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Vibrating infinite string under general observation conditions and minimally smooth force

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Abstract. Existence of the classical solution $u(x,t) \in C^2(\mathbb{R}^2)$ to the Problem (1), (2) (shortly Problem $\mathcal{A}$):

$$Lu := u_{tt}(x,t) - a^2 u_{xx}(x,t) = f(x,t), \qquad (x,t) \in \mathbb{R}^2,\ a > 0, \tag{1}$$

under the observation conditions (the observed states) given at $t_1,\ t_2 \in \mathbb{R}$ with variable coefficients $A_1,\ B_1,\ A_2,\ B_2$ such that

$$\begin{aligned} A_1(x)u|_{t=t_1} + B_1(x)u_t|_{t=t_1} &= g_1(x), \qquad x \in \mathbb{R}, \\ A_2(x)u|_{t=t_2} + B_2(x)u_t|_{t=t_2} &= g_2(x), \qquad x \in \mathbb{R}, \end{aligned} \tag{2}$$

is proved. Here the coefficients $A_i, B_i,\ i = 1,2$, and g_1, g_2 are given functions smooth enough, $f \in C(\mathbb{R}^2)$, the directional derivative $\partial f/\partial t$ exists and $\partial f/\partial t \in C(\mathbb{R}^2)$.

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1 Introduction

Preliminaries: Observation (observability) problems have origins in control theory, see e.g. [2–7,9–15], where among others, attainability results are given for some second order hyperbolic equations. Especially for a given T $(0<T<\infty)$ and given complete state $u|_{t=T} = f,\ u_t|_{t=T} = g$ (where $\underline{u}$ is the unknown solution) the initial data $u|_{t=0},\ u_t|_{t=0}$ were found, for which the prescribed state (f,g) at $t = T$ was attained. Sometimes approximate attainability was studied.

The simplest variant of the observation problems for the second order hyperbolic equations is to find the initial data $u|_{t=0},\ u_t|_{t=0}$ for which some prescribed (or observed) partial state conditions – e.g. $u|_{t=t_i} = f_i,\ i = 1,2$ – are satisfied at two time instants $t_1,\ t_2 \in (0,T)$.

Earlier mainly observability of the oscillations $u(x,t)$ of finite (bounded) objects were considered (strings, membranes, plates, beams). For example, in [20] four essential observation problems for the vibrating $[0,l]$ string described by the PDE

$$Lu := u_{tt} - a^2 u_{xx} = 0$$

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were considered:

$$\begin{aligned} u|_{t=t_1} &= f_1, & u|_{t=t_2} &= f_2, \\ u|_{t=t_1} &= f_1, & u_t|_{t=t_2} &= f_2, \\ u_t|_{t=t_1} &= f_1, & u|_{t=t_2} &= f_2, \\ u_t|_{t=t_1} &= f_1, & u_t|_{t=t_2} &= f_2. \end{aligned}$$

The observability was established supposing that the observation time instants are small enough, $0 < t_1 < t_2 < 2l/a$, and the initial data $u|_{t=0} = \varphi$, $u_t|_{t=0} = \psi$ were known beforehand on some subinterval $[h_1, h_2] \subset [0, l]$.

Another result is presented in [16] for the oscillations of the $[0, l]$ string described by the Klein–Gordon equation:

$$(L + c)u(x,t) = 0, \qquad (x,t) \in [0,l] \times \mathbb{R},\ 0 < c \in \mathbb{R},$$

under the same four essential observation conditions as in [20], but given at arbitrary $t_1, t_2 \in \mathbb{R}$. Here a sufficient condition of the observability is: $(t_1 - t_2)a/l$ is rational.

This result is generalized in paper [19], namely it is proved that outside of a set V (of the couples $(t_1, t_2) \in \mathbb{R}^2$) the observation problems of [16] can be solved, where the Lebesgue measure of V equals zero. Paper [19] also contains useful results on the observation problem for some equations of fourth order.

