[16] Let $h(t) \in C[0, 1]$ be a given function, then the boundary-value problem*

$$\begin{cases} \mathbf{D}_{0+}^{\alpha} u(t) + h(t) = 0, \quad 0 < t < 1, 2 \le n-1 < \alpha \le n, \\ u^{(i)}(0) = 0, \quad 0 \le i \le n-2, \\ u(1) = 0 \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s) h(s) ds, \tag{2.2}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0 \le t \le s \le 1. \end{cases} \tag{2.3}$$

Here $G(t, s)$ is called the Green's function for the boundary value problem (2.1).

Lemma 2.4 *[16] The Green's function $G(t,s)$ defined by* (2.3) *has the following properties:*

(R1) $G(t,s) = G(1-s, 1-t)$, *for* $t, s \in [0,1]$,

(R2) $\Gamma(\alpha)k(t)q(s) \le G(t,s) \le (\alpha-1)q(s)$, *for* $t, s \in [0,1]$,

(R3) $\Gamma(\alpha)k(t)q(s) \le G(t,s) \le (\alpha-1)k(t)$, *for* $t, s \in [0,1]$,

where

$$k(t) = \frac{t^{\alpha-1}(1-t)}{\Gamma(\alpha)}, \quad q(s) = \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha)}. \tag{2.4}$$

The following a nonlinear alternative of Leray-Schauder type and Krasnosel'skii's fixed-point theorems, will play major role in our next analysis.

Theorem 2.5 *[12] Let X be a Banach space with $\Omega \subset X$ be closed and convex. Assume U is a relatively open subsets of Ω with $0 \in U$, and let $S : \overline{U} \to \Omega$ be a compact, continuous map. Then either*

1. *S has a fixed point in $\overline{U}$, or*
2. *there exists $u \in \partial U$ and $\nu \in (0,1)$, with $u = \nu S u$.*

Theorem 2.6 *[8] Let X be a Banach space, and let $P \subset X$ be a cone in X. Assume Ω_1, Ω_2 are bounded open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $S : P \to P$ be a completely continuous operator such that, either*

1. *$\|Sw\| \le \|w\|$, $w \in P \cap \partial\Omega_1$, $\|Sw\| \ge \|w\|$, $w \in P \cap \partial\Omega_2$, or*
2. *$\|Sw\| \ge \|w\|$, $w \in P \cap \partial\Omega_1$, $\|Sw\| \le \|w\|$ $w \in P \cap \partial\Omega_2$.*

Then S has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3 Main Results

We make the following assumption:

(H_1) $f(t,z), g(t,z) \in C([0,1] \times [0,+\infty), (-\infty,+\infty))$, moreover there exists a function $e(t) \in L^1((0,1),(0,+\infty))$ such that $f(t,z) \ge -e(t)$ and $g(t,z) \ge -e(t)$, for any $t \in (0,1)$, $z \in [0,+\infty)$.

(H_1^*) $f(t,z), g(t,z) \in C((0,1) \times [0,+\infty), (-\infty,+\infty))$, f, g may be singular at $t = 0, 1$, moreover there exists a function $e(t) \in L^1([0,1],(0,+\infty))$ such that $f(t,z) \ge -e(t)$ and $g(t,z) \ge -e(t)$, for any $t \in (0,1)$, $z \in [0,+\infty)$.

(H_2) $f(t,0) > 0$ for $t \in [0,1]$; there exist $M > 0$, $\sigma > 0$ such that $\limsup\limits_{z \downarrow 0} \frac{g(t,z)}{z} < M$ for $t \in [0,1]$ and $g(t,z) > 0$ for $(t,z) \in [0,1] \times (0,\sigma]$.

(H_3) There exists $[\theta_1,\theta_2] \subset (0,1)$ such that $\lim\limits_{z \uparrow +\infty} \inf\limits_{t \in [\theta_1,\theta_2]} \frac{f(t,z)}{z} = +\infty$ and $\lim\limits_{z \uparrow +\infty} \inf\limits_{t \in [\theta_1,\theta_2]} \frac{g(t,z)}{z} = +\infty$.

(H_4) $\int_0^1 q(s)e(s)ds < +\infty$, $\int_0^1 q(s)f(s,z)ds < +\infty$ and $\int_0^1 q(s)g(s,z)ds < +\infty$ for any $z \in [0,m]$, $m > 0$ is any constant.

In fact, we only consider the boundary value problem

$$\begin{cases} -\mathbf{D}_{0+}^{\alpha} x = \lambda(f(t,[y(t)-w(t)]^*) + e(t)), & t \in (0,1), \lambda > 0, \\ -\mathbf{D}_{0+}^{\alpha} y = \lambda(g(t,[x(t)-w(t)]^*) + e(t)), & t \in (0,1), \\ x^{(i)}(0) = y^{(i)}(0) = 0, & 0 \le i \le n-2, \\ x(1) = y(1) = 0, \end{cases} \tag{3.1}$$

where

$$z(t)^* = \begin{cases} z(t), & z(t) \ge 0; \\ 0, & z(t) < 0. \end{cases}$$

and $w(t)=\lambda\int_0^1 G(t,s)e(s)ds$, which is the solution of the boundary value problem

$$\begin{cases} -\mathbf{D}_{0+}^{\alpha}w=\lambda e(t), \quad t\in(0,1),\\ w^{(i)}(0)=0, \quad 0\le i\le n-2,\\ w(1)=0.\end{cases}$$

We will show there exists a solution (x,y) for the boundary value problem (3.1) with $x(t)\ge w(t)$ and $y(t)\ge w(t)$ for $t\in[0,1]$. If this is true, then $u(t)=x(t)-w(t)$ and $v(t)=y(t)-w(t)$ is a nonnegative solution (positive on $(0,1)$) of the boundary value problem (1.1). Since for any $t\in(0,1)$,

$$\begin{aligned} -\mathbf{D}_{0+}^{\alpha}x&=-\mathbf{D}_{0+}^{\alpha}u+(-\mathbf{D}_{0+}^{\alpha}w)=\lambda[f(t,v)+e(t)],\\ -\mathbf{D}_{0+}^{\alpha}y&=-\mathbf{D}_{0+}^{\alpha}v+(-\mathbf{D}_{0+}^{\alpha}w)=\lambda[g(t,u)+e(t)],\end{aligned}$$

we have

$$-\mathbf{D}_{0+}^{\alpha}u=\lambda f(t,v) \text{ and } -\mathbf{D}_{0+}^{\alpha}v=\lambda g(t,u).$$

As a result, we will concentrate our study on the boundary value problem (3.1).

