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Existence and uniqueness of positive solutions for a third-order three-point problem on time scales*

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Abstract

In this paper, a class of third-order three-point boundary value problem on time scales is considered. Using monotone iterative technique and cone expansion and compression fixed point theorem of norm type, we do not only obtain the existence and uniqueness of positive solutions of the problem, but also establish the iterative schemes for approximating the solutions.

Keywords: Time scales; Uniqueness; Fixed point; Monotone iterative technique; Positive solution

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1. Introduction

In this paper, we are interested in the existence and uniqueness of positive solutions and establish the corresponding iterative schemes for the following third-order three-point boundary value problem (BVP) on time scales

$$\begin{cases} (px^{\Delta\Delta})^{\nabla}(t) + f(t, x(t)) = 0, \quad t \in [t_1, t_3]_{\mathbb{T}}, \\ x(\rho(t_1)) = 0 = x^{\Delta}(\rho(t_1)), \quad x^{\Delta}(\sigma(t_3)) = \alpha x^{\Delta}(t_2), \end{cases} \tag{1.1}$$

where p is a right-dense continuous, real-valued function with $0 < p(t) \leq 1$ on $\mathbb{T}$; $f : \mathbb{T} \times [0, +\infty) \longrightarrow [0, +\infty)$ is continuous; the boundary points from $\mathbb{T}$ satisfy $t_1 < t_2 < t_3$, with $t_2/\alpha \in \mathbb{T}$ such that the constants d and α satisfy

$$d := \int_{\rho(t_1)}^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)} - \alpha \int_{\rho(t_1)}^{t_2} \frac{\Delta\tau}{p(\tau)} > 0 \quad and \quad 1 < \alpha < \frac{\int_{\rho(t_1)}^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)}}{\int_{\rho(t_1)}^{t_2} \frac{\Delta\tau}{p(\tau)}}. \tag{1.2}$$

The theory of time scales was introduced and developed by Hilger [1] to unify continuous and discrete analysis. Time scales theory presents us with the tools necessary to understand and explain the mathematical structure underpinning the theories of discrete and continuous dynamic systems and allows us to connect them. On the other hand, the theory is widely applied to heat transfer, biology, epidemic models and stock market, for details, see [1-4] and references therein. Certain economically important phenomena contain processes that feature elements of

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both the continuous and the discrete. For example, a consumer wants to maximize his lifetime utility subject to certain constraints. During each period in his life a consumer has to make a decision concerning how much to consume and how much to spend. If the consumer consumes more today, he has to consume less tomorrow because of limited resource. In other words, he has to give up the utility he can derive from tomorrow consumption. So the solution is a function that describes optimal behavior for an individual, which shows how much one should consume each period to insure that he can maximum lifetime utility. So the lifetime utility is a function typically depending on consumption. One has to maximize the function in each period of lifetime, which can be regarded as a discrete model. We also consider the problem of maximization as a sum of instantaneous utilities, which can be described in a continuous model. While the time scales model can provide an unification from both discrete and continuous approaches subject to some constraints. Some definitions and conclusions on time scales can be found in [5-7].

In recent years, higher-order two-point boundary value problems on time scales have been studied extensively, see Boey and Wong [8], Sun [9], and Cetin and Topal [10-12]. At the same time, even-order multi-point boundary value problems on time scales have also attracted much attention, see Anderson and Avery [13], Anderson and Karaca [14], and Yaslan [15]. Third-order differential and difference equations, though less common in applications than even-order problems, nevertheless do appear, for example in the study of quantum fluids and gravity driven flows. Here we approach a third-order three-point problem on general time scales which has been considered in [16-18]. Note that boundary value problems on time scales that utilize both delta and nabla derivatives, such as the one here, were first introduced by Atici and Guseinov [5].

We would like to mention some results of Anderson and Hoffacker [16], Anderson and Smyrlis [17], and Sang and Wei [18], which motivated us to consider BVP (1.1). In [16], Anderson and Hoffacker were concerned with the existence and form of solutions to the following nonlinear third-order three-point boundary value problem on time scales:

$$\begin{cases} (px^{\Delta\Delta})^{\nabla}(t) + a(t)f(x(t)) = 0, \quad t \in [t_1, t_3]_{\mathbb{T}}, \\ x(\rho(t_1)) = 0 = x^{\Delta}(\rho(t_1)), \quad x^{\Delta}(\sigma(t_3)) = \alpha x^{\Delta}(t_2). \end{cases} \tag{1.3}$$

Using the corresponding Green's function, they proved the existence of at least one positive solution using the Guo-Krasnosel'skii fixed point theorem. Moreover, a third-order multi-point eigenvalue problem was formulated, and eigenvalue intervals for the existence of a positive solution were found.