Oscillations of a $[0, l]$ string satisfying a general wave equation with variable coefficients under a wide class of observation and boundary conditions are investigated in [17], where a sufficient condition of the observability is given in the terms of the asymptotics of the eigenvalues of the corresponding stationary boundary value problem. In the recent work [18] the observability is established in the case of inhomogeneous string equations for the infinite $(-\infty, \infty)$ and for the half infinite $[0, \infty)$ strings under the first observation conditions $u|_{t=t_i} = f_i$, $i = 1, 2$ (a generalization of Duhamel's principle is presented, too).

Finally, we refer to the works which are related to the observability problems for the equations describing oscillations of beams, plates and membranes. In chronological order they are [1], [8] and a part of [19].

In the present paper, at first (in Theorem 1.1) we consider Problem $\mathcal{A}$ under one of the following special cases of observation conditions (2):

(i)
$$\begin{gathered} A_1(x),\ A_2(x) \neq 0,\ x \in \mathbb{R}, \\ B_1,\ B_2 \equiv 0, \quad A_1,\ A_2,\ g_1,\ g_2 \in C^2(\mathbb{R}), \end{gathered}$$

(ii)
$$\begin{gathered} A_1(x),\ B_2(x) \neq 0,\ x \in \mathbb{R}, \\ A_2,\ B_1 \equiv 0, \quad A_1,\ g_1 \in C^2(\mathbb{R}), \quad B_2,\ g_2 \in C^1(\mathbb{R}), \end{gathered}$$

(iii)
$$\begin{gathered} A_2(x),\ B_1(x) \neq 0,\ x \in \mathbb{R}, \\ A_1,\ B_2 \equiv 0, \quad A_2,\ g_2 \in C^2(\mathbb{R}), \quad B_1,\ g_1 \in C^1(\mathbb{R}), \end{gathered}$$

(iv)
$$\begin{gathered} B_1(x),\ B_2(x) \neq 0,\ x \in \mathbb{R}, \\ A_1,\ A_2 \equiv 0, \quad B_1,\ B_2,\ g_1,\ g_2 \in C^1(\mathbb{R}). \end{gathered}$$

The most complicated case of conditions (2) is studied in Theorem 1.3.

Theorems 1.1 and 1.3 generalize the observability results of [18] related to the infinite string. Now, we formulate these theorems.

Theorem 1.1. *Let us suppose that one of the conditions* (i)–(iv) *is satisfied. Then Problem* $\mathcal{A}$ *has a classical* $C^2(\mathbb{R}^2)$ *solution.*

Remark 1.2. If the assumptions in the first lines of (i)–(iv) are violated, then Problem $\mathcal{A}$ can be solved if and only if the right hand sides of conditions (2) divided by the corresponding left hand side coefficients can be extended to $C^2(\mathbb{R})$ or $C^1(\mathbb{R})$ functions (depending on the case (i)–(iv)).

The statement of this remark can be easily derived from the proof of Theorem 1.1, where the given functions g_i appears only in the combinations of the type g_i/A_i or g_i/B_i, $i = 1, 2$.

In Theorem 1.1 the smoothness of the left and right hand side functions are the same (they are in accordance). As a complement to this theorem, we formulate Theorem 1.3, where at every $x \in \mathbb{R}$ the left hand side of the observation conditions contain two terms of different smoothness.

Theorem 1.3. *Let us suppose, that*

$$A_1(x), A_2(x), B_1(x), B_2(x) \neq 0,\ x \in \mathbb{R}, \quad A_1,\ A_2,\ B_1,\ B_2,\ g_1,\ g_2 \in C^1(\mathbb{R}).$$

Then Problem $\mathcal{A}$ *has a classical solution* $u(x,t) \in C^2(\mathbb{R}^2)$.

We emphasize, that Theorems 1.1 and 1.3 guarantee only the existence of the solutions $u(x,t) \in C^2(\mathbb{R}^2)$ to Problem $\mathcal{A}$. In fact the solution is not unique (a degree of freedom can be derived from the proofs).

2 Proof of Theorem 1.1

Before we investigate the four cases related to Theorem 1.1, let us formulate the following lemma.