We note that (3.1) is equal to

$$\begin{cases} x(t)=\lambda\int_0^1 G(t,s)(f(s,[y(s)-w(s)]^*)+e(s))ds\\ y(t)=\lambda\int_0^1 G(t,s)(g(s,[x(s)-w(s)]^*)+e(s))ds.\end{cases} \tag{3.2}$$

From (3.2) we have

$$x(t)=\lambda\int_0^1 G(t,s)(f(s,[\lambda\int_0^1 G(s,\tau)g(\tau,[x(\tau)-w(\tau)]^*)d\tau]^*)+e(s))ds. \tag{3.3}$$

For our constructions, we shall consider the Banach space $E=C[0,1]$ equipped with standard norm $\|x\|=\max\limits_{0\le t\le 1}|x(t)|, x\in X$. We define a cone P by

$$P=\{x\in X|x(t)\ge\frac{t^{\alpha-1}(1-t)}{p}\|x\|, \quad t\in[0,1],\alpha\in(n-1,n],n\ge 3\}.$$

Define an integral operator $T:P\to X$ by

$$Tx(t)=\lambda\int_0^1 G(t,s)(f(s,[\lambda\int_0^1 G(s,\tau)g(\tau,[x(\tau)-w(\tau)]^*)d\tau]^*)+e(s))ds.$$

Notice, from Lemma 2.3, we have $Tx(t)\ge 0$ on $[0,1]$ for $x\in P$ and

$$\begin{aligned} Tx(t)&=\lambda\int_0^1 G(t,s)(f(s,[\lambda\int_0^1 G(s,\tau)g(\tau,[x(\tau)-w(\tau)]^*)d\tau]^*)+e(s))ds\\ &\le\lambda\int_0^1(\alpha-1)q(s)(f(s,[\lambda\int_0^1 G(s,\tau)g(\tau,[x(\tau)-w(\tau)]^*)d\tau]^*)+e(s))ds,\end{aligned}$$

then $\|Tx\|\le\lambda\int_0^1(\alpha-1)q(s)(f(s,[\lambda\int_0^1 G(s,\tau)g(\tau,[x(\tau)-w(\tau)]^*)d\tau]^*)+e(s))ds.$

On the other hand, we have

$$\begin{aligned} Tx(t)&=\lambda\int_0^1 G(t,s)(f(s,[\lambda\int_0^1 G(s,\tau)g(\tau,[x(\tau)-w(\tau)]^*)d\tau]^*)+e(s))ds\\ &\ge\tfrac{t^{\alpha-1}(1-t)}{\alpha-1}\lambda\int_0^1(\alpha-1)q(s)(f(s,[\lambda\int_0^1 G(s,\tau)g(\tau,[x(\tau)-w(\tau)]^*)d\tau]^*)+e(s))ds\\ &\ge\tfrac{t^{\alpha-1}(1-t)}{\alpha-1}\|Tx\|.\end{aligned}$$

Thus, $T(P)\subset P$. In addition, standard arguments show that T is a compact, completely continuous operator.

Theorem 3.1 *Suppose that* (H_1) *and* (H_2) *hold. Then there exists a constant* $\overline{\lambda}>0$ *such that, for any* $0<\lambda\le\overline{\lambda}$, *the boundary value problem (1.1) has at least one positive solution.*

Proof Fix $\delta \in (0,1)$. From (H_2), let $0 < \varepsilon < \min\{1, \sigma\}$ be such that

$$f(t,z) \geq \delta f(t,0), \quad g(t,z) \leq Mz, \quad \text{for} \quad 0 \leq t \leq 1, \quad 0 \leq z \leq \varepsilon. \tag{3.4}$$

and

$$g(t,z) > 0, \text{ for } \quad 0 \leq t \leq 1, \quad 0 < z \leq \varepsilon.$$

Suppose

$$0 < \lambda < \min\{\frac{\varepsilon}{2c\overline{f}(\varepsilon)}, \frac{1}{Mc}\} := \overline{\lambda},$$

where $\overline{f}(\varepsilon) = \max\limits_{0 \leq t \leq 1, 0 \leq z \leq \varepsilon} \{f(t,z) + e(t)\}$ and $c = \int_0^1 (\alpha - 1) q(s) ds$. Since

$$\lim_{z \downarrow 0} \frac{\overline{f}(z)}{z} = +\infty$$

and

$$\frac{\overline{f}(\varepsilon)}{\varepsilon} < \frac{1}{2c\lambda},$$

then exists a $R_0 \in (0, \varepsilon)$ such that

$$\frac{\overline{f}(R_0)}{R_0} = \frac{1}{2c\lambda}.$$

Let $U = \{x \in P : \|x\| < R_0\}$, $x \in \partial U$ and $\nu \in (0,1)$ be such that $x = \nu T(x)$, we claim that $\|x\| \neq R_0$. In fact, for $x \in \partial U$ and $\|x\| \neq R_0$, we have

$$\begin{aligned}
&\lambda \textstyle\int_0^1 G(s,\tau) g(\tau, [x(\tau) - w(\tau)]^*) d\tau \\
&\leq \lambda \textstyle\int_0^1 (\alpha - 1) q(\tau) g(\tau, [x(\tau) - w(\tau)]^*) d\tau \\
&\leq \lambda \textstyle\int_0^1 (\alpha - 1) q(\tau) M [x(\tau) - w(\tau)]^* d\tau \\
&\leq \lambda \textstyle\int_0^1 (\alpha - 1) q(\tau) M R_0 d\tau \\
&\leq \lambda M \textstyle\int_0^1 (\alpha - 1) q(\tau) d\tau R_0 \\
&\leq R_0.
\end{aligned} \tag{3.5}$$