In [17], Anderson and Smyrlis applied Leray-Schauder nonlinear alternative to study the following third-order three-point boundary value problem on time scales:

$$\begin{cases} (px^{\Delta\Delta})^{\nabla}(t) + f(t, x(t), x^{\Delta}(t)) = 0, \quad t \in [t_1, t_3]_{\mathbb{T}}, \\ x(\rho(t_1)) = 0 = x^{\Delta}(\rho(t_1)), \quad x^{\Delta}(\sigma(t_3)) = \alpha x^{\Delta}(t_2), \end{cases} \tag{1.4}$$

where p is a right-dense continuous, real-valued function with $0 < p(t) \leq 1$ on $\mathbb{T}$. They obtained some sufficient conditions for the existence of at least one nontrivial solution of (1.4).

In [18], Sang and Wei considered the solutions and positive solutions of problem (1.1), the authors assumed that the nonlinear term f is bounded below, this implies that f is not necessarily nonnegative.

We note that Anderson and Karaca [14] were concerned with the dynamic three-point boundary value problem

$$\begin{cases} (-1)^n y^{\Delta^{2n}}(t) = f(t, y^\sigma(t)), \quad t \in [a,b]_{\mathbb{T}}, \\ \alpha_{i+1} y^{\Delta^{2i}}(\eta) + \beta_{i+1} y^{\Delta^{2i+1}}(a) = y^{\Delta^{2i}}(a), \quad \gamma_{i+1} y^{\Delta^{2i}}(\eta) = y^{\Delta^{2i}}(\sigma(b)), \quad 0 \le i \le n-1. \end{cases} \tag{1.5}$$

The monotone method was discussed to ensure the existence of solutions of BVP (1.5). The authors proved the existence theorem for solutions of BVP (1.5) which lie between the lower and upper solutions when they are given in the well order i.e., the lower solution is under the upper solution. Furthermore, Cetin and Topal [11] considered the nonlinear Lidstone boundary value problem

$$\begin{cases} (-1)^n y^{\Delta^{2n}}(t) = f(t, y^\sigma(t), y^{\Delta\Delta}(t), \cdots, y^{\Delta^{2(n-1)}}(t)), \quad t \in [0,1]_{\mathbb{T}}, \\ y^{\Delta^{2i}}(0) = y^{\Delta^{2i}}(\sigma(1)) = 0, \quad 0 \le i \le n-1. \end{cases} \tag{1.6}$$

The authors developed the monotone method which yields the solution of BVP (1.6), they gave the existence and uniqueness theorem for solution of BVP (1.6) when they are given in the well order. They claimed that ″This method is generally used to obtain the existence of solutions within specified bounds determined by the upper and lower solutions″.

It is also noted that the researchers mentioned above [16-18] only studied the existence and uniqueness of positive solutions. As a result, they failed to further provide the computational methods of positive solutions. Therefore, it is natural to consider the uniqueness and iteration of positive solutions to BVP (1.1).

In this paper, by considering the ″*heights*″ of the nonlinear term f on some bounded sets and applying monotone iterative techniques on a Banach space, we do not only obtain the existence and uniqueness of positive solutions for BVP (1.1), but also give the iterative schemes for approximating the solutions. In essence, we combine the method of lower and upper solutions with the cone expansion and compression fixed point theorem of norm type. The ideas of this paper come from Yao [19-21]. In order to obtain the uniqueness of positive solutions for BVP (1.1), we adopt some ideas established in [22].

2. Several lemmas

Let $\mathbb{T}$ be a time scale which has the subspace topology inherited from the standard topology on $\mathbb{R}$. For each interval I of $\mathbb{R}$, we define $I_{\mathbb{T}} = I \cap \mathbb{T}$.

Underlying our technique will be the Green's function for the homogeneous third-order three-point boundary value problem

$$\begin{cases} -(px^{\Delta\Delta})^\nabla(t) = 0, \quad t \in [t_1, t_3]_{\mathbb{T}}, \\ x(\rho(t_1)) = 0 = x^\Delta(\rho(t_1)), \quad x^\Delta(\sigma(t_3)) = \alpha x^\Delta(t_2), \end{cases} \tag{2.1}$$

The Green's function for (2.1) is well defined, nonnegative, and bounded above on $[\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}} \times [t_1, \sigma(t_3)]_{\mathbb{T}}$, as related in the following lemmas.