Lemma 2.1. *A delay differential equation with the form of*

$$H'(x+T) + c\,H'(x-T) = h(x), \qquad c = \pm 1,\ x \in \mathbb{R},\ h \in C^1(\mathbb{R}), \tag{2.1}$$

has a solution $H \in C^2(\mathbb{R})$. *The solution is not unique.*

Proof. Instead of directly finding a solution $H \in C^2(\mathbb{R})$ we will construct a function $H' \in C^1(\mathbb{R})$ which satisfies equation (2.1), from which we can define a solution $H(x) := \int_0^x H'(s)ds + C \in C^2(\mathbb{R})$ for any $C \in \mathbb{R}$.

First take an almost arbitrary function H' defined on $[-T,T]$, $H' \in C^1([-T,T])$ with the only restrictions that

$$H'(T) + c\,H'(-T) = h(0), \tag{2.2}$$
$$H''(T) + c\,H''(-T) = h'(0). \tag{2.3}$$

Such a function is for example

$$H'(x) = \frac{Th'(0) - h(0)}{4T^3}(x+T)^3 + \frac{3h(0) - 2Th'(0)}{4T^2}(x+T)^2, \qquad x \in [-T,T].$$

Starting with the function $H' \in C^1[-T,T]$, we define the extensions of H', H'' from $[-T,T]$ to

$$(T, 3T],\ (3T, 5T], \ldots \quad \text{and to} \quad [-3T, -T),\ [-5T, -3T), \ldots$$

in accordance with

$$H^{(j)}(x) := -c\,H^{(j)}(x-2T) + h^{(j-1)}(x-T), \quad j=1,2, \quad x \in \mathbb{R}; \tag{2.4}$$

$$H^{(j)}(x) := -c\,H^{(j)}(x+2T) + ch^{(j-1)}(x+T), \quad j=1,2, \quad x \in \mathbb{R} \tag{2.5}$$

as continuous functions on these intervals, thus step by step, we obtain the following representations:

$$\begin{gathered} H^{(j)}(x) = (-c)^n H^{(j)}(x-2nT) + \sum_{k=1}^{n} (-c)^{k-1} h^{(j-1)}(x-(2k-1)T), \\ x \in I_n := ((2n-1)T, (2n+1)T], \quad n \in \mathbb{N}, \ j=1,2, \end{gathered} \tag{2.6}$$

$$\begin{gathered} H^{(j)}(x) = (-c)^n H^{(j)}(x+2nT) + \sum_{k=1}^{n} c(-c)^{k-1} h^{(j-1)}(x+(2k-1)T), \\ x \in I_{-n} := [(-2n-1)T, (-2n+1)T), \quad n \in \mathbb{N}, \ j=1,2. \end{gathered} \tag{2.7}$$

The continuity of $H^{(j)}(x)$ inside the intervals I_n, I_{-n}, $n \in \mathbb{N}$ is trivial, as well as that $H'(x)$ satisfies equation (2.1).

To check the continuity of H', H'' at $x = T$ we substitute $x = T + 0$ into (2.4), and by using (2.2) and (2.3), we have

$$\begin{aligned} H^{(j)}(T+0) &= -c\,H^{(j)}(-T+0) + h^{(j-1)}(0+0) \\ &= -c\,H^{(j)}(-T) + h^{(j-1)}(0) = H^{(j)}(T), \quad j=1,2. \end{aligned} \tag{2.8}$$

Let us suppose, the continuity of $H^{(j)}$ at $x = (2n+1)T$, i.e.

$$H^{(j)}((2n+1)T+0) = H^{(j)}((2n+1)T). \tag{2.9}$$

Then to check the continuity of H', H'' at $x = (2n+3)T$, from (2.4) and (2.9) we have

$$\begin{aligned} H^{(j)}((2n+3)T+0) &= -c\,H^{(j)}((2n+1)T+0) + h^{(j-1)}((2n+2)T+0) \\ &= -c\,H^{(j)}((2n+1)T) + h^{(j-1)}((2n+2)T) \\ &= H^{(j)}((2n+3)T), \quad j=1,2. \end{aligned} \tag{2.10}$$

By induction, we have the continuity of $H^{(j)}(x)$, $j=1,2$ for all $x \in [-T,\infty)$.