It follows that

$$\begin{aligned}
x(t) &= \nu T x(t) \\
&\leq \nu \lambda \textstyle\int_0^1 (\alpha - 1) q(s) (f(s, [\lambda \int_0^1 G(s,\tau) g(\tau, [x(\tau) - w(\tau)]^*) d\tau]^*) + e(s)) ds \\
&\leq \lambda \textstyle\int_0^1 (\alpha - 1) q(s) (f(s, [\lambda \int_0^1 G(s,\tau) g(\tau, [x(\tau) - w(\tau)]^*) d\tau]^*) + e(s)) ds \\
&\leq \lambda \textstyle\int_0^1 (\alpha - 1) q(s) \max\limits_{0 \leq s \leq 1; 0 \leq z \leq R_0} [f(s,z) + e(s)] ds \\
&\leq \lambda \textstyle\int_0^1 (\alpha - 1) q(s) \overline{f}(R_0) ds \\
&\leq \lambda c \overline{f}(R_0),
\end{aligned}$$

that is

$$\frac{\overline{f}(R_0)}{R_0} \geq \frac{1}{c\lambda} > \frac{1}{2c\lambda} = \frac{\overline{f}(R_0)}{R_0},$$

which implies that $\|x\| \neq R_0$. By the nonlinear alternative of Leray-Schauder type, T has a fixed point $x \in \overline{U}$. Moreover, combing (3.4), (3.5) and the fact that $R_0 < \varepsilon$, we obtain

$$\begin{aligned}
x(t) &= \lambda \textstyle\int_0^1 G(t,s) (f(s, [\lambda \int_0^1 G(s,\tau) g(\tau, [x(\tau) - w(\tau)]^*) d\tau]^*) + e(s)) ds \\
&\geq \lambda \textstyle\int_0^1 G(t,s) [\delta f(s,0) + e(s)] ds \\
&\geq \lambda [\delta \textstyle\int_0^1 G(t,s) f(s,0) ds + \int_0^1 G(t,s) e(s) ds] \\
&> \lambda \textstyle\int_0^1 G(t,s) e(s) ds \\
&= w(t) \quad \text{for} \quad t \in (0,1).
\end{aligned}$$

Then T has a positive fixed point x and $\|x\| \leq R_0 < 1$.

On the other hand, from (3.2) and $w < x \le R_0 \le \varepsilon \le \sigma$, we have $g(s, x(s) - w(s)) > 0$. Then

$$\begin{aligned} y(t) &= \lambda \int_0^1 G(t,s)(g(s,[x(s)-w(s)]^*) + e(s))ds \\ &= \lambda \int_0^1 G(t,s)(g(s,x(s)-w(s)) + e(s))ds \\ &= \lambda[\int_0^1 G(t,s)g(s,x(s)-w(s))ds + \int_0^1 G(t,s)e(s)ds] \\ &> \lambda \int_0^1 G(t,s)e(s)ds \\ &= w(t) \quad \text{for} \quad t \in (0,1). \end{aligned}$$

Thus, (x,y) is positive solution (x,y) of the boundary value problem (3.1) with $x(t) \ge w(t)$ and $y(t) \ge w(t)$ for $t \in [0,1]$.

Let $u(t) = x(t) - w(t) > 0$ and $v(t) = y(t) - w(t) > 0$, then (u,v) is a nonnegative solution (positive on $(0,1)$) of the boundary value problem (1.1).

Theorem 3.2 *Suppose that* (H_1^*) *and* (H_3)-(H_4) *hold. Then there exists a constant* $\lambda^* > 0$ *such that, for any* $0 < \lambda \le \lambda^*$, *the boundary value problem (1.1) has at least one positive solution.*

Proof From (H_3), we choose $R_1 > \max\{1, r^2, (\frac{2(\alpha-1)}{\gamma})^2\}$ such that

$$\frac{g(t,z)}{z} > N_0, \quad \text{namely} \quad g(t,z) > N_0 z, \quad \text{for} \quad t \in [\theta_1, \theta_2], z > R_1^{\frac{1}{2}},$$

and $N_0 > 0$ satisfy

$$N_0 > \frac{r}{\rho},$$

where $r = \frac{\alpha-1}{\Gamma(\alpha)} \int_0^1 e(s)ds$, $\gamma = \min\limits_{\theta_1 \le t \le \theta_2} \{t^{\alpha-1}(1-t)\}$, and $\rho = \int_{\theta_1}^{\theta_2} q(s)ds$.

Let $\Omega_1 = \{x \in C[0,1] : \|x\| < R_1\}$ and

$$\lambda^* = \min\{\frac{1}{\alpha-1}, R_1[\int_0^1 (\alpha-1)q(s)[\max_{0\le z\le R} f(s,z) + g(s)]ds]^{-1}, \frac{R_1}{2(\alpha-1)r}\},$$

where $R = \int_0^1 (\alpha-1)q(\tau) \max\limits_{0\le z\le R_1} g(\tau,z)d\tau$ and $R > 0$.

Then for any $x \in P \cap \partial\Omega_1$, we have $\|x\| = R_1$ and $x(s) - w(s) \le x(s) \le \|x\|$,

$$\begin{aligned} \lambda \int_0^1 G(s,\tau)g(\tau,[x(\tau)-w(\tau)]^*)d\tau &\le \lambda \int_0^1 G(s,\tau) \max_{0\le z\le R_1} g(\tau,z)d\tau \\ &\le \int_0^1 (\alpha-1)q(\tau) \max_{0\le z\le R_1} g(\tau,z)d\tau = R. \end{aligned}$$

It follows that

$$\begin{aligned} \|Tx(t)\| &\le \lambda \int_0^1 (\alpha-1)q(s)(f(s,[\lambda \int_0^1 G(s,\tau)g(\tau,[x(\tau)-w(\tau)]^*)d\tau]^*) + e(s))ds \\ &\le \lambda \int_0^1 (\alpha-1)q(s)[\max_{0\le z\le R} f(s,z) + e(s)]ds \\ &\le R_1 = \|x\|. \end{aligned}$$

This implies

$$\|Tx\| \le \|x\|, x \in P \cap \partial\Omega_1.$$

On the other hand, choose a constant $N > 1$ such that

$$N \min\left\{\rho(r + \frac{1}{\lambda\gamma})^{-1}, \quad \frac{\gamma\rho}{2(\alpha-1)(1+r)}, \quad \lambda^2\gamma^2\rho\right\} \ge 1,$$

where $\gamma = \min\limits_{\theta_1 \le t \le \theta_2} \{t^{\alpha-1}(1-t)\}$.