Lemma 2.1. (See [16, 17].) *For $y \in C_{ld}[\rho(t_1), \sigma(t_3)]_{\mathbb{T}}$, the boundary value problem*

$$\begin{cases} (px^{\Delta\Delta})^\nabla(t) + y(t) = 0, \quad t \in [t_1, t_3]_{\mathbb{T}}, \\ x(\rho(t_1)) = 0 = x^\Delta(\rho(t_1)), \quad x^\Delta(\sigma(t_3)) = \alpha x^\Delta(t_2) \end{cases} \tag{2.2}$$

has a unique solution $x(t) = \int_{\rho(t_1)}^{\sigma(t_3)} G(t,s)y(s)\nabla s$, *where the Green's function corresponding to the problem* (2.1) *is given by*

$$G(t,s) = \begin{cases} \frac{1}{d}\left(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)} - \alpha \int_s^{t_2} \frac{\Delta\tau}{p(\tau)}\right) \int_{\rho(t_1)}^t \int_{\rho(t_1)}^{\xi} \frac{\Delta\tau}{p(\tau)}\Delta\xi - \int_s^t \int_s^{\xi} \frac{\Delta\tau}{p(\tau)}\Delta\xi : & s \le \min\{t_2, t\} \\ \frac{1}{d}\left(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)} - \alpha \int_s^{t_2} \frac{\Delta\tau}{p(\tau)}\right) \int_{\rho(t_1)}^t \int_{\rho(t_1)}^{\xi} \frac{\Delta\tau}{p(\tau)}\Delta\xi : & t \le s \le t_2 \\ \frac{1}{d}\left(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)}\right) \int_{\rho(t_1)}^t \int_{\rho(t_1)}^{\xi} \frac{\Delta\tau}{p(\tau)}\Delta\xi - \int_s^t \int_s^{\xi} \frac{\Delta\tau}{p(\tau)}\Delta\xi : & t_2 \le s \le t \\ \frac{1}{d}\left(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)}\right) \int_{\rho(t_1)}^t \int_{\rho(t_1)}^{\xi} \frac{\Delta\tau}{p(\tau)}\Delta\xi : & \max\{t_2, t\} \le s \end{cases} \tag{2.3}$$

for all $(t,s) \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}} \times [t_1, \sigma(t_3)]_{\mathbb{T}}$.

Lemma 2.2. (See [16].) *Assume* (1.2). *The Green's function* (2.3) *corresponding to the problem* (2.1) *satisfies*

$$0 \le G(t,s) \le g(s),$$

where g *is given by*

$$g(s) := \frac{1}{d}(\alpha+1)(\sigma^2(t_3) - \rho(t_1)) \left(\int_{\rho(t_1)}^{s} \frac{\Delta\tau}{p(\tau)}\right)\left(\int_s^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)}\right) \tag{2.4}$$

for all $(t,s) \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}} \times [t_1, \sigma(t_3)]_{\mathbb{T}}$.

Lemma 2.3. (See [16].) *Assume* (1.2). *The Green's function* (2.3) *corresponding to the problem* (2.1) *satisfies*

$$G(t,s) \ge \gamma g(s), \quad \gamma := \frac{\min\{\alpha-1, \alpha\} \int_{\rho(t_1)}^{t_2/\alpha} \int_{\rho(t_1)}^{u} \frac{\Delta\tau}{p(\tau)}\Delta u}{(\alpha+1)(\sigma^2(t_3) - \rho(t_1)) \int_{\rho(t_1)}^{\sigma(t_3)} \frac{\Delta\tau}{p(\tau)}} \in (0,1) \tag{2.5}$$

for all $(t,s) \in [t_2/\alpha, t_2]_{\mathbb{T}} \times [t_1, \sigma(t_3)]_{\mathbb{T}}$, *where* $g(s)$ *is given in* (2.4).

Let $\mathbb{B}$ denote the real Banach space $C[\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}$ with the supremum norm

$$\|x\| = \sup_{t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}} |x(t)|.$$

It is easy to see that BVP (1.1) has a solution $x = x(t)$ if and only if x is a fixed point of the following operator:

$$Fx(t) = \int_{\rho(t_1)}^{\sigma(t_3)} G(t,s)f(s,x(s))\nabla s, \quad t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}.$$

Set

$$P = \{x \in \mathbb{B} : \ x \textit{ is nonnegative, and } \min_{t \in [t_2/\alpha, t_2]_{\mathbb{T}}} x(t) \ge \gamma \|x\|\},$$

where γ is the same as in Lemma 2.3. By the proof of Section 3 in [16], we can obtain that $F(P) \subset P$ and $F: P \longrightarrow P$ is completely continuous.

3. Successive iteration and unique positive solution for (1.1)

For notational convenience, we denote

$$M = \left[\sup_{t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}} \int_{\rho(t_1)}^{\sigma(t_3)} G(t,s) \nabla s \right]^{-1}, \quad N = \left[\sup_{t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}} \int_{t_2/\alpha}^{t_2} G(t,s) \nabla s \right]^{-1}.$$

Constants $M,\ N$ are not easy to compute explicitly. For convenience, we can replace M by M', N by N', where

$$M' = \left[\int_{\rho(t_1)}^{\sigma(t_3)} g(s) \nabla s \right]^{-1}, \quad N' = \left[\gamma \int_{t_2/\alpha}^{t_2} g(s) \nabla s \right]^{-1}.$$

Obviously, $0 < M' < M < N < N'$.