Using an analogous procedure, the continuity of $H^{(j)}$, $j=1,2$ can be proved at the points $x = -3T, -5T, \dots$, and thus, the continuity of $H^{(j)}$, $j=1,2$ on the interval $(-\infty, T]$. So, the functions $H^{(j)} \in C(\mathbb{R})$, $j=1,2$ are constructed, especially $H^1 \in C^1(\mathbb{R})$, and with that $H \in C^2(\mathbb{R})$. □

Remark 2.2. Our construction for the solution H of (2.1) is valid not only for $c = \pm 1$, but also for any $c \in \mathbb{R}$ (replacing c with c^{-1} in (2.5)), but those values are not required for the proof of Theorem 1.1.

Note also, that in the proof, the interval $[x_0 - T, x_0 + T]$ with arbitrary $x_0 \in \mathbb{R}$ and arbitrary function $H' \in C^1([x_0 - T, x_0 + T])$ can be used instead of the initial interval $[-T, T]$ and initial function $H' \in C^1([-T, T])$, if

$$\begin{aligned} H'(x_0+T) + cH'(x_0-T) &= h(x_0), \\ H''(x_0+T) + cH''(x_0-T) &= h'(x_0). \end{aligned}$$

A more complicated replacement of the initial data is the following: give a system of disjoint intervals $[a_i, b_i]$, for which $\sum(b_i - a_i) = 2T$ and prescribe $H'|_{[a_i,b_i]}$ such that equalities (2.4), (2.5) define H', H'' as $C(\mathbb{R})$ functions.

The traditional approach on observation problems for the string equation is to find directly the initial data $u|_{t=0}$, $u_t|_{t=0}$ using the observation conditions. Our goal is to describe the whole oscillation process, i.e. to find $u(x,t)$ for all $(x,t) \in \mathbb{R}^2$ using indirectly the complete state (u, u_t) of the string at the time instant $t = t_1$. To this effect, we will use the following representation of the solutions of string vibration (1):

$$u(x,t) = F(x - a(t - t_1)) + G(x + a(t - t_1)) + v(x,t), \tag{2.11}$$

where v is the solution of Cauchy problem (2.12):

$$\begin{gathered} v \in C^2(\mathbb{R}^2), \quad Lv := v_{tt}(x,t) - a^2 v_{xx}(x,t) = f(x,t), \quad (x,t) \in \mathbb{R}^2, \\ v(x,t_1) = v_t(x,t_1) = 0, \quad x \in \mathbb{R}. \end{gathered} \tag{2.12}$$

The existence of the solution v to Problem (2.12) is proved in [18, Theorem B].

After defining the left travelling function F and a right travelling function G for all $x \in \mathbb{R}$ (of course we need that $F,\ G \in C^2$) the traditional initial data $(u|_{t=0}, u_t|_{t=0})$ can be found by substituting $t = 0$ into (2.11) and into the derivative of (2.11).

Let us introduce the notation $T := a(t_2 - t_1)$. Using the representation (2.11), the observation conditions (2) obtain the form of

$$A_1(x)(F(x) + G(x)) + aB_1(x)(-F'(x) + G'(x)) = g_1(x), \qquad x \in \mathbb{R}, \tag{2.13}$$

$$\begin{aligned} &A_2(x)(F(x - T) + G(x + T) + v(x,t_2)) \\ &\quad + aB_2(x)(-F'(x - T) + G'(x + T) + v_t(x,t_2)) = g_2(x), \end{aligned} \qquad x \in \mathbb{R}. \tag{2.14}$$

In Theorem 1.1, we consider the cases, when two of the coefficients $A_1,\ A_2,\ B_1,\ B_2$ are constant zeroes.