By the assumption (H_3), there exists a constant $B > R_1$ such that

$$\frac{f(t,z)}{z} > N, \quad \text{namely} \quad f(t,z) > Nz, \quad \text{for} \quad t \in [\theta_1, \theta_2], z > B;$$

and

$$\frac{g(t,z)}{z} > N, \quad \text{namely} \quad g(t,z) > Nz, \quad \text{for} \quad t \in [\theta_1, \theta_2], z > B.$$

Choose $R_2 = \max\{R_1 + 1, 2\lambda(\alpha-1)r, \frac{2(\alpha-1)(B+1)}{\gamma}\}$, and let $\Omega_2 = \{x \in C[0,1] : \|x\| < R_2\}$. Then for any $x \in P \cap \partial\Omega_2$, we have

$$\begin{aligned} x(t) - w(t) &= x(t) - \lambda \int_0^1 G(t,s)e(s)ds \\ &\geq x(t) - \tfrac{\alpha-1}{\Gamma(\alpha)} t^{\alpha-1}(1-t)\lambda \int_0^1 e(s)ds \\ &\geq x(t) - t^{\alpha-1}(1-t)\lambda r \\ &\geq x(t) - \tfrac{(\alpha-1)x(t)}{\|x\|}\lambda r \\ &\geq x(t) - \tfrac{(\alpha-1)x(t)}{R_2}\lambda r \\ &\geq (1 - \tfrac{(\alpha-1)\lambda r}{R_2})x(t) \\ &\geq \tfrac{1}{2}x(t) \geq 0, \ t \in [0,1]. \end{aligned}$$

And then

$$\begin{aligned} \min_{\theta_1 \leq t \leq \theta_2}\{x(t) - w(t)\} &\geq \min_{\theta_1 \leq t \leq \theta_2}\{\tfrac{1}{2}x(t)\} \geq \min_{\theta_1 \leq t \leq \theta_2}\{\tfrac{1}{2(\alpha-1)} t^{\alpha-1}(1-t)\|x\|\} \\ &= \tfrac{1}{2(\alpha-1)} R_2 \min_{\theta_1 \leq t \leq \theta_2}\{t^{\alpha-1}(1-t)\} \geq B + 1 > B. \end{aligned}$$

It follows that

$$\begin{aligned} &\lambda \int_0^1 G(s,\tau)g(\tau, [x(\tau) - w(\tau)]^*)d\tau \\ &\quad = \lambda(\int_0^1 G(s,\tau)(g(\tau, [x(\tau) - w(\tau)]^*) + e(\tau))d\tau - \int_0^1 G(s,\tau)e(\tau)d\tau) \\ &\quad \geq \lambda(s^{\alpha-1}(1-s)\int_0^1 q(\tau)(g(\tau, [x(\tau) - w(\tau)]^*) + e(\tau))d\tau - \tfrac{\alpha-1}{\Gamma(\alpha)} s^{\alpha-1}(1-s))\int_0^1 e(\tau)d\tau) \\ &\quad \geq \lambda s^{\alpha-1}(1-s)(\int_{\theta_1}^{\theta_2} q(\tau)(g(\tau, [x(\tau) - w(\tau)]^*) + e(\tau))d\tau - \tfrac{\alpha-1}{\Gamma(\alpha)}\int_0^1 e(\tau)d\tau) \\ &\quad \geq \lambda\gamma(\int_{\theta_1}^{\theta_2} q(\tau)g(\tau, [x(\tau) - w(\tau)]^*)d\tau - r) \\ &\quad \geq \lambda\gamma(\int_{\theta_1}^{\theta_2} q(\tau)\tfrac{N}{2}x(\tau)d\tau - r) \\ &\quad \geq \lambda\gamma(\int_{\theta_1}^{\theta_2} q(\tau)NBd\tau - r) \\ &\quad \geq \lambda\gamma(NB\rho - r) > B, \quad s \in [\theta_1, \theta_2]. \end{aligned}$$

In fact, from

$$N\rho(\tfrac{1}{\lambda\gamma} + r)^{-1} \geq 1 \Leftrightarrow N\rho \geq \tfrac{1}{\lambda\gamma} + r \Leftrightarrow N\rho - r \geq \tfrac{1}{\lambda\gamma},$$

we have

$$B(N\rho - r) \geq \tfrac{B}{\lambda\gamma} \Rightarrow NB\rho - r \geq \tfrac{B}{\lambda\gamma} \Leftrightarrow \lambda\gamma(NB\rho - r) > B.$$

Thus

$$\begin{aligned} &f(s, [\lambda \int_0^1 G(s,\tau)g(\tau, [x(\tau) - w(\tau)]^*)d\tau]^*) \\ &\quad \geq N\lambda \int_0^1 G(s,\tau)g(\tau, [x(\tau) - w(\tau)]^*)d\tau \\ &\quad \geq N\lambda\gamma(\int_{\theta_1}^{\theta_2} q(\tau)\tfrac{N}{2}x(\tau)d\tau - r) \\ &\quad \geq N\lambda\gamma(\int_{\theta_1}^{\theta_2} q(\tau)\tfrac{N}{2(\alpha-1)}\tau^{\alpha-1}(1-\tau)\|x\|d\tau - r) \\ &\quad \geq N\lambda\gamma(\tfrac{N}{2(\alpha-1)}\gamma\int_{\theta_1}^{\theta_2} q(\tau)R_2 d\tau - r) \\ &\quad \geq N\lambda\gamma(\tfrac{N}{2(\alpha-1)}\gamma\rho - r)R_2 \\ &\quad \geq N\lambda\gamma R_2, \quad s \in [\theta_1, \theta_2]. \end{aligned}$$