Theorem 3.1. *Assume there exist two positive numbers a, b with $b < a$ such that*
(H_1) $\max\{f(t,a):\ t \in \mathbb{T}\} \le aM,\ \ \min\{f(t,\gamma b):\ t \in [t_2/\alpha, t_2]_{\mathbb{T}}\} \ge bN$;
(H_2) *$f(t,x_1) \le f(t,x_2)$ for any $t \in \mathbb{T},\ 0 \le x_1 \le x_2 \le a$;*
(H_3) *for any $x \in [0,a]$ and $r \in (0,1)$, there exists $\eta = \eta(x,r) > 0$ such that*

$$f(t, rx) \ge [1 + \eta(x,r)] r f(t,x), \quad t \in \mathbb{T}.$$

Then BVP (1.1) has a unique positive solution x^* such that $b \le \|x^*\| \le a$ and $\lim_{n\to\infty} F^n \tilde{x} = x^*$, i.e., $F^n \tilde{u}$ converges uniformly to x^* in $[\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}$, where $\tilde{x}(t) \equiv a,\ t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}$.

Remark 3.1. The iterative scheme in Theorem 3.1 is $x_1 = F\tilde{x},\ x_{n+1} = F x_n,\ n = 1, 2, \cdots$. It starts off with constant function $\tilde{x}(t) \equiv a,\ t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}$.

Proof of Theorem 3.1. Set $P[b,a] = \{u \in P:\ b \le \|u\| \le a\}$. If $u \in P[b,a]$, then

$$\max_{t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}} x(t) \le a, \qquad \min_{t \in [t_2/\alpha, t_2]_{\mathbb{T}}} x(t) \ge \gamma \|x\| \ge \gamma b.$$

According to Assumptions (H_1) and (H_2), we have

$$f(t, x(t)) \le f(t,a) \le aM, \quad t \in [\rho(t_1), \sigma(t_3)]_{\mathbb{T}};$$

$$f(t, x(t)) \ge f(t, \gamma b) \ge bN, \quad t \in [t_2/\alpha, t_2]_{\mathbb{T}}.$$

It follows that

$$\begin{aligned} \|Fx\| &= \sup_{t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}} \left| \int_{\rho(t_1)}^{\sigma(t_3)} G(t,s) f(s, x(s)) \nabla s \right| \\ &\le aM \sup_{t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}} \int_{\rho(t_1)}^{\sigma(t_3)} G(t,s) \nabla s = a; \\ \|Fx\| &\ge \sup_{t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}} \int_{t_2/\alpha}^{t_2} G(t,s) f(s, x(s)) \nabla s \\ &\ge bN \sup_{t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}} \int_{t_2/\alpha}^{t_2} G(t,s) \nabla s = b. \end{aligned}$$

Thus, we assert that $F : P[b, a] \longrightarrow P[b, a]$.

Let $\tilde{x}(t) \equiv a,\ t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}$, then $\tilde{x} \in P[b, a]$. Let $x_1 = F\tilde{x}$, thus $x_1 \in P[b, a]$. Set $x_{n+1} = Fx_n,\ n = 1, 2, \cdots$. Since $F(P[b, a]) \subset P[b, a]$, we have $x_n \in F(P[b, a]) \subset P[b, a],\ n = 1, 2, \cdots$, which together with the complete continuity of F implies that $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ and there exists $x^* \in P[b, a]$, such that $x_{n_k} \longrightarrow x^*$.

Now, it follows from $x_1 \in P[b, a]$ that

$$x_1(t) \le \|x_1\| \le a = \tilde{x}(t), \quad t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}.$$

By Assumption (H_2), we have

$$\begin{aligned} x_2(t) &= Fx_1(t) \\ &= \int_{\rho(t_1)}^{\sigma(t_3)} G(t, s) f(s, x_1(s)) \nabla s \\ &\le \int_{\rho(t_1)}^{\sigma(t_3)} G(t, s) f(s, \tilde{x}(s)) \nabla s \\ &= F\tilde{x}(t) = x_1(t). \end{aligned}$$

By mathematical induction, we obtain

$$x_{n+1}(t) \le x_n(t), \quad t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}}, \quad n = 1, 2, \cdots.$$

Hence, $F^n\tilde{x} = x_n \longrightarrow x^*$. It is easy to know from the continuity of F and $x_{n+1} = Fx_n$ that $Fx^* = x^*$.

In the following, we show that x^* is the unique fixed point of F. In fact, suppose $\overline{x}$ is a fixed point of F. We can know that there exists $\lambda > 0$ such that $\overline{x} \ge \lambda x^*$.