- The case (i), i.e. when $B_1 \equiv B_2 \equiv 0$ can be found in [18].
- When (ii) holds, then we need to solve the following system of equations for the functions F and G:

  $$\begin{aligned} A_1(x)(F(x) + G(x)) &= g_1(x), \qquad x \in \mathbb{R}, \\ aB_2(x)(-F'(x - T) + G'(x + T) - v_t(x,t_2)) &= g_2(x), \qquad x \in \mathbb{R}. \end{aligned}$$

  After differentiating with respect to x and shifting, we can express $G'(x + T)$ from the first equation, and substituting it into the second, we get that

  $$F'(x + T) + F'(x - T) = v_t(x,t_2) - \frac{g_2(x)}{aB_2(x)} + \left(\frac{g_1(x + T)}{A_1(x + T)}\right)', \quad x \in \mathbb{R}.$$

  This differential equation fits into the statement of Lemma 2.1 with

  $$F = H,\ c = 1 \quad \text{and} \quad v_t(x,t_2) - \frac{g_2(x)}{aB_2(x)} - \left(\frac{g_1(x + T)}{A_1(x + T)}\right)' = h(x), \quad x \in \mathbb{R},$$

  so it can be solved, and with the help of the solution $F \in C^2(\mathbb{R})$ we can also determine the function $G \in C^2(\mathbb{R})$, thus solving Problem $\mathcal{A}$ in this special case.

- The instance of (iii) is essentially the same as the previous case after interchanging the notations t_1 and t_2, A_1 and A_2, B_1 and B_2, g_1 and g_2 in conditions (2).

- In the case when (iv) is satisfied, we need to solve the following system of equations for the functions F and G:

$$\begin{aligned} aB_1(x)(-F'(x)+G'(x)) &= g_1(x), \qquad & x\in\mathbb{R},\\ aB_2(x)(-F'(x-T)+G'(x+T)+v_t(x,t_2)) &= g_2(x), \qquad & x\in\mathbb{R}. \end{aligned}$$

  After shifting, we can express $G'(x+T)$ from the first equation, and substituting it into the second, we get that

$$F'(x+T)-F'(x-T)=\frac{g_2(x)}{aB_2(x)}-v_t(x,t_2)-\frac{g_1(x+T)}{B_1(x+T)}, \quad x\in\mathbb{R}.$$

  This differential equation fits into the statement of Lemma 2.1 with

$$F=H,\ c=-1 \quad \text{and} \quad \frac{g_2(x)}{aB_2(x)}-v_t(x,t_2)-\frac{g_1(x+T)}{B_1(x+T)}=h(x), \quad x\in\mathbb{R},$$

  so it can be solved, and with the help of the solution $F\in C^2(\mathbb{R})$ we can also determine the function $G'\in C^1(\mathbb{R})$ from the first observation condition. Finally, the pair (F,G) – with any primitive function G of G' – define a solution u of Problem $\mathcal{A}$ with the help of (2.11).

3 Proof of Theorem 1.3

In the case of Theorem 1.3 our aim is the same as in Theorem 1.1: to describe the whole oscillation process, i.e. to find $u(x,t)$ for all $(x,t)\in\mathbb{R}^2$ using the complete state $(u,u_t)=(\varphi,\psi)$ of the string at the time instant $t=t_1$, because the usage of the travelling wave representation of the solutions of (1) is not enough effective. So with the help of D'Alembert's formula and Duhamel's principle, the solutions of (1) can be written in the form of

$$u(x,t)=\frac{\varphi(x-a(t-t_1))+\varphi(x+a(t-t_1))}{2}+\frac{1}{2a}\int_{x-a(t-t_1)}^{x+a(t-t_1)}\psi(s)ds+v(x,t), \tag{3.1}$$

$$\begin{aligned} u_t(x,t)=&\,a\frac{-\varphi'(x-a(t-t_1))+\varphi'(x+a(t-t_1))}{2}\\ &+\frac{\psi(x+a(t-t_1))+\psi(x-a(t-t_1))}{2}+v_t(x,t), \end{aligned} \tag{3.2}$$

where again, v is the solution of Cauchy problem (2.12).

After defining (φ,ψ) for all $x\in\mathbb{R}$ (of course we need that $\varphi\in C^2$, $\psi\in C^1$), the traditional initial data $(u|_{t=0},u_t|_{t=0})$ can be found by substituting $t=0$ into (3.1) and (3.2).