This implies

$$\begin{aligned} \|Tx(t)\| &\geq \max_{0 \leq t \leq 1} \lambda \int_0^1 G(t,s)(f(s, [\lambda \int_0^1 G(s,\tau)g(\tau, [x(\tau) - w(\tau)]^*)d\tau]^*) + e(s))ds \\ &\geq \max_{0 \leq t \leq 1} \lambda t^{\alpha-1}(1-t)\int_{\theta_1}^{\theta_2} q(s)f(s, [\lambda \int_0^1 G(s,\tau)g(\tau, [x(\tau) - w(\tau)]^*)d\tau]^*)ds \\ &\geq \lambda \min_{\theta_1 \leq t \leq \theta_2} t^{\alpha-1}(1-t)\int_{\theta_1}^{\theta_2} q(s)f(s, [\lambda \int_0^1 G(s,\tau)g(\tau, [x(\tau) - w(\tau)]^*)d\tau]^*)ds \\ &\geq \lambda\gamma \int_{\theta_1}^{\theta_2} q(s)N\lambda\gamma R_2 ds \\ &\geq \lambda^2\gamma^2 N\rho R_2 \\ &\geq R_2 = \|x\| \end{aligned}$$

and

$$\|Tx\| \geq \|x\|, x \in P \cap \partial\Omega_2.$$

Condition (2) of Krasnoesel'skii's fixed-point theorem is satisfied. So T has a fixed point x with $r \leq R_1 < \|x\| < R_2$.

Since $r < R_1 < \|x\|$,

$$\begin{aligned} x(t) - w(t) &\geq \tfrac{1}{\alpha-1} t^{\alpha-1}(1-t)\|x\| - \lambda \int_0^1 G(t,s)e(s)ds \\ &\geq \tfrac{1}{\alpha-1} t^{\alpha-1}(1-t)\|x\| - \tfrac{\alpha-1}{\Gamma(\alpha)} t^{\alpha-1}(1-t)\lambda \int_0^1 e(s)ds \\ &\geq \tfrac{1}{\alpha-1} t^{\alpha-1}(1-t)\|x\| - t^{\alpha-1}(1-t)\lambda r \\ &\geq \tfrac{1}{\alpha-1} t^{\alpha-1} r - t^{\alpha-1}(1-t)\lambda r \\ &\geq (\tfrac{1}{\alpha-1} - \lambda) t^{\alpha-1}(1-t) r \\ &> 0, \ \ t \in (0,1). \end{aligned}$$

On the other hand, according to the choice of λ^* and R_1, we have

$$\begin{aligned} x(s) - w(s) &\geq x(s) - \tfrac{(\alpha-1)x(s)}{\|x\|}\lambda r \\ &\geq x(s) - \tfrac{(\alpha-1)x(s)}{R_1}\lambda r \\ &\geq (1 - \tfrac{\lambda(\alpha-1)r}{R_1}) x(s) \\ &\geq \tfrac{1}{2} x(s) \\ &\geq \tfrac{1}{2(\alpha-1)} s^{\alpha-1}(1-s)\|x\| \\ &\geq \tfrac{1}{2(\alpha-1)} \gamma R_1 \\ &\geq R_1^{\frac{1}{2}}, \ \ t \in [\theta_1, \theta_2]. \end{aligned}$$

This implies

$$g(s, [x(s) - w(s)]^*) \geq N_0 R_1^{\frac{1}{2}}, \ \ s \in [\theta_1, \theta_2].$$

This together with the choice of N_0, for $\|x\| \geq R_1$, we have

$$\begin{aligned} &\lambda \int_0^1 G(t,s) g(s, [x(s) - w(s)]^*) ds \\ &\quad = \lambda(\int_0^1 G(t,s)(g(s, [x(s) - w(s)]^*) + e(s))ds - \int_0^1 G(s,s)e(s)ds) \\ &\quad \geq \lambda t^{\alpha-1}(1-t)(\int_{\theta_1}^{\theta_2} q(s)(g(s, [x(s) - w(s)]^*) + e(s))ds - \tfrac{\alpha-1}{\Gamma(\alpha)} \int_0^1 e(s)ds) \\ &\quad \geq \lambda\gamma(\int_{\theta_1}^{\theta_2} q(s) g(s, [x(s) - w(s)]^*)ds - r) \\ &\quad \geq \lambda\gamma(\int_{\theta_1}^{\theta_2} q(s) N_0 R_1^{\frac{1}{2}} ds - r) \\ &\quad \geq \lambda\gamma(N_0 R_1^{\frac{1}{2}} \rho - r) \\ &\quad \geq \lambda\gamma(\rho N_0 - r) R_1^{\frac{1}{2}} > 0, \quad t \in [0,1]. \end{aligned}$$

It follows that

$$\begin{aligned} y(t) &= \lambda \int_0^1 G(t,s)(g(s, [x(s) - w(s)]^*) + e(s))ds \\ &= \lambda(\int_0^1 G(t,s)(g(s, x(s) - w(s))ds + \int_0^1 G(t,s)e(s))ds) \\ &> \lambda \int_0^1 G(t,s)e(s)ds \\ &= w(t) \quad \text{for} \quad t \in (0,1). \end{aligned}$$

Thus, (x, y) is positive solution (x, y) of the boundary value problem (3.1) with $x(t) \geq w(t)$ and $y(t) \geq w(t)$ for $t \in [0,1]$.

Let $u(t) = x(t) - w(t) > 0$ and $v(t) = y(t) - w(t) > 0$, then (u, v) is a nonnegative solution (positive on $(0,1)$) of the boundary value problem (1.1).