Let

$$c_1 = \sup\{c > 0 | \overline{x} \ge cx^*\}.$$

Evidently, $0 < c_1 < +\infty$, $\overline{x} \ge c_1 x^*$. Furthermore, we can prove that $c_1 \ge 1$. If $0 < c_1 < 1$, from (H_3), there exists $\eta_0 = \eta_0(x^*, c_1) > 0$, such that

$$f(s, c_1 x^*) \ge (1 + \eta_0) c_1 f(s, x^*).$$

It follows that

$$\begin{aligned} \overline{x} &= F\overline{x} \ge F(c_1 x^*) \\ &= \int_{\rho(t_1)}^{\sigma(t_3)} G(t, s) f(s, c_1 x^*(s)) \nabla s \\ &\ge \int_{\rho(t_1)}^{\sigma(t_3)} G(t, s)(1 + \eta_0) c_1 f(s, x^*(s)) \nabla s \\ &\ge [1 + \inf_{s \in [\rho(t_1), \sigma(t_3)]_{\mathbb{T}}} \eta_0(x^*(s), c_1)] c_1 x^*. \end{aligned}$$

Since $[1 + \inf_{s \in [\rho(t_1), \sigma(t_3)]_{\mathbb{T}}} \eta_0(x^*(s), c_1)] c_1 > c_1$, this contradicts with the definition of c_1. Hence, $c_1 \ge 1$, and then we obtain that $\overline{x} \ge c_1 x^* \ge x^*$. Similarly, we can prove that $x^* \ge \overline{x}$, thus $\overline{x} = x^*$. Therefore, F has a unique fixed point x^*. Therefore, we can conclude that x^* is a unique positive solution of BVP (1.1).

Corollary 3.1. *Assume the following conditions are satisfied*
(H_1') $\overline{\lim\limits_{l\to 0}} \min\limits_{t\in[t_2/\alpha,t_2]_{\mathbb{T}}} f(t,l)/l > \gamma^{-1}N,\quad \underline{\lim\limits_{l\to+\infty}} \max\limits_{t\in[\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}} f(t,l)/l < M$;
(H_2') $f(t,x_1)\le f(t,x_2)$ *for any* $t\in\mathbb{T},\ x_1\le x_2,\ x_1,\ x_2\in[0,+\infty)$;
(H_3') *for any* $x\in[0,a]$ *and* $r\in(0,1)$, *there exists* $\overline{\eta}=\overline{\eta}(x,r)>0$ *such that*

$$f(t,rx)\ge[1+\overline{\eta}(x,r)]rf(t,x),\quad t\in\mathbb{T}.$$

Then BVP (1.1) *has a unique positive solution* $x^*\in P$ *and there exists a positive number* a *such that* $\lim\limits_{n\to\infty} F^n\tilde{x}=x^*$, *i.e.,*

$$\lim_{n\to\infty}\sup_{t\in[\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}}|F^n\tilde{x}(t)-x^*(t)|=0,$$

where $\tilde{x}(t)\equiv a,\ t\in[\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}$.

Theorem 3.2. *Assume the following conditions are satisfied*
(C_1) *there exists* $a>0$ *such that* $f(t,\cdot):\ [0,a]\longrightarrow(0,+\infty)$ *is nondecreasing for any* $t\in\mathbb{T}$ *and* $\max\{f(t,a):\ t\in\mathbb{T}\}\le aM$;
(C_2) $f(t,0)>0$, *for any* $t\in\mathbb{T}$.
Then BVP (1.1) *has one positive solution* x^* *such that* $0<\|x^*\|\le a$ *and* $\lim\limits_{n\to\infty} F^n0=x^*$, *i.e.,* F^n0 *converges uniformly to* x^* *in* $[\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}$. *Furthermore, if there exists* $0<\kappa<1$ *such that*

$$|f(t,l_2)-f(t,l_1)|\le\kappa M|l_2-l_1|,\quad t\in\mathbb{T},\quad 0\le l_1,\ l_2\le a.$$

Then $\|F^{n+1}0-x^*\|\le\frac{\kappa^n}{1-\kappa}\|F0\|$.

Proof. Set $P[0,a]=\{x\in P:\ \|x\|\le a\}$. Similarly to the proof of Theorem 3.1, we can know that $F:P[0,a]\longrightarrow P[0,a]$. Let $\tilde{x}_1=F0$, then $\tilde{x}_1\in P[0,a]$. Denote $\tilde{x}_{n+1}(t)=F\tilde{x}_n,\quad n=1,2,\cdots$. Copying the corresponding proof of Theorem 3.1, we can show that

$$\tilde{x}_{n+1}(t)\ge\tilde{x}_n(t),\quad t\in[\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}},\quad n=1,2,\cdots.$$

Since F is completely continuous, we can obtain that there exists $x^*\in P[0,a]$ such that $\tilde{x}_n\longrightarrow x^*$. The continuity of F and $\tilde{x}_{n+1}(t)=F\tilde{x}_n$ lead to $Fx^*=x^*$. We note that $f(t,0)>0,\ \forall\ t\in\mathbb{T}$, it implies that the zero function is not the solution of problem (1.1). Therefore, x^* is a positive solution of problem (1.1).