After substituting $t=t_2$ into (3.1) and (3.2), we get the following equalities:

$$u(x,t_2)=\frac{\varphi(x-T)+\varphi(x+T)}{2}+\frac{\Psi(x+T)-\Psi(x-T)}{2a}+v(x,t_2), \tag{3.3}$$

$$u_t(x,t_2)=a\frac{-\varphi'(x-T)+\varphi'(x+T)}{2}+\frac{\psi(x+T)+\psi(x-T)}{2}+v_t(x,t_2), \tag{3.4}$$

where $\Psi(x) = \int_0^x \psi(s)ds$ and we used the notation $T := a(t_2 - t_1)$ for the sake of transparency.

Now, the observation conditions (2) obtain the following form:

$$A_1(x)\varphi(x) + B_1(x)\psi(x) = g_1(x), \tag{3.5}$$

$$\begin{aligned} A_2(x)\left(\frac{\varphi(x+T)+\varphi(x-T)}{2} + \frac{1}{2a}\int_{x-T}^{x+T}\psi(s)ds + v(x,t_2)\right) \\ + B_2(x)\left(a\frac{\varphi'(x+T)-\varphi'(x-T)}{2} + \frac{\psi(x+T)+\psi(x-T)}{2} + v_t(x,t_2)\right) = g_2(x). \end{aligned} \tag{3.6}$$

By equation (3.5), we have

$$\begin{aligned} \psi(x) &= \frac{g_1(x)}{B_1(x)} - \frac{A_1(x)}{B_1(x)}\varphi(x), \qquad x \in \mathbb{R}, \\ \psi'(x) &= \left(\frac{g_1(x)}{B_1(x)}\right)' - \left(\frac{A_1(x)}{B_1(x)}\right)'\varphi(x) - \frac{A_1(x)}{B_1(x)}\varphi'(x), \qquad x \in \mathbb{R}. \end{aligned} \tag{3.7}$$

After dividing both sides of (3.6) by $A_2(x)$ and differentiating with respect to x we get

$$\begin{aligned} &\left(\frac{\varphi(x+T)+\varphi(x-T)}{2} + \frac{1}{2a}\int_{x-T}^{x+T}\psi(s)ds + v(x,t_2)\right)' \\ &\quad + \left(\frac{B_2(x)}{A_2(x)}\left(a\frac{\varphi'(x+T)-\varphi'(x-T)}{2} + \frac{\psi(x+T)+\psi(x-T)}{2} + v_t(x,t_2)\right)\right)' \\ &= \left(\frac{g_2(x)}{A_2(x)}\right)'. \end{aligned} \tag{3.8}$$

By substituting (3.7) into (3.8), we get a second order differential equation for φ of the form:

$$\begin{aligned} &D_1(x)\varphi''(x+T) + D_2(x)\varphi'(x+T) + D_3(x)\varphi(x+T) \\ &\quad = E_1(x)\varphi''(x-T) + E_2(x)\varphi'(x-T) + E_3(x)\varphi(x-T) + h(x), \qquad x \in \mathbb{R}, \end{aligned} \tag{3.9}$$

where the coefficients $E_i(x), D_i(x)$, $i = 1,2,3$ and the function $h(x)$ are known continuous functions for all $x \in \mathbb{R}$. The exact formulae for $E_i(x), D_i(x)$, $i = 1,2,3$ and for $h(x)$ are not important for the proof, but if the reader is interested, they are the following:

$$D_1(x) = E_1(x) = \frac{aB_2(x)}{2A_2(x)} \neq 0, \quad x \in \mathbb{R},$$

$$D_2(x) = \frac{1}{2} + \left(\frac{aB_2(x)}{2A_2(x)}\right)' - \frac{B_2(x)A_1(x+T)}{2A_2(x)B_1(x+T)},$$

$$E_2(x) = -\frac{1}{2} + \left(\frac{aB_2(x)}{2A_2(x)}\right)' + \frac{B_2(x)A_1(x-T)}{2A_2(x)B_1(x-T)},$$

$$D_3(x) = -\frac{A_1(x+T)}{2aB_1(x+T)} - \left(\frac{B_2(x)}{A_2(x)}\right)'\frac{A_1(x+T)}{2B_1(x+T)} - \frac{B_2(x)}{2A_2(x)}\left(\frac{A_1(x+T)}{B_1(x+T)}\right)',$$