Since condition (H_1) implies conditions (H_1^*) and (H_4), then from proof of Theorem 3.1 and 3.2, we immediately have the following theorem:

Theorem 3.3 *Suppose that (H_1)-(H_3) hold. Then the boundary value problem (1.1) has at least two positive solutions for $\lambda > 0$ sufficiently small.*

In fact, let $0 < \lambda < \min\{\overline{\lambda}, \lambda^*\}$, then the boundary value problem (1.1) has at least two positive solutions.

4 Example

To illustrate the usefulness of the results, we give some examples.

Example 4.1 Consider the boundary value problem

$$\begin{cases} -\mathbf{D}_{0+}^{\alpha}u = \lambda(v^{\alpha} + \frac{1}{(t-t^2)^{\frac{1}{2}}}\cos(2\pi v)), \quad t \in (0,1), \lambda > 0, \\ -\mathbf{D}_{0+}^{\alpha}v = \lambda(u^{\beta} + \frac{1}{(t-t^2)^{\frac{1}{2}}}\sin(2\pi u)), \\ u^{(i)}(0) = v^{(i)}(0) = 0, \quad 0 \le i \le n-2, \\ u(1) = v(1) = 0, \end{cases} \tag{4.1}$$

where $a > 1$. Then, if $\lambda > 0$ is sufficiently small, (4.1) has a positive solutions (u, v) with $u > 0, v > 0$ for $t \in (0,1)$.

To see this we will apply Theorem 3.2 with

$$\begin{aligned} &f(t,z) = z^{\alpha} + \tfrac{1}{(t-t^2)^{\frac{1}{2}}}\cos(2\pi z), \quad g(t,z) = z^{\beta} + \tfrac{1}{(t-t^2)^{\frac{1}{2}}}\sin(2\pi z), \\ &e(t) = \tfrac{2}{(t-t^2)^{\frac{1}{2}}}. \end{aligned}$$

Clearly, for $t \in (0,1)$,

$$\begin{aligned} &f(t,z) + e(t) \ge z^{\alpha} + 1 > 0, \quad g(t,z) + e(t) \ge z^{\beta} + 1 > 0, \text{ for } t \in (0,1); \\ &\lim_{z\uparrow+\infty}\inf \tfrac{f(t,z)}{z} = +\infty, \quad \lim_{z\uparrow+\infty}\inf \tfrac{g(t,z)}{z} = +\infty, \text{ for } \forall\, t \in [\theta_1, \theta_2] \subset (0,1), \end{aligned}$$

for $u > 0$. Namely (H_1^*) and (H_3)-(H_4) hold. From $r = \int_0^1 \frac{2}{(s-s^2)^{\frac{1}{2}}}ds = \pi$, let $[\theta_1, \theta_2] \in (0,1)$, $R_1 = 17 + (\frac{2}{C_0\gamma})^2 + (m_0 + \frac{4}{C_0\rho})^{\frac{2}{\beta-1}}$ and $N_0 = \frac{4}{C_0\rho}$.

Then, we have

$$R_1 > 17 + \left(\frac{2}{C_0\gamma}\right)^2 > 1 + r^2 + \left(\frac{2}{C_0\gamma}\right)^2 > \max\left\{1, r^2, \left(\frac{2}{C_0\gamma}\right)^2\right\}, \quad N_0 > \frac{r}{C_0\rho}.$$

When $z > R_1^{\frac{1}{2}} > (m_0 + \frac{4}{C_0\rho})^{\frac{1}{\beta-1}}$, we have

$$\frac{g(t,z)}{z} > z^{\beta-1} - m_0 > \frac{4}{C_0\rho} \quad \text{for} \quad t \in [\theta_1, \theta_2],$$

where $m_0 = \max\limits_{0<\theta_1\le t\le\theta_2<1}\{\frac{2}{(t-t^2)^{\frac{1}{2}}}\}$. So

$$\frac{g(t,z)}{z} > N_0 \quad \text{for} \quad t \in [\theta_1, \theta_2], z > R_1^{\frac{1}{2}}.$$

We have

$$R = \int_0^1 pq(\tau)(\max_{0\le z\le R_1}\{z^{\beta} + \frac{1}{(\tau-\tau^2)^{\frac{1}{2}}}\sin(2\pi z)\} + e(\tau))d\tau \le (R_1^{\beta} + \pi)\int_0^1 pq(\tau)d\tau$$

and

$$\int_0^1 pq(s)(\max_{0\le z\le R}\{z^{\alpha} + \frac{1}{(s-s^2)^{\frac{1}{2}}}\cos(2\pi z)\} + e(s))ds \le (R^{\alpha} + \pi)\int_0^1 pq(s)ds.$$

Let

$$\lambda^* = \min\{1, R_1[(R^{\alpha} + 3)\int_0^1 pq(s)ds]^{-1}, \frac{C_0R_1}{2r}\}.$$

Now, if $\lambda < \lambda^*$, Theorem 3.2 guarantees that (4.1) has a positive solutions u with $\|u\| \ge 2$.

Example 4.2 Consider the boundary value problem

$$\begin{cases} -\mathbf{D}_{0+}^{\alpha} u = \lambda(v-\alpha)(v-\beta), & t \in (0,1), \lambda > 0, \\ -\mathbf{D}_{0+}^{\alpha} v = \lambda u(u-a)(u-b), \\ u^{(i)}(0) = v^{(i)}(0) = 0, & 0 \le i \le n-2, \\ u(1) = v(1) = 0, \end{cases} \tag{4.2}$$

where $\beta > \alpha > 0$, $b > a > 0$. Then, if $\lambda > 0$ is sufficiently small, (4.2) has two solutions (u_1, v_1), (u_2, v_2) with $u_i(t) > 0, v_i(t) > 0$ for $t \in (0,1), i = 1,2$.