Now, since

$$|f(t,l_2)-f(t,l_1)|\le\kappa M|l_2-l_1|,\quad t\in\mathbb{T},\quad 0\le l_1,\ l_2\le a.$$

If $x_1,\ x_2\in P[0,a]$ and $x_2(t)\ge x_1(t)$, $t\in[\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}$, then

$$\begin{aligned}\|Fx_2-Fx_1\| &= \sup_{t\in[\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}}\left|\int_{\rho(t_1)}^{\sigma(t_3)}G(t,s)[f(s,x_2(s))-f(s,x_1(s))]\nabla s\right|\\ &\le\kappa M\sup_{t\in[\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}}\int_{\rho(t_1)}^{\sigma(t_3)}G(t,s)|x_2(s)-x_1(s)|\nabla s\\ &\le\kappa M\|x_2-x_1\|M^{-1}\\ &=\kappa\|x_2-x_1\|.\end{aligned}$$

Hence, we can deduce that

$$\|\tilde{x}_{n+2} - \tilde{x}_{n+1}\| = \|F\tilde{x}_{n+1} - F\tilde{x}_n\| \le \kappa^n \|F0 - 0\| = \kappa^n \|F0\|,$$

$$\|\tilde{x}_{n+k+2} - \tilde{x}_{n+1}\| \le (\kappa^{n+k} + \kappa^{n+k-1} + \cdots + \kappa^n)\|F0\| < \frac{\kappa^n}{1-\kappa}\|F0\|.$$

It implies that

$$\|F^{n+1}0 - x^*\| \le \frac{\kappa^n}{1-\kappa}\|F0\|.$$

The proof is completed.

4. Existence of n positive solutions

Theorem 4.1. *Assume there exist $2n$ positive numbers $a_1, \cdots, a_n, b_1, \cdots, b_n$ with $b_1 < a_1 < b_2 < a_2 < \cdots < b_n < a_n$ such that*
(E_1) $\max\{f(t, a_i) :\ t \in \mathbb{T}\} \le a_i M,\ \ \min\{f(t, \gamma b_i) :\ t \in [t_2/\alpha, t_2]_{\mathbb{T}}\} \ge b_i N,\ i = 1, 2, \cdots, n;$
(E_2) *$f(t, x_1) \le f(t, x_2)$ for any $t \in \mathbb{T},\ 0 \le x_1 \le x_2 \le a_n$.*
Then BVP (1.1) has n positive solutions $x_i^*,\ i = 1, 2, \cdots, n$ such that $b_i \le \|x_i^*\| \le a_i$ and $\lim_{n\to\infty} F^n \tilde{x}_i = x_i^*$, i.e.,

$$\lim_{n\to\infty} \sup_{t\in[\rho(t_1),\sigma^2(t_3)]_{\mathbb{T}}} |F^n\tilde{x}_i(t) - x_i^*(t)| = 0,$$

where $\tilde{x}_i(t) \equiv a_i,\ t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}},\ i = 1, 2, \cdots, n.$

Corollary 4.1. *Assume that (H_1')-(H_2') hold, and the following condition is satisfied*
(E') there exist $2(n-1)$ positive numbers $a_1 < b_2 < a_2 < \cdots < b_{n-1} < a_{n-1} < b_n$ such that

$$\max\{f(t, a_i) :\ t \in \mathbb{T}\} < a_i M, \quad i = 1, \cdots, n-1,$$

$$\min\{f(t, \gamma b_i) :\ t \in [t_2/\alpha, t_2]_{\mathbb{T}}\} > b_i N, \quad i = 2, \cdots, n.$$

Then BVP (1.1) has n positive solutions $x_i^*,\ i = 1, 2, \cdots, n$, and there exists a positive number a_n with $a_n > b_n$ such that $\lim_{n\to\infty} F^n\tilde{x}_i = x_i^*$, where $\tilde{x}_i(t) \equiv a_i,\ t \in [\rho(t_1), \sigma^2(t_3)]_{\mathbb{T}},\ i = 1, 2, \cdots, n.$

5. Examples

Example 5.1. Let $\mathbb{T} = [0, \frac{1}{7}] \cup [\frac{1}{4}, 1]$. Considering the following BVP:

$$\begin{cases} (x^{\Delta\nabla})^{\nabla}(t) + f(t, x) = 0, \quad t \in [0, 1]_{\mathbb{T}}, \\ x(\rho(0)) = 0 = x^{\Delta}(\rho(0)), \quad x^{\Delta}(\sigma(1)) = 2x^{\Delta}(1/4), \end{cases} \tag{5.1}$$

where $f(t, x) = \frac{2717}{21591}x^2 + 1$. It is easy to check that $f(t, 0) = 1 > 0$, for any $t \in [0, 1]_{\mathbb{T}}$.
By direct calculation, we have

$$d := \int_0^1 \Delta\tau - 2\int_0^{\frac{1}{4}} \Delta\tau = \frac{1}{2}, \quad 1 < 2 = \alpha < \frac{\int_0^1 \Delta\tau}{\int_0^{\frac{1}{4}} \Delta\tau} = 4,$$