$$E_3(x) = -\frac{A_1(x-T)}{2aB_1(x-T)} + \left(\frac{B_2(x)}{A_2(x)}\right)'\frac{A_1(x-T)}{2B_1(x-T)} + \frac{B_2(x)}{2A_2(x)}\left(\frac{A_1(x-T)}{B_1(x-T)}\right)',$$

$$\begin{aligned} h(x) := &\left(-\frac{1}{a} - \left(\frac{B_2(x)}{A_2(x)}\right)'\right)\left(\frac{g_1(x+T)}{2B_1(x+T)} - \frac{g_1(x-T)}{2B_1(x-T)}\right) \\ &- (v(x,t_2))' - \left(\frac{B_2(x)}{A_2(x)}\right)'(v_t(x,t_2))'. \end{aligned}$$

Due to the conditions of Theorem 1.3, they are continuous functions defined on whole $\mathbb{R}$.

To solve the delay differential equation (3.9) we will use the method of step by step extensions. To this effect, we prove the following lemma.

Lemma 3.1. *If there is an $x_0 \in \mathbb{R}$ and a given $\Phi_1 \in C^2([x_0 - T, x_0 + T])$ such that*

$$\begin{aligned} &D_1(x_0)\Phi_1''(x_0+T) + D_2(x_0)\Phi_1'(x_0+T) + D_3(x_0)\Phi_1(x_0+T) \\ &\quad = E_1(x_0)\Phi_1''(x_0-T) + E_2(x_0)\Phi_1'(x_0-T) + E_3(x_0)\Phi_1(x_0-T) + h(x_0), \end{aligned} \tag{3.10}$$

then there is a $\Phi_2 \in C^2([x_0 + T, x_0 + 3T])$ such that

$$\varphi(x) := \begin{cases} \Phi_1(x), & \text{if } x \in [x_0 - T, x_0 + T] \\ \Phi_2(x), & \text{if } x \in [x_0 + T, x_0 + 3T] \end{cases}$$

satisfies (3.9) for all $x \in [x_0, x_0 + 2T]$ and $\varphi \in C^2([x_0 - T, x_0 + 3T])$.

Proof. The function φ will satisfy (3.9) for all $x \in [x_0, x_0 + 2T]$, if we find a function Φ_2 such that

$$\begin{aligned} &D_1(x)\Phi_2''(x+T) + D_2(x)\Phi_2'(x+T) + D_3(x)\Phi_2(x+T) \\ &\quad = E_1(x)\Phi_1''(x-T) + E_2(x)\Phi_1'(x-T) + E_3(x)\Phi_1(x-T) + h(x) \end{aligned} \tag{3.11}$$

holds for all $x \in [x_0, x_0 + 2T]$. Equation (3.11) can be written as the following second order linear differential equation for Φ_2:

$$\widetilde{D_1}(x)\Phi_2''(x) + \widetilde{D_2}(x)\Phi_2'(x) + \widetilde{D_3}(x)\Phi_2(x) = \widetilde{h}(x), \qquad x \in [x_0 + T, x_0 + 3T] \tag{3.12}$$

where the right-hand side

$$\widetilde{h}(x) = E_1(x-T)\Phi_1''(x-2T) + E_2(x-T)\Phi_1'(x-2T) + E_3(x-T)\Phi_1(x-2T) + h(x-T)$$

is a known $C([x_0 + T, x_0 + 3T])$ function and

$$\widetilde{D_1}(x) = D_1(x-T), \quad \widetilde{D_2}(x) = D_2(x-T), \quad \widetilde{D_3}(x) = D_3(x-T).$$

It is well known, that (3.12) can be uniquely solved with the initial conditions

$$\Phi_2(x_0+T) = \Phi_1(x_0+T), \quad \Phi_2'(x_0+T) = \Phi_1'(x_0+T) \tag{3.13}$$

and the solution $\Phi_2 \in C^2([x_0 + T, x_0 + 3T])$.