To see this we will apply Theorem 3.3 with

$$f(t,z) = z^2 - (\alpha+\beta)z + \alpha\beta \quad \text{and} \quad g(t,z) = z^3 - (a+b)z^2 + abz \quad \text{for} \quad z \ge 0.$$

Clearly, there exists a constant $e(t) = M_0 > 0$ such that

$$f(t,z) + e(t) > 1, \quad g(t,z) + e(t) > 0$$

and

$$f(t,0) = \alpha\beta > 0, \quad g(t,z) > 0 \text{ for } 0 < z < a, \quad \lim_{z \downarrow 0} \frac{g(t,z)}{z} = ab < M,$$

where $M = (a+1)(b+1)$.

Since $g(t,z)$ increase to z for $0 \le t \le 1, \quad 0 \le z \le (a+b) - (a^2+b^2)^{\frac{1}{2}}$, $f(t,z)$ decrease to z for $0 \le t \le 1, \quad 0 \le z \le \alpha$. Let $\delta = \frac{\alpha}{4\beta}$, $\varepsilon = \frac{1}{4}\min\{1, \alpha, (a+b) - (a^2+b^2)^{\frac{1}{2}}\}$ and $c = \int_0^1 pq(s)ds$. We have

$$f(t,z) \ge \delta f(t,0), \quad 0 < g(t,z) \le Mz, \quad \text{for} \quad 0 \le t \le 1, \quad 0 \le z \le \varepsilon.$$

Namely (H_1)-(H_2) hold. We choose

$$\overline{\lambda} = \min\{\frac{1}{\alpha\beta + M_0}, \frac{1}{Mc}\}. \tag{4.3}$$

Now, if $\lambda < \overline{\lambda}$, Theorem 3.1 guarantees that (4.2) has a positive solutions (u_1, v_1) with $\|u_1\| \le \frac{1}{4}$.

On the other hand,

$$\lim_{z \uparrow +\infty} \inf \tfrac{f(t,z)}{z} = +\infty, \quad \lim_{z \uparrow +\infty} \inf \tfrac{g(t,z)}{z} = +\infty \quad \text{for} \quad t \in (0,1).$$

Namely (H_1)-(H_4) hold and $r = M_0$. Next, let $[\theta_1, \theta_2] \in (0,1)$, $R_1 > 1 + M_0^2 + (\frac{2}{C_0\gamma})^2$ such that $g(t,z) > N_0 z \quad \text{for} \quad z > R_1^{\frac{1}{2}}$ and $N_0 = \frac{M_0+1}{C_0\gamma}$. We have

$$R = \int_0^1 pq(\tau)(\max_{0 \le z \le R_1}\{z(z-a)(z-b)\} + M_0)d\tau$$

and

$$\lambda^* = \min\{1, R_1[\int_0^1 pq(s)(\max_{0 \le z \le R}\{(z-\alpha)(z-\beta)\} + M_0)ds]^{-1}, \frac{C_0 R_1}{2r}\}.$$

Now, if $0 < \lambda < \lambda^*$, Theorem 3.2 guarantees that (4.2) has a positive solutions (u_2, v_2) with $\|u_2\| \ge 1$.

Since all the conditions of Theorem 3.3 are satisfied , if $\lambda < \min\{\overline{\lambda}, \lambda^*\}$, Theorem 3.3 guarantees that (4.2) has two solutions u_i with $u_i(t) > 0$ for $t \in (0,1), i = 1,2$.

Example 4.3 Consider the boundary value problem

$$\begin{cases} -\mathbf{D}_{0+}^{\alpha} u = \lambda(v^{\alpha} + \cos(2\pi v)), & t \in (0,1), \lambda > 0, \\ -\mathbf{D}_{0+}^{\alpha} v = \lambda(u^{\beta} + \sin(2\pi u)), \\ u^{(i)}(0) = v^{(i)}(0) = 0, & 0 \le i \le n-2, \\ u(1) = v(1) = 0, \end{cases} \tag{4.4}$$

where $\alpha, \beta > 1$. Then, if $\lambda > 0$ is sufficiently small, (4.4) has two solutions (u_1, v_1), (u_2, v_2) with $u_i(t) > 0, v_i(t) > 0$ for $t \in (0,1), i = 1,2$.

To see this we will apply Theorem 3.3 with

$$f(t,z) = z^\alpha(t) + \cos(2\pi z), \quad g(t,z) = z^\beta + \sin(2\pi z), \quad e(t) = 2.$$

Clearly, for $t \in (0,1)$,

$$\begin{aligned} &f(t,z) + e(t) \geq z^\alpha + 1 > 0, \quad g(t,z) + e(t) \geq z^\beta + 1 > 0, \\ &f(t,0) = 1 > 0, \quad g(t,z) > 0 \text{ for } 0 < z < \tfrac{1}{2}, \\ &\lim_{z\uparrow+\infty} \inf \tfrac{f(t,z)}{z} = +\infty, \quad \lim_{z\downarrow 0} \sup \tfrac{g(t,z)}{z} = 2\pi < M, \quad \lim_{z\uparrow+\infty} \inf \tfrac{g(t,z)}{z} = +\infty, \end{aligned}$$

where $M = 2\pi + 1$.

Namely (H_1)-(H_4) hold. Let $\delta = \frac{1}{2}$, $\varepsilon = \frac{1}{8}$ and $c = \int_0^1 pq(s)ds$. We have

$$\frac{\varepsilon}{2c(\max\limits_{0\leq x\leq\varepsilon} f(t,x) + 2)} \geq \frac{1}{16c(2+3)} = \frac{1}{90c}. \tag{4.5}$$

Let $\overline{\lambda} = \min\{\frac{1}{90c}, \frac{1}{Mc}\}$. Now, if $0 < \lambda < \overline{\lambda}$ then $0 < \lambda < \frac{\varepsilon}{2c(\max\limits_{0\leq x\leq\varepsilon} f(t,x)+2)}$, Theorem 3.1 guarantees that (4.4) has a positive solutions (u_1, v_1) with $\|u_1\| \leq \frac{1}{8}$.

Next, from $r = \int_0^1 e(s)ds = 2$, let $[\theta_1, \theta_2] \in (0,1)$, $R_1 = 5 + (\frac{2}{C_0\gamma})^2 + (1 + \frac{3}{C_0\rho})^{\frac{2}{\beta-1}}$ and $N_0 = \frac{3}{C_0\rho}$.