and

$$
\begin{aligned}
M' &= \left[\int_0^1 g(s)\nabla s\right]^{-1}\\
&= \left[6\int_0^1 s(1-s)\nabla s\right]^{-1}\\
&= \left[6\int_0^{\frac{1}{7}} s(1-s)ds + 6\int_{\rho(\frac{1}{4})}^{\frac{1}{4}} s(1-s)\nabla s + 6\int_{\frac{1}{4}}^1 s(1-s)ds\right]^{-1}\\
&= \left[\frac{19}{7^3} + 6\left(\frac{1}{4}-\frac{1}{7}\right)\frac{1}{4}\left(1-\frac{1}{4}\right) + \frac{54}{4^3}\right]^{-1}\\
&= \tfrac{1372}{1399}.
\end{aligned}
$$

Choose $a = 3$, it is easy to check that $f(t,\cdot) : [0,3] \longrightarrow [0,+\infty)$ is nondecreasing for fixed $t \in [0,1]_{\mathbb{T}}$ and

$$
\max_{t\in[0,1]_{\mathbb{T}}} f(t,3) = \frac{2717\times 9}{21591} + 1 \le 3\cdot\frac{1372}{1399}.
$$

Let $\tilde{x}_0(t) \equiv 0$, for $n = 0,1,2,\cdots$, we have

$$
\begin{aligned}
\tilde{x}_{n+1}(t) &= 2\left[\int_0^{\frac{1}{4}}\left(\frac{1}{2}+s\right)\left(\frac{2717}{21591}\tilde{x}_n(s)+1\right)\nabla s + \int_{\frac{1}{4}}^1 (1-s)\left(\frac{2717}{21591}\tilde{x}_n(s)+1\right)\nabla s\right]\int_0^t u\Delta u\\
&\quad - \int_0^t\left(\int_s^t (u-s)\Delta u\right)\left(\frac{2717}{21591}\tilde{x}_n(s)+1\right)\nabla s.
\end{aligned}
$$

By Theorem 3.2, BVP (5.1) has one positive solution x^* such that $0 < \|x^*\| \le 3$ and $F^n 0 \longrightarrow x^*$. On the other hand, for any $0 \le x_1,\ x_2 \le 3$, we have

$$
\begin{aligned}
|f(x_1) - f(x_2)| &= \tfrac{2717}{21591}|x_1^2 - x_2^2|\\
&\le \tfrac{16302}{21591}|x_1 - x_2| = \tfrac{1372}{1399}\cdot\tfrac{1399}{1372}\tfrac{16302}{21591}|x_1 - x_2|\\
&= \tfrac{22806498}{29622852}M'|x_1 - x_2|.
\end{aligned}
$$

Then

$$
\|F^{n+1}0 - x^*\| \le \frac{(\frac{22806498}{29622852})^n}{1-\frac{22806498}{29622852}}\|F0\|.
$$

The first and second terms of this scheme are as follows.

$$
\tilde{x}_0(t) = 0.
$$

For $t \in [0,1/7]$,

$$
\begin{aligned}
\tilde{x}_1(t) &= t^2\left[\int_0^{\frac{1}{7}}\left(\frac{1}{2}+s\right)ds + \int_{\frac{1}{7}}^{\frac{1}{4}}\left(\frac{1}{2}+s\right)\nabla s + \int_{\frac{1}{4}}^1 (1-s)ds\right] - \tfrac{1}{2}\int_0^t (t-s)^2 ds\\
&= \tfrac{695}{1568}t^2 - \tfrac{t^3}{6}.
\end{aligned}
$$

For $t \in [1/4,1]$, since

$$
\begin{aligned}
\int_s^t (u-s)\Delta u &= \int_s^{\frac{1}{7}}(u-s)ds + \int_{\frac{1}{7}}^{\frac{1}{4}}(u-s)\Delta s + \int_{\frac{1}{4}}^t (u-s)du\\
&= \tfrac{-9}{2\times 4^2\times 7^2} + \tfrac{(s-t)^2}{2}.
\end{aligned}
$$

Thus

$$\begin{aligned}
\int_0^t \left(\int_s^t (u-s)\Delta u\right)\nabla s &= \int_0^{\frac{1}{7}} \left[\frac{-9}{2\times 4^2\times 7^2} + \frac{(s-t)^2}{2}\right] ds + \int_{\frac{1}{7}}^{\frac{1}{4}} \left[\frac{-9}{2\times 4^2\times 7^2} + \frac{(s-t)^2}{2}\right] \nabla s \\
&+ \int_{\frac{1}{4}}^{t} \left[\frac{-9}{2\times 4^2\times 7^2} + \frac{(s-t)^2}{2}\right] ds \\
&= \tfrac{27}{21952} - \tfrac{9t}{784} + \tfrac{t^3}{6}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\tilde{x}_1(t) &= \tfrac{695}{784}\left(\int_0^{\frac{1}{7}} udu + \int_{\frac{1}{7}}^{\frac{1}{4}} u\Delta u + \int_{\frac{1}{4}}^{t} udu\right) - \int_0^t \left(\int_s^t (u-s)\Delta u\right)\nabla s \\
&= \tfrac{695}{784}\left(\tfrac{-9}{2\times 4^2\times 7^2} + \tfrac{t^2}{2}\right) - \tfrac{27}{21952} + \tfrac{9t}{784} - \tfrac{t^3}{6} \\
&= -\tfrac{t^3}{6} + \tfrac{695t^2}{1568} + \tfrac{9t}{784} - \tfrac{23301}{3687936}.
\end{aligned}$$