With this Φ_2, the function φ satisfies (3.9) and it is obviously C^2 smooth in the intervals $[x_0 - T, x_0 + T]$ and $[x_0 + T, x_0 + 3T]$. Due to the initial conditions (3.13), φ is well-defined and continuously differentiable in $x_0 + T$. It only remains to show, that φ is continuously differentiable two times in $x_0 + T$, namely that

$$\lim_{r \to 0^+} \varphi''(x_0 + T + r) = \lim_{r \to 0^-} \varphi''(x_0 + T + r), \tag{3.14}$$

but this easily follows from (3.10), (3.11) and (3.13). □

Note, that the statement of Lemma 3.1 remains true with the previous function $\Phi_1 \in C^2([x_0 - T, x_0 + T])$ and here also exists $\Phi_2 \in C^2([x_0 - 3T, x_0 - T])$ such that

$$\varphi(x) := \begin{cases} \Phi_1(x), & \text{if } x \in [x_0 - T, x_0 + T] \\ \Phi_2(x), & \text{if } x \in [x_0 - 3T, x_0 - T] \end{cases}$$

satisfies (3.9) for all $x \in [x_0 - 2T, x_0]$ and $\varphi \in C^2([x_0 - 3T, x_0 + T])$. This statement easily follows from the rearranging of (3.11).

To solve our observation problem, let us choose a $\varphi \in C^2([-T, T])$ such that

- φ satisfies (3.6) at $x = 0$ (using the first equation of (3.7)), i.e.:

$$\begin{aligned} &a\frac{B_2(0)}{A_2(0)}\frac{\varphi'(T) - \varphi'(-T)}{2} \\ &\quad = \frac{g_2(0)}{A_2(0)} - \frac{\varphi(T) + \varphi(-T)}{2} - \frac{1}{2a}\int_{-T}^{T}\left(\frac{g_1(s)}{B_1(s)} - \frac{A_1(s)}{B_1(s)}\varphi(s)\right) ds - v(0, t_2) \\ &\qquad - \frac{B_2(0)}{A_2(0)}\left(\frac{g_1(T)}{2B_1(T)} - \frac{A_1(T)}{2B_1(T)}\varphi(T) + \frac{g_1(-T)}{2B_1(-T)} - \frac{A_1(-T)}{2B_1(-T)}\varphi(-T) + v_t(0, t_2)\right). \end{aligned} \tag{3.15}$$

- the values of $\varphi''(T)$ and $\varphi''(-T)$ are such that (3.9) also holds at $x = 0$, that is:

$$\begin{aligned} &D_1(0)\varphi''(T) - E_1(0)\varphi''(-T) \\ &\quad = E_2(0)\varphi'(-T) - D_2(0)\varphi'(T) + E_3(0)\varphi(-T) - D_3(0)\varphi(T) + h(0). \end{aligned} \tag{3.16}$$

Then by using Lemma 3.1 and the method of step by step extensions, we get a solution $\varphi \in C^2(\mathbb{R})$ of (3.9). We define the initial speed ψ according to (3.7):

$$\psi(x) = \frac{g_1(x)}{B_1(x)} - \frac{A_1(x)}{B_1(x)}\varphi(x), \qquad x \in \mathbb{R}.$$

Here $(\varphi, \psi) \in C^2(\mathbb{R}) \times C^1(\mathbb{R})$ and they satisfy system (3.5), (3.6) because of (3.15) and (3.16). Thus we got a solution of the observation Problem $\mathcal{A}$. □

Remark 3.2. The proof of Theorem 1.3 works also under the weaker conditions

$$\begin{gathered} A_i, B_i, g_i \in C^1, \quad i = 1, 2, \\ A_2 \not\equiv 0, \; B_1 \not\equiv 0, \\ \frac{g_1}{B_1}, \frac{A_1}{B_1}, \frac{g_2}{A_2} \text{ can be extended to } C^1(\mathbb{R}), \\ \frac{B_2}{A_2} \text{ can be extended to a function in } C^1(\mathbb{R}) \text{ having no zeros.} \end{gathered}$$

This comment can be easily verified, because we use only these four fractions during the proof of Theorem 1.3 (via the functions $D_1, D_2, D_3, E_1, E_2, E_3, h$).

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