Then, we have

$$R_1 > 5 + (\frac{2}{C_0\gamma})^2 > 1 + r^2 + (\frac{2}{C_0\gamma})^2 > \max\{1, r^2, (\frac{2}{C_0\gamma})^2\}, \quad N_0 > \frac{r}{C_0\rho}.$$

When $z > R_1^{\frac{1}{2}} > (1 + \frac{3}{C_0\rho})^{\frac{1}{\beta-1}}$, we have

$$\frac{g(t,z)}{z} > z^{\beta-1} - 1 > \frac{3}{C_0\rho}.$$

So,

$$\frac{g(t,z)}{z} > N_0 \quad \text{for} \quad z > R_1^{\frac{1}{2}}.$$

we have

$$R = \int_0^1 pq(\tau)(\max_{0\leq z\leq R_1}\{z^\beta + \sin(2\pi z)\} + e(\tau))d\tau \leq (R_1^\beta + 3)\int_0^1 pq(\tau)d\tau$$

and

$$\int_0^1 pq(s)(\max_{0\leq z\leq R}\{z^\alpha + \cos(2\pi z)\} + M_0)ds \leq (R^\alpha + 3)\int_0^1 pq(s)ds.$$

Let

$$\lambda^* = \min\{1, R_1[(R^\alpha + 3)\int_0^1 pq(s)ds]^{-1}, \frac{C_0R_1}{2r}\}.$$

Now, if $0 < \lambda < \lambda^*$ then $0 < \lambda < R_1(\int_0^1 pq(s)(\max\limits_{0\leq z\leq R}\{z^\alpha + \cos(2\pi z)\} + M_0)ds)^{-1}$, Theorem 3.2 guarantees that (4.4) has a positive solutions (u_2, v_2) with $\|u_2\| \geq 1$.

So, if $\lambda < \min\{\overline{\lambda}, \lambda^*\}$, Theorem 3.3 guarantees that (4.4) has two solutions (u_1, v_1) and (u_2, v_2) with $u_i, v_i > 0$ for $t \in (0,1), i = 1,2$.

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References

[1] Asghar Ghorbani, *Toward a new analytical method for solving nonlinear fractional differential equations*, Computer Methods in Applied Mechanics and Engineering, **197**(2008), 4173-4179

[2] I. Podlubny, *Fractional Differential Equations, Mathematics in Science and Engineering*, vol, 198, Academic Press, New York/London/Toronto, 1999.

[3] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives (Theory and Applications).* Gordon and Breach, Switzerland, 1993.

[4] A. Ashyralyev, *A note on fractional derivatives and fractional powers of operators*, J. Math. Anal. Appl. **357**(2009), 232-236.

[5] S.P. Mirevski, L. Boyadjiev, R. Scherer, *On the Riemann-Liouville fractional calculus, g-Jacobi functions and F-Gauss functions*, Appl. Math. Comp., **187**(2007), 315-325

[6] A. Mahmood, S. Parveen, A. Ara, N.A. Khan, *Exact analytic solutions for the unsteady flow of a non-Newtonian fluid between two cylinders with fractional derivative model*, Communications in Nonlinear Science and Numerical Simulation, **14**(2009), 3309-3319.

[7] Guy Jumarie, *Table of some basic fractional calculus formulae derived from a modified Riemann-Liouville derivative for non-differentiable functions*, Appl. Math. Lett. **22**(2009), 378-385.

[8] M. A. Krasnosel'skii, *Positive solutions of operator equations*, Noordhoff Gronigen, Netherland, 1964.

[9] V. Lakshmikantham, S. Leela, *A Krasnoselskii-Krein-type uniqueness result for fractional differential equations*, Nonlinear Anal., **71**(2009), 3421-3424.

[10] Yong Zhou, Feng Jiao, Jing Li, *Existence and uniqueness for fractional neutral differential equations with infinite delay*, Nonlinear Anal., **71**(2009), 3249-3256.

[11] Nickolai Kosmatov, *Integral equations and initial value problems for nonlinear differential equations of fractional order*, Nonlinear Anal., **70**(2009), 2521-2529

[12] Chuanzhi Bai, *Positive solutions for nonlinear fractional differential equations with coefficient that changes sign*, Nonlinear Anal., **64**(2006), 677-685

[13] Shuqin Zhang, *Existence of Positive Solution for some class of Nonlinear Fractional Differential Equations*, J. Math. Anal. Appl. **278**(2003), 136-148.

[14] M. Benchohra, S. Hamani, S.K. Ntouyas, *Boundary value problems for differential equations with fractional order and nonlocal conditions*, Nonlinear Anal. TMA, **71**(2009), 2391-2396.

[15] Zhanbing Bai, Haishen Lü, *Positive solutions for boundary-value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl. **311**(2005), 495-505.

[16] Chengjun Yuan, *Multiple positive solutions for (n-1, 1)-type semipositone conjugate boundary value problems of nonlinear fractional differential equations*, Electronic Journal of Qualitative Theory of Differential Equations, **36**(2010), 1-12.

[17] Chengjun Yuan, Daqing Jiang, and Xiaojie Xu, *Singular positone and semipositone boundary value problems of nonlinear fractional differential equations*, Mathematical Problems in Engineering, Volume 2009, Article ID 535209, 17 pages.

[18] Xinwei Su, *Boundary value problem for a coupled system of nonlinear fractional differential equations*, Appl. Math. Lett. **22**(2009), 64-69.

[19] Bashir Ahmad, S. Sivasundaram, *Existence results for nonlinear impulsive hybrid boundary value problems involving fractional differential equations*, Nonlinear Anal. Hybrid Systems, **3**(2009), 251-258.

[20] Mouffak Benchohra, Samira Hamani, *The method of upper and lower solutions and impulsive fractional differential inclusions*, Nonlinear Anal. Hybrid Systems, **3** (2009), 433-440.

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