Example 5.2. Let $\mathbb{T} = \{0, \frac{1}{4}, \frac{1}{3}\} \cup [\frac{1}{2}, 1]$. Considering the following BVP on $\mathbb{T}$

$$\begin{cases} (x^{\Delta\nabla})^{\nabla}(t) + \sqrt[3]{x(t)} = 0, \quad t \in [\frac{1}{4}, 1]_{\mathbb{T}}, \\ x(\rho(\frac{1}{4})) = 0 = x^{\Delta}(\rho(\frac{1}{4})), \quad x^{\Delta}(\sigma(1)) = \frac{3}{2}x^{\Delta}(\frac{1}{2}). \end{cases} \tag{5.2}$$

Some calculations lead to $d = \frac{1}{4},\ 1 < \alpha = \frac{3}{2} < 2,\ \gamma = \frac{1}{240}$.

By Example 5.1 in [18], we can know that

$$M = \left[\sup_{t\in[\rho(\frac{1}{4}),\sigma^2(1)]_{\mathbb{T}}} \int_{\rho(\frac{1}{4})}^{\sigma(1)} G(t,s)\nabla s\right]^{-1} = \frac{3456}{1877},$$

$$N = \left[\sup_{t\in[\rho(\frac{1}{4}),\sigma^2(1)]_{\mathbb{T}}} \int_{\frac{1}{3}}^{\frac{1}{2}} G(t,s)\nabla s\right]^{-1} = \frac{48}{7}.$$

Choose $a = 64,\ b = \frac{\sqrt{39}}{1911}$, it is easy to see that the nonlinear term f possesses the following properties

(a) $f : [0, +\infty) \longrightarrow [0, +\infty)$ is continuous;

(b) $f(x_1) \le f(x_2)$ for any $0 \le x_1 \le x_2 \le 64$;

(c) for any $r \in (0, 1)$, there exists $\eta_0 > 0$, such that

$$\sqrt[3]{rx} \ge (1+\eta_0)r\sqrt[3]{x}, \quad x \in [0, 64];$$

(d) $\max\{f(64)\} = \sqrt[3]{64} \le 64 \times \frac{3456}{1877} = aM$, $\min\{f(\frac{1}{240}\frac{\sqrt{39}}{1911})\} = \sqrt[3]{\frac{1}{240}\frac{\sqrt{39}}{1911}} \ge \frac{\sqrt{39}}{1911} \times \frac{48}{7} = bN$.

By Theorem 3.1, BVP (5.2) has a unique positive solution x^* such that $\frac{\sqrt{39}}{1911} \le \|x^*\| \le 64$ and $\lim_{n\to\infty} F^n\tilde{x} = x^*$, where $\tilde{x}(t) \equiv 64,\ t \in [0, 1]_{\mathbb{T}}$.

Let $x_0(t) \equiv 64,\ t \in [0, 1]_{\mathbb{T}}$. For $n = 0, 1, 2, \cdots$, we have

$$\begin{aligned}
\tilde{x}_{n+1}(t) &= 2\left[\int_0^{\frac{1}{4}} \left(\frac{1}{2} + s\right) \sqrt[3]{x_n(s)}\nabla s + \int_{\frac{1}{4}}^{1} (1-s)\sqrt[3]{x_n(s)}\nabla s\right] \int_0^t u\Delta u \\
&- \int_0^t \left(\int_s^t (u-s)\Delta u\right) \sqrt[3]{x_n(s)}\nabla s.
\end{aligned}$$

Remark 5.1. We note that $f(0) = 0$ in Example 5.2, however, the condition (C_2) in Theorem 3.2 asserts that $f(0) > 0$, we cannot solve Example 5.2 by use of Theorem 3.2. Thus, Theorem 3.1 and Theorem 3.2 do not contain each other. Furthermore, by Theorem 3.2 in [16], Theorem 3.2 and Theorem 3.4 in [17], and Theorem 3.1 in [18], the existence and uniqueness of positive solutions for BVP (5.1) and (5.2) can be obtained, however, we cannot give a way to find the solutions which will be useful from an application viewpoint. Therefore, our theorems improve and extend the main results of [16-18].

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